# PION PHOTOPRODUCTION AND FIXED-t-DISPERSION 

RELATIONS

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#### Abstract

We review the dispersion theoretic models of the isobar approximation which predict single pion photoproduction in the region of the $\Delta(1236)$-resonance. The differences between the various models are explained and their consequences discussed. Considerable efforts are made to look for further improvements of the theory from the confrontation of theory and experiment. Anticipating the development of experimental techniques we discuss finally an example for a more complete photoproduction experiment. Specifically, we study the usefulness of a $\pi^{0}$-photoproduction experiment with polarized $\gamma^{\prime}$ s and/or polarized target for the region of the $\Delta(1236)$ resonance.


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## INTRODUCTION

In this paper we continue our analysis of single pion photoproduction in the region of the $\Delta(1236)$-resonance by means of fixed-t-dispersion relations. The main idea of a previous paper ${ }^{1}$ was to establish in the whole region of the first pion nucleon resonance $\Delta(1236)$ the ansatz of $\mathrm{Ball}^{2}$ as a zero order approximation for the total amplitudes. Ball retained only the large $\mathrm{M}_{1+}^{3 / 2}$-contribution of $\Delta(1236)$ in the dispersion integral. To get then an absolute prediction for the angular distributions, it was necessary to know the resonant multipole $\mathrm{M}_{1+}^{3 / 2}$ taken in Refs. 1 and 2 from the work of Chew, et al. ${ }^{3}$ Also the pion nucleon scattering phase shifts were needed in order to apply the Watson theorem for the calculation of the imaginary parts of the amplitudes. Because of the dominance of the $\Delta(1236)$ resonance, this isobar-approximation works surprisingly well even at higher energies if it is applied there to the slowly varying real background amplitude. ${ }^{4,5}$

Since the publication of Ref. 1 - four years ago - the same subject has been treated by several authors again using fixed-t-dispersion relations and again exploiting some type of isobar approximation. These new efforts have had three general objectives:

1. to improve the theory of the $\Delta(1236)$ resonance in photoproduction, especially the prediction of the small $\mathrm{E}_{1+}^{3 / 2} 6,7,8,9,10 \mathrm{a}, 11,12,13$
2. to retain also some of the smaller $s, p$ imaginary parts under the dispersion integral ${ }^{10 a, 13}$; and
3. to compare systematically the theoretical predictions with the experimental data available, in order to look for deficiencies of the theory. ${ }^{14,15,16,17}$

The achievement of this work was a refinement of the calculations, partially through the more involved use of computers. New ideas, which would supplement the isobar approximation essentially, were not presented. For example, in the present formulation of the isobar approximation the question of high energy contributions arising from the dispersion integrals is completely unsettled. This question deserves particular attention, if more refined calculations of partial amplitudes are started (see, e.g., Ref. 12).

In this paper we consider in Section I some typical isobar approximations for the evaluation of the lowest partial amplitudes. We then turn in Section II to a comparison with the experiments, where we concentrate on situations, which are particularly sensitive to the present approximations. Lastly we discuss in Section III possible new experiments in the region of the $\Delta(1236)$ resonance, which should prove useful for the further development of the theory.

A word to motivate our method of confronting theory and experiment may be useful. There are at least two ways to discuss the error in the experimental prediction - implied by the uncertainty of the theory. One can include all known systematic errors in order to obtain the total error of the experimental prediction. The result is represented by an error band. The uncertainty of the theory is thus demonstrated in a very obvious way (see, e.g., 10a). Also one can calculate the effect of each known systematic error in order to identify its particular contribution in the crror band of the first method. In this way - which we prefer - one can try to resolve a discrepancy in terms of the main errors of the theory. One can then check if a possible explanation is consistent with all available information. This check is not possible with the first method. One may then overestimate the agreement between theory and experiment. The danger of the second method lies in underestimating the
agreement. But it seems to be the only systematic way to analyze the experimental data. ${ }^{15}$ One can learn from a recent note ${ }^{10 \mathrm{~b}}$ that both methods may lead to quite different conclusions.

Finally we mention that we use in the formulae units such that $\hbar=c=m_{\pi}$ $=1$. Amplitudes are usually given in units $10^{-2} \lambda=10^{-2} \pi / \mathrm{m}_{\pi} \mathrm{c}$, so that, e.g., $\operatorname{Im} M_{1+}^{3 / 2}$ is of the order 1 around the resonance. Cross sections, etc. , are given in units $\mu \mathrm{b} /$ ster.

## I. EVALUATION OF PARTIAL AMPLITUDE DISPERSION RELATIONS

## IN PION-PHOTOPRODUCTION

## A. Dispersion Relations

At present partial amplitude dispersion relations are the best tool in analysing pion photoproduction in the region of the first pion nucleon resonance, since they allow us to incorporate most efficiently our phenomenological knowledge about the pion nucleon final state. Consider therefore a dispersion relation for the parity conserving helicity amplitudes $H_{\lambda}^{\mathrm{J} \pm}(\mathrm{W})$

$$
\begin{gathered}
\operatorname{Re~}_{\lambda}^{\mathrm{J} \mathrm{ \pm}, \mathrm{I}}(\mathrm{~W})=\mathrm{H}_{\lambda, \mathrm{inh}}^{\mathrm{J} \mathrm{ \pm} \mathrm{I}}(\mathrm{~W})+\frac{1}{\pi} \mathrm{P} \int_{\mathrm{M}+1}^{\infty} \mathrm{dW} \frac{\operatorname{Im} \mathrm{H}_{\lambda}^{\mathrm{J}, \mathrm{I}}\left(\mathrm{~W}^{\mathrm{t}}\right)}{\mathrm{W}^{\mathrm{l}}-\mathrm{W}} \\
\lambda=1 / 2,3 / 2
\end{gathered}
$$

The relation of these amplitudes to the conventional multipoles $\mathrm{E}_{\ell \pm}^{\mathrm{I}}, \mathrm{M}_{\ell \pm}^{\mathrm{I}}{ }^{3}$ is given in the appendix. From fixed-t-dispersion relations ${ }^{2}$ one obtains an
explicit result for the inhomogeneous term $H_{\lambda, \text { inh }}^{\mathrm{J} \pm, \mathrm{I}}(\mathrm{W}) .{ }^{20}$
$\mathrm{H}_{\lambda, \text { inh }}^{\mathrm{J} \pm, \mathrm{I}}(\mathrm{W})=$ p.t.c. $+\frac{1}{\pi} \int_{M+1}^{\infty} d W^{\prime} \frac{\operatorname{Im} H_{\lambda}^{\mathrm{JF}, \mathrm{I}}\left(\mathrm{W}^{\prime}\right)}{\mathrm{W}^{\mathrm{I}}+\mathrm{W}}$

$$
\begin{equation*}
+\sum_{J^{\prime} I^{\prime} \lambda^{i}=\frac{1}{2}}^{3 / 2}\left(\operatorname{Im} H_{\lambda^{\prime}}^{J^{\prime}-, I^{!}}\left(W^{\prime}\right) G_{\lambda \lambda^{\prime}}^{J I, J^{\prime} I^{\prime}}\left(\mp W, W^{t}\right)-\operatorname{Im} H_{\lambda^{\prime}}^{J^{\prime}+,} I^{\prime}\left(W^{\imath}\right) G_{\lambda \lambda^{i}}^{\left.J, J^{\prime} I^{\prime}\left(\mp W,-W^{t}\right)\right)}\right. \tag{I.1b}
\end{equation*}
$$

In this equation p.t.c. denotes the well-known pole term contribution. ${ }^{20}$ $G_{\lambda \lambda^{\prime}}^{J, J^{\prime} I^{\eta}}\left(W, W^{\mathbf{\prime}}\right)$ are known kinematical functions which represent the coupling of the different partial amplitudes following from fixed-t-dispersion relations. The sum in (I. 1b) - infinite in principle - has been truncated at $J^{\prime}=3 / 2$, so that the system I. 1 is finite. It is expected to be valid in the region of the first resonance, where the convergence of the series in (I.1b) is guaranteed, if the Mandelstam representation is valid. ${ }^{2}$

The largest contributions to the integrals in (I.1) arise from the imaginary parts of the first resonance, i.e., $\operatorname{Im} H_{\lambda}^{3 / 2-}, \lambda=1 / 2,3 / 2$. Therefore the evaluation of ( 1.1 ) has always been based on some type of isobar approximation, of which we consider three examples in the following. In this discussion we do not treat the partial amplitudes of the first resonance, since they deserve more sophisticated techniques. We shall rely on the work in Ref. 12 for the theory of the first resonance.

## B. Pure Isobar Approximation

We shall understand as "pure isobar approximations" if only $\operatorname{Im} H_{\lambda}^{3 / 2-}$ is retained in the integrands of the set (I.1). One then obtains from (I.1) for $\mathrm{J}, \mathrm{I} \neq 3 / 2, \mathrm{P} \neq+1$

$$
\begin{equation*}
\operatorname{Re} H_{\lambda}^{J \pm}(W) \approx H_{\lambda, \mathrm{inh}^{\prime}}^{J \pm}(W) \tag{I.2}
\end{equation*}
$$

In this approximation the remaining integrals in (I.1b) can be very well evaluated by the narrow width approximation yielding the convenient expressions

$$
\begin{equation*}
\mathrm{H}_{\lambda, \mathrm{inh}}^{\mathrm{J} \pm, \mathrm{I}}(\mathrm{~W})=\text { p.t.c. }+\sum_{\lambda^{\prime}=1 / 2}^{3 / 2} \mathrm{~g}_{\lambda^{\prime}} \mathrm{G}_{\lambda \lambda^{\prime}} \mathrm{J}^{\frac{3}{2}} \frac{3}{2}\left(\mp \mathrm{~W}, \mathrm{~W}_{\mathrm{R}}\right) \tag{I.3a}
\end{equation*}
$$

where $W_{R}$ is the resonance energy and $g_{\lambda^{\prime}}$ are the coupling constants of the first resonance defined by

$$
\begin{equation*}
g_{\lambda^{\prime}}=\frac{1}{\pi} \int_{M+1}^{\infty} \mathrm{dW}^{\prime} \operatorname{Im} H_{\lambda^{i}}^{3 / 2-}\left(\mathrm{W}^{\prime}\right)\left(\mathrm{g}_{1 / 2} \approx 1.6 \mathrm{~g}_{3 / 2} \approx-6.4 \mathrm{~m}_{\pi}=\mathrm{m}_{\pi \mathrm{o}}\right) \tag{I.3b}
\end{equation*}
$$

The approximation (I.3) for (I.1b) has been treated in Ref. 18 and has been shown to be of reasonable accuracy for practical applications. Under the further simplifying assumption that in $\operatorname{Im} H_{\lambda}^{3 / 2-}$ the $\operatorname{Im} \mathrm{E}_{1+}^{3 / 2}$-contribution can be neglected (see (A.1)), the partial amplitudes have been discussed in Part I. The differences arising from the inclusion of $\operatorname{Im} E_{1+}^{3 / 2}$ are only markable for the $J=1 / 2$ multipoles $\operatorname{Re} E_{0+}^{\pi^{+}}$, $\operatorname{Re} M_{1-}^{\pi^{+}}$. Results for the real parts of the $J=1 / 2,3 / 2$ multipoles are shown in Fig. 1.

From the Watson theorem

$$
\begin{equation*}
\operatorname{Im} H_{\lambda}^{J \pm}(W)=\operatorname{tg} \delta^{J \pm}(W) \operatorname{Re} H_{\lambda}^{J \pm}(W) \tag{I.4}
\end{equation*}
$$

with $\operatorname{Re} H_{\lambda}^{\mathrm{J} \pm}(\mathrm{W})$ taken from (I.3) and the pion nucleon scattering phase shift $\delta^{\mathrm{J} \pm}$ taken from experiment ${ }^{19}$ - one finds values for the imaginary parts $\operatorname{Im} H_{\lambda}^{\mathrm{J} \pm}(\mathrm{W})$ in the region of the first resonance, which are negligibly small except for the s-waves $\operatorname{Im} \mathrm{E}_{0+}^{1 / 2,3 / 2}$, which are always of the order of $1 / 10 \cdot \operatorname{Im} M_{1+}^{3 / 2}\left(W_{R}\right)$ 。 Therefore the inclusion of this imaginary part in the integrals of (I.1) is a presently possible and perceivable improvement of the pure isobar approximation.

## C. Isobar Approximation

In the "isobar approximation" apart from $\operatorname{Im} H_{\lambda}^{3 / 2-}(W)$, the s-wave multipole $\operatorname{Im} \mathrm{H}_{1 / 2}^{1 / 2-}$ is also retained in the integrands of the set (I. 1). $\operatorname{Im} H_{1 / 2}^{1 / 2-}$ is taken from (I.4) with $\operatorname{ReH} H_{1 / 2}^{1 / 2-}$ taken in the narrow width approximation (I.3).

The multipoles in this approximation (Fig。1) differ markably only for $\operatorname{Re} E_{0+}^{1 / 2,3 / 2}$, because of contributions arising mainly from the principal value integrals in (I.1). But the relative changes are noticeable only in $\pi^{\circ}$-photoproduction, because of a very sensitive cancellation of certain $\mathrm{I}=1 / 2$ and $3 / 2$ contributions. In $\pi^{\circ}$ production, the new contributions reduce the modul of $\operatorname{Re} E_{0+}^{\pi 0}$ further if compared to the results of the pure isobar approximation. This reduction is very critical, since the result for $\operatorname{Re} E_{0+}^{\pi^{\circ}}$ is of the order or possibly smaller than unknown high energy contributions so that the present results for $\operatorname{Re} E_{0+}^{\pi^{0}}$ are very ambiguous.

The neglection of the imaginary parts at high energies is one of the most serious sources for systematical errors. To estimate the uncertainty arising from this ignorance, we considered the following contribution to the rescattering terms

$$
\begin{equation*}
\left.\Delta \operatorname{Re} H_{\lambda}^{J \pm, I}(W)=\frac{1}{\pi} P \int_{W_{c}}^{\infty} d W^{\ell} \operatorname{Im} H_{\lambda}^{J \pm, I}\left(W^{\prime}\right)\left[\frac{1}{W^{\prime}-W} \mp G_{\lambda \lambda}^{J, ~} \Pi_{(\mp W,} \mp W^{\prime}\right)\right] \tag{I.5}
\end{equation*}
$$

where $W_{c}$ is an energy between the first and second resonance, above which $\operatorname{Im} H_{\lambda}^{\mathrm{J} \pm, \mathrm{I}}$ is either unknown or at least uncertain. An upper limit for the modul of (I.5) follows from

$$
\begin{equation*}
\left|\Delta \operatorname{ReH} H_{\lambda}^{\mathrm{J} \pm}(\mathrm{W})\right| \leq\left|\operatorname{Im} \mathrm{H}_{\lambda}^{\mathrm{J} \pm, \mathrm{I}}(\overline{\mathrm{~W}})\right| \mathrm{S}_{\lambda}^{\mathrm{J} \pm, \mathrm{I}}(\mathrm{~W}) \tag{I.6a}
\end{equation*}
$$

with

$$
\begin{equation*}
\mathrm{s}_{\lambda}^{J \pm, I}(\mathrm{~W})=\frac{1}{\pi} \int_{W_{c}}^{\infty} d W^{\prime}\left|\frac{1}{W^{t}-W^{\prime}} \mp G_{\lambda \lambda}^{\left.J, J_{(\mp W}, \mp W^{\prime}\right)}\right| \tag{I.6b}
\end{equation*}
$$

$\overline{\mathrm{W}}$ is a suitable mean value parameter from the interval $\mathrm{W}_{\mathrm{c}} \leq \overline{\mathrm{W}} \leq \infty$. The corresponding result for the multipoles $M_{\ell \pm}^{\mathrm{I}}$, can be written

$$
\begin{equation*}
\left|\Delta \operatorname{Re} \mathrm{M}_{\ell \pm}^{\mathrm{I}}(\mathrm{~W})\right| \leq\left|\operatorname{Im} \mathrm{M}_{\ell \pm}^{\mathrm{I}}(\overline{\mathrm{~W}})\right| \mathrm{T}_{\ell \pm}^{\mathrm{I}}(\mathrm{~W}) . \tag{I.7}
\end{equation*}
$$

The functions $T_{\ell \pm}^{I}(W)$ are shown in Table $I$, with $W_{c}=10.15$ corresponding to a photon energy $E_{c}=600 \mathrm{MeV}$. The integrands in (I. 6b) decrease very rapidly with increasing energy $W^{\mathbf{\prime}}$. From the unitarity of the Compton scattering one concludes that

$$
\begin{equation*}
\mathrm{W} \longrightarrow+\infty: \mathrm{M}_{\ell \pm}(\mathrm{W}) \sim \frac{\text { const }}{\mathrm{W}} \tag{I.8}
\end{equation*}
$$

Therefore one expects that the main contribution to the integrals ( 1.5 ) comes from values $W^{\prime}$ around $W_{c}$, i.e., from the low energy part of the integration interval. At these energies one can assume that the order 1 is a safe upper limit for all $\left|\operatorname{Im~}_{\ell \pm}(\bar{W})\right|$, which in fact should be reached only by the $J=1 / 2$ multipoles Re $\mathrm{E}_{0+}$, $\operatorname{Re~M} 1_{1-}$. The results in Table I for $\mathrm{T}_{\ell \pm}^{\mathrm{I}}$ (W) indicate therefore that by neglecting the high energy contributions (I.5), appreciable errors can only appear in the $J=1 / 2$ multipoles. On the other hand, it has been shown in Ref. 18 that the coupling of the other partial amplitudes to the $J=1 / 2$ multipoles is negligibly small apart from the first resonance, the effect of which can be calculated rather reliably. So one possibility to overcome the present difficulties in the calculation of the $J=1 / 2$ multipoles would be subtractions, if the subtraction constants would be known from other sources, which in principle could be low energy theorems but in practice presumably only multipole analyses of experimental data ( $\mathrm{E}_{0+}^{\pi^{0}},{ }^{21} \mathrm{M}_{1-}^{\pi^{\mathrm{o}}}$, see Section II). Otherwise one can try to solve the present discrepancies between theory and experiment by a suitable choice of the subtraction constants.
D. Isobar Approximation in Subtracted Dispersion Relations

Consider the subtracted dispersion relations following from (I. 1a)

$$
\begin{align*}
\operatorname{Re} H_{\lambda}^{J \pm, I}(W)=\operatorname{Re} H_{\lambda}^{J \pm, I}\left(W_{0}\right) & +H_{\lambda, \text { inh }}^{J \pm, I}(W)-H_{\lambda, \text { inh }}^{J \pm, I}\left(W_{0}\right) \\
& +\frac{\left(W-W_{0}\right)}{\pi} P \int_{\mathrm{M}+1}^{W_{c}} \mathrm{dW}^{\prime} \frac{\operatorname{Im} H_{\lambda}^{J \pm, I}\left(W^{\prime}\right)}{\left(W^{\top}-W\right)\left(W^{\prime}-W_{0}\right)} \tag{I.9}
\end{align*}
$$

where $W_{0}$ is the subtraction energy and $W_{c}$ a cutoff energy $\left(W_{c}=10.15\right.$, $\Delta \mathrm{E}_{\mathrm{c}}=600 \mathrm{MeV}$.

We shall call isobar approximation of (I.9) if

1. in the calculation of the difference
$\Delta H_{\lambda, \mathrm{inh}}^{\mathrm{J} \pm, \mathrm{I}}(\mathrm{W})=\mathrm{H}_{\lambda, \mathrm{inh}}^{\mathrm{J} \pm, \mathrm{I}}(\mathrm{W})-\mathrm{H}_{\lambda, \mathrm{inh}}^{\mathrm{J} \pm, \mathrm{I}}\left(\mathrm{W}_{0}\right)$
according to ( I .1 b ) only the pole term contribution and the contribution of the first resonance is retained; and
2. in the principal value integrals of (I.9) Im $H_{\lambda}^{J \pm}$ is taken
from the isobar approximation Section I. B.
The omission of all imaginary parts except those of the first resonance in (I.10) causes a negligible error. Their contribution - already quite small in ( $\mathrm{I}, 1 \mathrm{~b}$ ) - partly cancels in the difference because of its slow energy dependence. To check this we write ( $\mathrm{I}, 10$ ) for the multipoles $\mathrm{M}_{\ell \pm}$ in the form

$$
\begin{equation*}
\Delta M_{\ell \pm, \operatorname{inh}}^{\mathrm{I}}(\mathrm{~W})=\sum_{\ell^{\prime} \pm, \mathrm{I}^{\prime}} \frac{1}{\pi} \int_{\mathrm{M}+1}^{\infty} d W^{\prime} \operatorname{Im} M_{\ell^{\prime} \pm}^{\mathrm{I}^{\prime}}\left(\mathrm{W}^{\prime}\right) r\left(M_{\ell \pm}^{\mathrm{I}}, M_{\ell^{\prime} \pm}^{\mathrm{I}^{\prime}}, W_{0} ; W, W^{\prime}\right) \tag{I.11}
\end{equation*}
$$

As a bound for (I.11), one obtains

$$
\begin{equation*}
\left|\Delta M_{l \pm, i n h}^{I}(W)\right| \leq \sum_{l^{\prime} \pm, I^{\mathrm{I}}}^{\mathrm{I}}\left|\operatorname{Im} M_{\ell^{\prime} \pm}^{\mathrm{I}^{\prime}}(\overline{\mathrm{W}})\right| R\left(\mathrm{M}_{\ell \pm}^{\mathrm{I}}, \mathrm{M}_{\ell^{\prime} \pm}^{\mathrm{I}^{\mathrm{I}}}, \mathrm{~W}_{0} ; \mathrm{W}\right) \tag{I.12a}
\end{equation*}
$$

with

$$
\begin{equation*}
R\left(M_{\ell \pm}^{I}, M_{\ell^{\prime} \pm}^{I^{\prime}}, W_{0} ; W\right)=\frac{1}{\pi} \int_{W_{i}}^{\infty} d W^{\prime}\left|r\left(M_{\ell \pm}^{I}, M_{\ell^{\prime} \pm}^{I^{\prime}}, W_{0} ; W, W^{\prime}\right)\right| \tag{I.12b}
\end{equation*}
$$

To avoid the threshold singularity $\left(q^{\prime}\right)^{-l}$ in $r$ at $W^{\prime}=(M+1)$ we started the integration somewhat above threshold at $\mathrm{W}_{\mathrm{i}}=8.0\left(\mathrm{E}_{\mathrm{i}}=200 \mathrm{MeV}\right)$. Since the contribution from the interval $M+1 \leq W^{\prime} \leq 8.0$ is small, (I.12) is still a realistic bound for (I. 11). In (I. 11) this threshold singularity is compensated by the threshold factor $\left(q^{\circ}\right)^{2 \ell+1}$ of $\operatorname{Im} M_{\ell \pm}$.

In the sum (I.12a) the contribution of $\operatorname{Im} M_{\ell \pm}^{I}$ is excluded. It is discussed together with the neglected rescattering term in (I.9), but will be taken into account only for $W^{\prime} \geq W_{c}$. This contribution is written in the form

$$
\begin{equation*}
\Delta M_{\ell \pm}^{I}(W)=\frac{1}{\pi} \int_{W_{c}}^{\infty} d W^{\prime} \operatorname{Im~M} M_{\ell \pm}^{I}\left(W^{\prime}\right) \tilde{r}\left(M_{\ell \pm}^{I}, W_{0} ; W, W^{\prime}\right) \tag{I.13}
\end{equation*}
$$

with the bound

$$
\begin{equation*}
\left|\Delta \mathrm{M}_{\ell \pm}^{\mathrm{I}}(\mathrm{~W})\right| \leq\left|\operatorname{Im} \mathrm{M}_{\ell \pm}^{\mathrm{I}}(\overline{\mathrm{~W}})\right| \widetilde{\mathrm{R}}\left(\mathrm{M}_{\ell \pm}^{\mathrm{I}}, \mathrm{~W}_{0} ; \mathrm{W}\right) \tag{I.14a}
\end{equation*}
$$

where

$$
\begin{equation*}
\widetilde{R}\left(M_{\ell \pm}^{I}, W_{0}, W\right)=\frac{1}{\pi} \int_{W_{c}}^{\infty} d W^{\prime}\left|\widetilde{\mathrm{r}}\left(M_{\ell \pm}^{I}, W_{0} ; W, W^{\prime}\right)\right| \tag{I.14b}
\end{equation*}
$$

The results in Table 2 for $R$ indicate that the largest contributions to (I. 12) come from the imaginary parts of the first resonance, which are supposedly taken into account in (I.10) or (I.11). All other imaginary parts yield contributions, which, to the accuracy presently required, are negligibly small. Either the kinematical coupling factors $R$ are too small or the imaginary parts themselves $\left(\operatorname{Im} E_{1+}^{1 / 2,0}\right)$. From the results in Table 3 for $\widetilde{R}$ follows the important fact that even for $J=1 / 2$ the rescattering terms ( I .13 ) now can be neglected.

From this analysis we conclude that the present limits on handling the system (I.1) are very clearly manifested in the form of the subtracted dispersion
relations ( $I, 9$ ). Our ignorance can thus be related to the uncertainty in calculating the subtraction constants $\operatorname{Re} H_{\lambda}^{J \pm}\left(W_{0}\right)$ in the low energy region. For $J=1 / 2$ especially, these constants are largely determined by unknown highenergy contributions.

Finally, we should like to mention that the successful use of subtracted dispersion relations depends on the right choice of the kinematical factor, which is separated from the physical amplitudes $H_{\lambda}^{J \pm, ~} \mathrm{I}(\mathrm{W})$ in (A.3). In that equation, the threshold factor (qk) ${ }^{\ell}$ is separated. This choice is motivated from the results in Ref. 20, where it is shown that the kernels $G_{\lambda \lambda^{\prime}}$ of (I. 1b) are slowly varying functions only after separation of the threshold factor (qk) ${ }^{\ell}$. The difference (I.10) will therefore be small as long as $\mathrm{W}, \mathrm{W}_{0}$ belong to the region of the threshold and the $\Delta(1236)$ resonance.

## E. Application of Subtracted Dispersion Relations

Predictions with subtracted dispersion relations for $J=1 / 2$. The higher resonances or the high energy contributions yield in the region of the first resonance slowly varying modifications to the isobar approximation of the multipoles in section I. B. According to our previous discussion, one expects that these changes are especially markable for the $J=1 / 2$ partial amplitudes. In Fig. 2 we compare therefore the results for $\operatorname{Re} E_{0+}$ and $\operatorname{Re} M_{1-}$ in the isobar approximation for $\pi^{\circ}$ - and $\pi^{+}$-production with the predictions resulting from subtracted dispersion relations of section I.C. The subtraction constants, our additional input, is taken from experiment in $\pi^{\circ}$-production: $\operatorname{Re} M_{1-}^{\pi^{0}}$ ( $\mathrm{E}=360 \mathrm{MeV}$ ) from the measurements of the recoil polarization (see Section $\Pi$ ); $\operatorname{Re} E_{0+}^{\pi^{\mathrm{o}}}(E=180 \mathrm{MeV})$ from a recent multipole analysis of $\pi^{\circ}$-angular distributions at low energies. ${ }^{21}$ In $\pi^{+}$-production we do not have corresponding information, so that we assumed some reasonable changes of the isobar
approximation at $E=400 \mathrm{MeV}$ in order to see what the effect could be. The results in Fig. 2 for $\mathrm{J}=1 / 2$ show that the main effect of the subtraction in (I.1) is only a parallel shift of the original isobar approximation. The change of the functional behavior is only slight in all cases considered. This is at present the most important consequence with respect to the use of subtracted dispersion relations.

In this connection we should like to point out that the results in Fig. 2 exclude the possibility of a phenomenological solution for $\operatorname{Re} \mathrm{E}_{0+}^{\pi^{0}}$ following from the multipole analysis ${ }^{21}$ of recent data by Govorkov, et al. ${ }^{22}$ This solution of $\operatorname{Re} E_{0+}^{\pi^{0}}$ goes from negative values at threshold with positive slope to positive values crossing zero at $\mathrm{E} \approx 210 \mathrm{MeV}$.

Comparison with the results from conformal mapping techniques. ${ }^{10 \mathrm{a}}$ In this section we consider the numerical differences between the isobar approximation (Section I. B) and the results derived for the $J=1 / 2,3 / 2$ multipoles by Behrends, Donnachie and Weaver. ${ }^{10}$ These authors use conformal mapping techniques in order to solve the coupled system (I.1). They retain the $P_{11}$ and $D_{13}$ resonances in the dispersion integrals in addition to the effects, which are already taken into account in the isobar approximation (Section I.B). Therefore one should expect markable changes at least in the partial amplitudes leading into the $P_{11}$ and $D_{13}$ final states.

At $\underline{E=200 \mathrm{MeV}}$ the results for $\mathrm{J}=3 / 2$ of both approximations are either identical or practically negligible. For $J=1 / 2$ the differences amount to $0.2 \ldots$ 0.01 , which are all of the order $10 \%$. At $\mathrm{E}=400 \mathrm{MeV}$ the differences are in some cases clearer: for $E_{0+}^{3 / 2}, M_{1-}^{1 / 2}, M_{1-}^{0}, M_{1+}^{1 / 2}, E_{2-}^{1 / 2}$ and $E_{2-}^{0}$ they are $-37 \%$, $57 \%,-51 \%, 27 \%, 22 \%$ and $34 \%$ compared to the isobar approximation. In all other cases they are smaller than $15 \%$. Effects of this order in small multipoles should
not be taken very seriously in the existing models, since they are in the limits of other unknown contributions. They can come as well by a different choice of the cutoff parameters in the integrals or from different extrapolations of $\operatorname{Im} M_{\ell \pm}(W)$ to high energies.

Finally we tried to parametrize the results of Ref. 10a by the isobar approximation in the subtracted version ( 1.9 ). We took as subtraction energy the point $E=400 \mathrm{MeV}$ and used as subtraction constants the results of Ref. 10a. The predictions at $\mathrm{E}=200 \mathrm{MeV}$ are shown in Table 4 together with the original result of Ref. 10a. The agreement is not completely satisfying. The differences are in some cases of the order $5 \%$. They should be taken seriously in the case of $\operatorname{Re} E_{0+}^{1 / 2}$ and $\operatorname{Re} E_{0+}^{3 / 2}$, which largely cancel each other in $\operatorname{Re} E_{0+}^{\pi^{0}}$. We checked that the results are negligibly affected by the present uncertainty of $\operatorname{Im} \mathrm{M}_{1+}^{3 / 2}$ and $\operatorname{Im} E_{1+}^{3 / 2}$ used in (I.9). The reasons for the discrepancy are not known. They might be connected with some of the approximations, which are made in the application of the conformal mapping method and which might not be justified from a physical point of view.

## II. COMPARISON WITH EXPERIMENTS

As has been shown in Ref. 12 there exists no sufficiently precise prediction for the partial amplitudes of the first resonance. The present ignorance can be characterized by one parameter for each multipole $M_{1+}^{3 / 2}$ and $E_{1+}^{3 / 2}$, e.g. by $\operatorname{Im} M_{1+}^{3 / 2}\left(W_{R}\right)$ and $\operatorname{Im} E_{1+}^{3 / 2}\left(W_{R}\right)$. The theory can predict $\operatorname{Im} M_{1+}^{3 / 2}\left(W_{R}\right)$ only with large systematical errors of the order of about $\pm 10 \%$. For the considerably smaller multipole $E_{1+}^{3 / 2}$ even a statement on the sign of $\operatorname{Im} E_{1+}^{3 / 2}\left(W_{R}\right)$ is impossible. Presently therefore these parameters have to be fixed by experimental data, which determine them within the limits shown in Fig. 3. At the moment
it is not reasonable to introduce further parameters. These would yield smaller effects in our energy region and cannot be identified uniquely because of the uncertainty of other partial amplitudes. If not otherwise stated these amplitudes are calculated in the isobar approximation of Section (I.C).
A. $\pi^{\mathrm{o}}$-Production

The information on the angular distributions in $\pi^{0}$-production is conveniently expressed by the coefficients $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \ldots, \mathrm{I}_{0}, \ldots$ of the expansion

$$
\begin{align*}
\frac{k}{q} \frac{d \sigma}{d \Omega}(E, \theta, \phi)= & \frac{k}{q} \frac{d \sigma}{d \Omega}(E, \theta)_{\text {unpol. }}+\sin ^{2} \theta \cos 2 \phi \cdot I(E, \theta)= \\
& =A(E)+B(E) \cos \theta+C(E) \cos ^{2} \theta+D(E) \cos ^{3} \theta+\ldots  \tag{II.1}\\
& +\sin ^{2} \theta \cos 2 \phi\left\{I_{0}(E)+I_{1}(E) \cos \theta+\ldots\right\}
\end{align*}
$$

In (II. 1) the term I(E, $\theta$ ) represents the information coming from the use of plane polarized $\gamma^{\prime}$ s. $\phi$ is the azimuthal angle between the polarization-plane and the reaction plane. We shall also use the information from measurements of the recoil polarization $P_{z}$ with unpolarized $\gamma^{\prime}$ s

$$
\begin{equation*}
\frac{1}{\sin \theta} \frac{\mathrm{k}}{\mathrm{q}} \frac{\mathrm{~d} \sigma(\mathrm{E}, \theta)}{\mathrm{d} \Omega} \mathrm{P}_{\mathrm{z}}(\mathrm{E}, \theta)=\mathrm{P}_{2}(\mathrm{E}, \theta)_{\text {unpol. }}=\mathrm{a}(\mathrm{E})+\mathrm{b}(\mathrm{E}) \cos \theta+\ldots \tag{II.2}
\end{equation*}
$$

The coefficient $A(E)$. The present uncertainty of theoretical predictions for the partial amplitudes of the first resonance is demonstrated by $A(E)$. Its prediction with an unitary ansatz for the total amplitudes - was based mainly on a result for $M_{1+}^{3 / 2}$, which deviates by +2 to $5 \%$ from the old result for $M_{1+}^{3 / 2}$ of Chew, Goldberger, Low and Nambu。 ${ }^{3}$ The values for $A(E)$ seemed to be in excellent agreement (e.g., Part I and Ref. 16a) with the experiments. But new experimental results ${ }^{23,} 24$ give an $A(E)$ which is higher by $10-15 \%$ around the maximum at 320 MeV . These discrepancies
question thus seriously the old predictions for $M_{1+}^{3 / 2}$, since $A(E)$ depends strongly on $M_{1+}$. Therefore we shall use $A(E)$ to fix the constant $\operatorname{Im} M_{1+}^{3 / 2}\left(W_{R}\right)$. The results for $A$ in Fig. 4 are calculated with four solutions for $M_{1+}^{3 / 2}, E_{1+}^{3 / 2}-$ characterized by the four points $1,2,3$ and 5 in the $\operatorname{Im} M_{1+}\left(W_{R}\right)-\operatorname{Im} E_{1+}\left(W_{R}\right)-$ plane (see Fig. 3). The different choices for $\operatorname{Im} E_{1+}^{3 / 2}\left(W_{R}\right)$ follow from the discussion below.

The ratio $I_{0} / \mathrm{C}$. The ratio $\mathrm{I}_{0} / \mathrm{C}$ depends very strongly on the prediction of the ratio $E_{1+} / M_{1+}$. At the moment it is therefore the best source to fix $\operatorname{Im} E_{1+}^{3 / 2}\left(W_{R}\right)$ as has been discussed in Ref. 32. Figure 5a shows that solutions which fit the experiments for $230<\mathrm{E}<350 \mathrm{MeV}$ yield values for $\mathrm{I}_{0} / \mathrm{C}$ around $\mathrm{E}=400 \mathrm{MeV}$, which are too large and show a systematical energy dependence different from the experimental data.

In Fig. 5a are also shown the various ratios $\mathrm{E}_{1+}^{3 / 2} / \mathrm{M}_{1+}^{3 / 2}$ for the solutions $1,2,3,5$, with which $I_{0} / \mathrm{C}$ has been calculated. Typical for these solutions are the small values for $\mathrm{E}_{1+}^{3 / 2} / \mathrm{M}_{1+}^{3 / 2}$ above $\mathrm{E}=350 \mathrm{MeV}$ and the small slope, with which these solutions change sign. One should note that because of the small values for $\mathrm{E}_{1+}^{3 / 2} / \mathrm{M}_{1+}^{3 / 2}$ the results are very sensitive to the approximations of the theory, particularly those which have to be applied to high energy contributions. In Fig. 5 b are also shown the experimental values for $\mathrm{C} / \mathrm{A}$ with which the experimental data for $I_{0} / \mathrm{A}$ were multiplied to obtain $\mathrm{I}_{0} / \mathrm{C}$.

The conclusion to be drawn from the ratio $\mathrm{I}_{0} / \mathrm{C}$ is that below the resonance the experimental results together with the results from the multipole analysis near threshold are in fair agreement with the chosen solutions for $E_{1+} / M_{1+}{ }^{\circ}$ Since there is a very critical cancellation of several multipole contributions in $I_{0}$ and $C$ near threshold, with a zero in $I_{0}$ and possibly also $C$, present predictions for $I_{0} / C$ below $E=210 \mathrm{MeV}$ are not reliable. Above $E=400 \mathrm{MeV}$ the
experimental results seem to indicate a different energy dependence for $\mathrm{E}_{1+} / \mathrm{M}_{1+}$ than predicted by our solutions. In this connection we should like to mention that the Bonn group cites $\mathrm{E}_{1+}^{3 / 2}\left(\mathrm{~W}_{\mathrm{R}}\right) / \mathrm{M}_{1+}^{3 / 2}\left(\mathrm{~W}_{\mathrm{R}}\right)=-0.004 .{ }^{7 \mathrm{~b}}$ But no results for $\mathrm{I}_{0} / \mathrm{C}$ have been published up to now.

The measurement of the recoil polarization. The most valuable information obtained from present measurements of the proton recoil polarization (II. 2) is represented in the parameter $b$. In terms of multipoles $b$ is given by

$$
\begin{equation*}
\mathrm{b}=3 \operatorname{Im} \mathrm{M}_{1-}^{*}\left(\mathrm{M}_{1+}+3 \mathrm{E}_{1+}\right)+\ldots \tag{II.3}
\end{equation*}
$$

It is dominated by the contribution ${ }^{14}$

$$
\begin{equation*}
\mathrm{b} \approx 3 \operatorname{Re} \mathrm{M}_{1-} \operatorname{Im}\left(\mathrm{M}_{1+}+3 \mathrm{E}_{1+}\right)+\ldots \tag{II.4}
\end{equation*}
$$

The inclusion of $\operatorname{Im} \mathrm{E}_{1+} \approx 2 / 3 \operatorname{Im} \mathrm{E}_{1+}^{3 / 2}$ in $b$ according to the solutions $1,2,3,5$ becomes not effective because of the simultaneous change in $\mathrm{M}_{1+}$. At 360 MeV the new measurements ${ }^{34}$ of the polarization are in fair agreement with the theoretical calculations whereas at 300 MeV the experimental value of b is $>0$, what is in complete disagreement with the present theoretical predictions.

Unfortunately it is not possible at the moment to analyze in detail the discrepancy in $a(E)$. The errors can arise in various large multipole contributions to $\mathrm{a}(\mathrm{E})^{14}$ and the present experimental errors do not allow to detect a typical energy dependence for the discrepancy.

Results for the coefficients $\mathrm{a}(\mathrm{E})$ and $\mathrm{b}(\mathrm{E})$ are presented in Fig. 6.
The coefficients B, D and C. Not many conclusions can be drawn at present from the comparison of the asymmetry coefficients B and D. The recent values for $B$ at threshold up to $E=210 \mathrm{MeV}$ serve more or less only as a check for the cut off energy $W_{c}$ in the dispersion relation (I.1a) for $\operatorname{Re} E_{0+}^{\pi^{o}}$. Above $E=210$ MeV the experimental situation is presently completely unclear. Fits to angular distributions, which neglect the coefficient $D$ are dubious at least above $E=300$

MeV as the results in Fig. 4 for D indicate. There is the possibility that B is rather small in the region 220 to 340 MeV . In this case D should also be taken into account in fits at lower energies.

Presently the most obvious discrepancies are connected with the coefficient $C$. This is particularly true at low energies where $C$ is not dominated by $\mathrm{M}_{1+}$. These deviations could be removed by noteworthy small contributions to $\mathrm{E}_{1+}$.

Summary to Section II. A. Very recent results in backward direction ${ }^{15 \mathrm{~b}, 24}$ confirm the present discrepancies in C and presumably also in B (Fig. 7). It is not possible to explain them by reasonable changes in $\operatorname{Re} \mathrm{E}_{0+}$ and $\operatorname{Re} \mathrm{M}_{1-}$ as they are suggested by subtracted dispersion relations (Section I.D and Fig.2) for $\operatorname{Re} E_{0+}^{\pi^{0}}$ and $\operatorname{Re} M_{1-}^{\pi^{0}}$. One has to alter the amplitudes in such a way that the changes in the excitation curves have on both sides of the resonance a different sign particularly at $\theta=180^{\circ}$. Because of the discrepancies in $C$ and $I_{0} / C$ we believe at the moment that the multipole $\mathrm{E}_{1+}$ observes further detailed considerations. But since the alterations of $E_{1+}$ should be rather energy dependent the $E_{1+}^{3 / 2}$ part is the most doubtful one. In the framework of the dispersion relations (I. 1) it seems physically impossible to find a corresponding change of the other isospin parts. B. $\pi^{+}$-Production

In $\pi^{+}$-production we shall discuss the discrepancies in the angular distributions (Fig. 9) and excitation curves (Fig. 8) for unpolarized or plane-polarized $\gamma^{\boldsymbol{\prime}} \mathrm{S}$.
$\theta=90^{\circ} \pi^{+}$- excitation curves. The results for unpolarized $\gamma^{\prime} \mathrm{s}$ with solutions $2,3,5$ and 6 yield at $\theta=90^{\circ}$ (Fig. 8) better agreement with experiments than in former calculations (see Part I, Fig. 5). Especially the typical discrepancy is smaller on the low energy side of the resonance, where $\operatorname{Re} M_{1+}^{3 / 2}$ has its maximum
( $\mathrm{E} \approx 280$ ). With the new results the prediction at threshold is about $10 \%$ higher than the values in part I. The difference arises from the inclusion of $\operatorname{Im} E_{0+}^{1 / 2,3 / 2}$ and particularly of $\operatorname{Im} \mathrm{E}_{1+}^{3 / 2}$ in the dispersion integrals. The experimental data suggest values which are $5 \%$ smaller up to $\mathrm{E}=190 \mathrm{MeV}$. Presently it cannot be decided whether this discrepancy arises by unknown high-energy contributions or e.g., a too large $f^{2}=0.080$.

For (A-I $\mathrm{I}_{0}$ (Fig. 8) following from measurements with polarized $\gamma^{\prime} \mathrm{s}$ the discrepancy around $\mathrm{E}=280 \mathrm{MeV}$ is very markable. Because of its energy dependence it must mainly arise in the p-wave multipoles $\mathrm{M}_{1-}, \mathrm{M}_{1+}$ - as already mentioned in part I. With the new experimental results for ( $\mathrm{A}-\mathrm{I}_{0}$ ) in $\pi^{+}$- and $\pi^{\circ}$-production ${ }^{35}$ it now becomes possible to compare also the sum of both

$$
\begin{align*}
S(E)= & \left\{A(E)-I_{0}(E)\right\}^{\pi^{+}}+\left\{A(E-7 \mathrm{MeV})-I_{0}(E-7 \mathrm{MeV})\right\}^{0} \\
= & \frac{2}{3}\left|\mathrm{E}_{0+}^{3 / 2}+\mathrm{E}_{2-}^{3 / 2}\right|^{2}+\frac{2}{3}\left|2 \mathrm{M}_{1+}^{3 / 2}+\mathrm{M}_{1-}^{3 / 2}\right|^{2}+\frac{1}{3}\left|\mathrm{E}_{0+}^{1 / 2}+\mathrm{E}_{2-}^{1 / 2}\right|^{2}  \tag{III.5}\\
& +\frac{1}{3}\left|2 \mathrm{M}_{1+}^{1 / 2}+\mathrm{M}_{1-}^{1 / 2}\right|^{2}+\ldots
\end{align*}
$$

as suggested in (Part I Section V). In (II. 5) the interference terms between the multipoles leading into the isospin $I=1 / 2$ and $3 / 2$ states cancel (see (5.1a) in Part I). The results in Fig. 9 for the difference ( $\mathrm{S}_{\exp }$ - $\mathrm{S}_{\text {theor }}$ ) point clearly to a discrepancy, which because of its energy dependence should mainly arise in the $I=3 / 2$ part of the real parts of the p-wave multipoles $\mathrm{M}_{1-}$. Assuming only changes $\Delta$ in $\operatorname{Re} M_{1-}^{3 / 2}$ one obtains for $S$ from (II. 5 )

$$
\begin{equation*}
\Delta S(F)=\frac{8}{3} \operatorname{Re} M_{1+}^{3 / 2} \cdot \Delta \operatorname{Re} M_{1-}^{3 / 2} \tag{II.6}
\end{equation*}
$$

The result for $\Delta S$ in Fig. 9 was obtained with the rather large value $\Delta \operatorname{Re} \mathrm{M}_{1-}^{3 / 2}=0.45$ (in our units $10^{-2} \pi$ ). This would amount to a $60 \%$ change in Re $\mathrm{M}_{1-}^{3 / 2}$ at $E=300 \mathrm{MeV}$. In $\pi^{+}$- and $\pi^{\circ}$-photoproduction it would yield the changes
$\Delta M_{1-}^{\pi^{+}}=-\sqrt{2 / 3} \operatorname{Re~} M_{1-}^{3 / 2}=-0.21$ and $\Delta M_{1-}^{\pi^{0}}=+2 / 3 \operatorname{Re~} M_{1-}^{3 / 2}=+0.30$. Changes of this order one might expect according to Fig. 2. But in $\pi^{\circ}$-photoproduction changes $\Delta M_{1-}^{\pi^{\circ}}=0.30$ would spoil the good agreement for $b$ (II. 4) at 360 MeV (see Section II. A). To get a consistent result one obviously has therefore to consider corrections in $\operatorname{Re} M_{1-}^{3 / 2}$ and $\operatorname{Re} M_{1-}^{1 / 2}$ as well as $\operatorname{Re} M_{1-}^{0}$.
$\pi^{+}$-angular distributions. In the angular distributions for unpolarized $\gamma^{\text {s }} \mathrm{S}$ one finds reasonably good agreement in forward direction up to $\theta=50^{\circ}$ as can be seen, e.g., in Fig. 10. Neither the new ${ }^{37 \mathrm{a}}$ nor the old ${ }^{37 \mathrm{~b}}$ data indicate a peak at $\theta=10^{\circ}$ which is predicted in Ref. 10a. The discrepancy at lower angles can very well be explained by reasonable alterations in the $s$ - and p-wave multipoles as already discussed in Ref. 15a. But one can show that a single multipole alone is not the reason for the total discrepancy. Unfortunately a more definite conclusion cannot be drawn from the present data. The same agreement holds also for angular distributions with plane polarized $\gamma^{\prime}$ s, e.g., if the polarization vector points perpendicular to the reaction plane ( $\phi=90^{\circ}$ ) (Fig. 10)

$$
\begin{equation*}
\frac{\mathrm{d} \sigma}{\mathrm{~d} \Omega}\left(\theta, \phi=90^{\circ}, E\right)=\frac{\mathrm{d} \sigma_{1}}{\mathrm{~d} \Omega}(\theta, \mathrm{E})=\left(\mathrm{A}-\mathrm{I}_{0}\right) \frac{\mathrm{q}}{\mathrm{k}} \tag{III.7}
\end{equation*}
$$

Summary to Section II. B. At $\theta=0^{\circ}$ the theory is very sensitive to small alterations in all multipoles. Therefore the partly remarkable agreoment in Fig. 8 is noteworthy. On the other hand mcasurements of the asymmetry ratio

$$
A_{R}(\theta, E)=\frac{d \sigma_{\perp}-d \sigma_{11}}{d \sigma_{\perp}+d \sigma_{11}}
$$

for plane-polarized $\gamma^{\prime}$ s indicate near $\mathrm{E}=250 \mathrm{McV}$ and $\theta=30^{\circ}$ a completely different encrgy-dependence (Fig. 11), which is very hard to understand. But better statistics is needed for a definite conclusion.

It was mainly our aim to exhibit the typical discrepancies in $\pi^{+}$-production. In our opinion these cannot be overcome purely by a refinement of the present methods. Presumably one needs more physical information which effects several multipoles in a correlated way. At very low energies this has been tried recently, e.g., in $\pi^{0}$-photoproduction by the inclusion of the $\omega$ and B-meson resonances. ${ }^{38}$ But these results are not very conclusive. In any case above $\mathrm{E}=200 \mathrm{MeV}$ the authors expect that the effects of the meson resonances are of minor importance. There is at the moment also agreement that the effects of the $\rho$-resonance cannot account for the main discrepancies. 39,13

## III. PREDICTIONS FOR NEW KINDS OF EXPERIMENTS

$$
\text { IN } \pi^{\circ} \text {-PHOTOPRODUCTION }
$$

## A. General Discussion

In this section we discuss predictions of the isobar approximation for some more general experiments, in which, e.g., linear polarized $\gamma^{\mathbf{\prime}}$ s together with a polarized target are used. The immediate question is then, which new information on the low partial amplitudes follows from these more difficult-experiments. Also as soon as more general experiments become possible one might think of the possibility to perform a complete partial amplitude analysis or to determine the full helicity amplitudes from experimental data. ${ }^{40}$ But this program can be successful only if the new experiments are sensitive to contributions of the helicity amplitudes which are poorly known since they are otherwise hardly detectable. To decide on the usefulness of a new type of experiment purely kinematic considerations are insufficient. Some dynamical input is needed. For this we shall use the isobar approximation. As an example for this kind of discussion we choose $\pi^{\circ}$-photoproduction on the proton.

Assuming plane-polarized photons and a polarized nucleon target, we generalize (II. 1) and (II.2). The result for the cross section is

$$
\begin{aligned}
\frac{\mathrm{k}}{\mathrm{q}} \frac{\mathrm{~d} \sigma}{\mathrm{~d} \Omega}=\mathrm{S}_{1}+ & {\left[-\kappa \cos 2 \phi \sin \theta \operatorname{Re} \mathrm{~L}_{8}+\mathrm{p}_{2} \operatorname{Im}_{6}-\right.} \\
& \left.-\mathrm{p}_{1} \kappa \sin 2 \phi \operatorname{Im} \mathrm{~L}_{9}+\mathrm{p}_{2} \kappa \cos 2 \phi \cdot \operatorname{Im} \mathrm{~L}_{10}+\mathrm{p}_{3} \kappa \sin 2 \phi \sin \theta \operatorname{Im} \mathrm{~L}_{8}\right] \sin \theta
\end{aligned}
$$

This form shows explicitly the dependence on the polarization angle $\phi$ (Fig. 12) and on the components $p_{x}, p_{y}, p_{z}$ of the polarization vector of the nucleon target. $\kappa$ denotes the polarization degree of the plane-polarized photons. Furthermore, for the recoil-polarization $P_{i}$ of the outgoing nucleon one obtains

$$
\begin{align*}
& \frac{\mathrm{k}}{\mathrm{q}} \frac{\mathrm{~d} \sigma}{\mathrm{~d} \Omega} \mathrm{P}_{\mathrm{z}}=\left[\kappa \sin 2 \phi \sin \theta \operatorname{Im} \mathrm{~L}_{7}+\mathrm{p}_{1} \operatorname{Re} \mathrm{~L}_{5}\right] \sin \theta+\mathrm{p}_{3} \mathrm{~S}_{2}+\ldots  \tag{III.2}\\
& \frac{\mathrm{k}}{\mathrm{q}} \frac{\mathrm{~d} \sigma}{\mathrm{~d} \Omega} \mathrm{P}_{\mathrm{x}}=\left[-\kappa \sin 2 \phi \operatorname{Im} \mathrm{~L}_{5}+\mathrm{p}_{1} \sin \theta \operatorname{Re} \mathrm{~L}_{7}+\mathrm{p}_{3} \operatorname{Re} \mathrm{~L}_{9}+\ldots\right] \sin \theta  \tag{III.3}\\
& \frac{\mathrm{k}}{\mathrm{q}} \frac{\mathrm{~d} \sigma}{\mathrm{~d} \Omega} \mathrm{P}_{\mathrm{y}}=\left[-\operatorname{Im} \mathrm{L}_{10}-\kappa \cos 2 \phi \cdot \operatorname{Im} \mathrm{~L}_{6}+\mathrm{p}_{2} \sin \theta \operatorname{Re} \mathrm{~L}_{8}+\ldots\right] \sin \theta \tag{III.4}
\end{align*}
$$

The neglected terms in (III. 2) - (III.4) contribute only at measurements, where polarized $\gamma^{\prime}$ 's and a polarized target are used simultaneously. For a derivation of (III.1) - (II. 4) see, e.g., Ref. 20.

In (III. 1) - (III. 4), the real functions $\mathrm{S}_{\mathrm{i}}(\mathrm{E}, \cos \theta)[\mathrm{i}=1,2]$ and the complex functions $L_{i}(E, \cos \theta)[i=5$ to 10$]$ appear. (The notation has been taken from Ref. 41.) The quantities

$$
S_{1}=\frac{k}{q} \frac{d \sigma}{d \Omega}(E, \cos \theta)_{\text {unpol. }}, \operatorname{Re} L_{8}=-I(E, \cos \theta)
$$

and

$$
\operatorname{Im}_{10}=P_{2}(E, \cos \theta)_{\text {unpol }}
$$

are already measured (see (II. 1) and (II. 2)) and therefore shall not be discussed here. To determine $S_{2}, \operatorname{Re}_{5}, \operatorname{Re} L_{7}$ and $\operatorname{Re} L_{9}$ by a measurement of the recoil polarization, one necessarily needs a polarized target. To obtain $\operatorname{Im} L_{5}, \operatorname{Im} L_{6}$ and $\operatorname{Im} L_{7}$, plane-polarized $\gamma^{\prime}$ s can be used. We restrict our discussion in the following mainly to these two groups (1 and 2) of observables.

To go into more detail we shall assume an expansion in powers of $\cos \theta$ analogously as done in (II.1) and (II.2)

$$
\begin{gather*}
S_{i}(E, \cos \theta)=A_{i}(E)+B_{i}(E) \cos \theta+C_{i}(E) \cos ^{2} \theta+D_{i} \cos ^{3} \theta+\ldots \\
i=1,2 \tag{III.5}
\end{gather*}
$$

$L_{i}(E, \cos \theta)=A_{i}(E)+B_{i}(E) \cos \theta+C_{i}(E) \cos ^{2} \theta$

$$
\begin{equation*}
i=5, \ldots 10 \tag{III.6}
\end{equation*}
$$

Note that one conventionally writes for $i=1: A_{1}=A, B_{1}=B$, etc.

## B. Group 1: $\mathrm{S}_{2}, \mathrm{Re}_{5}, \mathrm{Re}_{7}$ and $\operatorname{Re} \mathrm{L}_{9}$

In Tables 5 and 6, we present numerical results for the coefficients of this group at three energies using the isobar approximation (solution 5). For a better understanding of these results we further expand the coefficients $A_{i}, B_{i}, C_{i}$ and $D_{i}$ into multipoles. Neglecting all multipoles with $\ell \geq 2$ one obtains ${ }^{41}$

$$
\begin{align*}
& A_{2}=\operatorname{Re} E_{0+}^{*}\left(2 M_{1-}+M_{1+}+3 E_{1+}\right)  \tag{III.7}\\
& B_{2}=-\left|E_{0+}\right|^{2}+\operatorname{ReM} M_{1-}^{*}\left(-M_{1-}+6 E_{1+}+2 M_{1+}\right)+3 \operatorname{Re} E_{1+}^{*}\left(\frac{9}{2} E_{1+}+M_{1+}\right)  \tag{III.8}\\
& C_{2}=\operatorname{Re} B_{9}=-3 \operatorname{Re} E_{0+}^{*}\left(M_{1+}+3 E_{1+}\right)  \tag{III.9}\\
& D_{2}=\operatorname{ReC}_{9}=-9\left[\operatorname{Re} M_{1+}^{*}\left(E_{1+}+\frac{1}{2} M_{1+}\right)+\frac{5}{2}\left|E_{1+}\right|^{2}\right]  \tag{III.10}\\
& \operatorname{Re} A_{5}=-\operatorname{Re} A_{8}=+I_{0}=-3 \operatorname{Re} M_{1-}^{*}\left(M_{1+}-E_{1+}\right)- \\
& \quad-\frac{3}{2}\left[\operatorname{Re} M_{1+}^{*}\left(2 E_{1+}+M_{1+}\right)-3\left|E_{1+}\right|^{2}\right]
\end{align*}
$$

$\operatorname{Re} B_{5}=\operatorname{Re} A_{7}=3 \operatorname{Re} E_{0+}^{*}\left(M_{1+}-E_{1+}\right)$
$\operatorname{Re} \mathrm{C}_{5}=\operatorname{Re} \mathrm{B}_{7}=-3 \operatorname{Re} \mathrm{~A}_{5}-9 \operatorname{Re} \mathrm{M}_{1-}^{*}\left(\mathrm{M}_{1+}-\mathrm{E}_{1+}\right)$
$\operatorname{Re} \mathrm{C}_{7}=0$
$\operatorname{Re} A_{9}=A-2\left|E_{0+}\right|^{2} \quad$ (see Eq. (II. 1)).

It follows from these equations that the coefficients in Table 5 are dominated by $s-p$ interference terms and those in Table 6 by p-p interference terms with the exception of $\mathrm{B}_{2}$ and Re $\mathrm{A}_{9}$ which contain also $\left|\mathrm{E}_{0+}\right|^{2}$. In the following discussion we shall assume that the resonant multipoles $\mathrm{M}_{1+}$ and $\mathrm{E}_{1+}$ are already determined e.g., by the measurement of $S_{1}$ and $\operatorname{Re} L_{8}$.

We should like to note:

1. From (III. 9), (III. 10), (III. 12) and (III. 13) one should expect that certain pairs of coefficients are almost equal. The results in Tables 5 and 6, which take into account all multipoles, show that this is generally true - apart at $\mathrm{E}=280$ in Table 5 - indicating the minor importance of the higher multipoles in these and similar cases.
2. At the first look $P_{x}(E, \cos \theta=0)$ might appear very suitable for a determination of $\mathrm{E}_{0+}$. One might expect from (III.12) that the s-p interference term $\operatorname{Re} \mathrm{E}_{0+}^{*}\left(\mathrm{M}_{1+}-\mathrm{E}_{1+}\right)$, is best investigated by measuring $A_{7}$. This term can be directly measured in contrast to B and $\operatorname{Re} \mathrm{B}_{5}$. But from (III. 1), (III. 3), (III.5), (III. 6) and the values in Table 5, one obtains

$$
\begin{array}{r}
P_{x}(E=340 \mathrm{MeV}, \cos \theta=0)=\mathrm{p}_{\mathrm{x}} \frac{\operatorname{ReA}_{7}(\mathrm{E}=340 \mathrm{MeV})}{\mathrm{A}(\mathrm{E}=340 \mathrm{MeV})}=0.24 \mathrm{p}_{\mathrm{x}} \\
\mathrm{p}_{\mathrm{z}}=\kappa=0 \tag{III.16}
\end{array}
$$

a result, which should represent the order of magnitude correctly. One sees from (III. 16) that only with a fairly high degree of polarization $p_{x}$, will one be able to beat the present limitations on the accuracy of $B$, which around the resonance is typically $25 \%$ (see Fig. 4) and which presumably will be improving. With the same experimental arrangement but a different direction of the initial polarization vector, one obtains also

$$
\begin{array}{r}
\mathrm{P}_{\mathrm{x}}(\mathrm{E}, \cos \theta=0)=\mathrm{p}_{\mathrm{z}} \frac{\operatorname{Re} \mathrm{~A}_{9}(\mathrm{E})}{\mathrm{A}(\mathrm{E})}=\mathrm{p}_{\mathrm{z}}\left(1-\frac{2\left|\mathrm{E}_{0+}\right|^{2}}{\mathrm{~A}}+\ldots\right) \\
\kappa=\mathrm{p}_{\mathrm{x}}=0 \tag{III.17}
\end{array}
$$

For the last equality we used the approximate relation (III. 15). The quantity $\left(1-2\left|E_{0+}\right|^{2} / \mathrm{A}\right)$ has a very peculiar energy behavior. It increases from its value -1 at threshold, where $A=\left|E_{0+}\right|^{2}$, to $\approx+1$ within 20 MeV , since A increases very rapidly in contrast to $\left|\mathrm{E}_{0+}\right|^{2}$. Throughout the region of the first resonance, where $2\left|\mathrm{E}_{0+}\right|^{2} / \mathrm{A} \approx 1 / 30$, a measurement of (III. 17) would therefore need an accuracy $<1 \%$ to yield any reliable information on $\left|\mathrm{E}_{0+}\right|^{2}$.
3. From the angular distribution of $\mathrm{Re}_{5}$, which appears in $\mathrm{P}_{\mathrm{z}}$ (III. 2), one obtains the coefficients $\mathrm{A}_{5}$ and $\mathrm{C}_{5}$ 。 According to (III. 13)

$$
\begin{equation*}
\operatorname{Re}\left(A_{5}+\frac{1}{3} C_{5}\right)=-3 \operatorname{Re} M_{1-}^{*}\left(M_{1+}-E_{1+}\right)+\ldots \tag{III.18}
\end{equation*}
$$

Since in our energy region $\operatorname{Im} \mathrm{M}_{1-} \approx 0$ (III. 18) is sensitive to Re $\mathrm{M}_{1-}$ apart from the immediate neighborhood of the resonance where also $\operatorname{Re}\left(\mathrm{M}_{1+}-\mathrm{E}_{1+}\right) \approx 0$. This is in contrast to (II.4) from which at present our best knowledge of $\operatorname{Re} \mathrm{M}_{1-}$ follows.

## C. Group 2: $\operatorname{Im} L_{i}, i=5,6,7$

The quantity $\operatorname{Im} \mathrm{L}_{6}$ should be the easiest obtainable in this group since only a measurement of the cross section with a polarized target is necessary (see (III. 1)). As in part III. B we present in Table 7 numerical results for some of the relevant coefficients. Neglecting again all multipoles with $\ell \geq 2$, one obtains from ${ }^{41}$

$$
\begin{align*}
& \operatorname{Im} A_{5}=-\operatorname{Im} A_{8}=3 \operatorname{Im} \mathrm{M}_{1-}\left(\mathrm{M}_{1+}-\mathrm{E}_{1+}\right)^{*}+6 \operatorname{Im} \mathrm{E}_{1+} \mathrm{M}_{1+}^{*}  \tag{III.19}\\
& \operatorname{Im} \mathrm{~B}_{5}=\operatorname{Im} \mathrm{A}_{6}=\operatorname{Im} \mathrm{A}_{7}=3 \operatorname{Im} \mathrm{E}_{0+}\left(\mathrm{E}_{1+}-\mathrm{M}_{1+}\right)^{*}  \tag{III.20}\\
& \operatorname{Im} \mathrm{C}_{5}=\operatorname{Im} \mathrm{B}_{7}=-18 \operatorname{Im} \mathrm{E}_{1+} \mathrm{M}_{1+}^{*}  \tag{III.21}\\
& \operatorname{Im} \mathrm{~B}_{6}=3 \operatorname{Im~M}  \tag{III.22}\\
& 1- \\
& \left(\mathrm{M}_{1+}-\mathrm{E}_{1+}\right)^{*}-12 \operatorname{Im} \mathrm{E}_{1+} \mathrm{M}_{1+}^{*} \\
& \operatorname{Im} \mathrm{C}_{6}=\operatorname{Im} \mathrm{I}_{7}=0
\end{align*}
$$

We should like to note:

1. The results in Table 7 show that the neglection of higher multipoles is in Group 2 more severe. This is particularly true for $\operatorname{Im} A_{6}$, $\operatorname{Im} A_{7}$ and $\operatorname{Im} B_{5}$.
2. From (III. 21) it follows that $\operatorname{Im} \mathrm{B}_{7}$ or $\operatorname{Im} \mathrm{C}_{5}$ allow a very direct determination of $\mathrm{E}_{1+}$. From (III. 2) it follows that

$$
\begin{equation*}
\mathrm{P}_{\mathrm{z}}=\kappa \sin 2 \phi \sin ^{2} \theta \frac{\operatorname{Im} \mathrm{~L}_{7}}{\mathrm{~S}_{1}} \quad \mathrm{p}_{\mathrm{x}}=\mathrm{p}_{\mathrm{z}}=0 \tag{III.24}
\end{equation*}
$$

With the values of Tables 6 and 7, one obtains then at $\mathrm{E}=340 \mathrm{MeV}$

$$
P_{z}=\kappa \sin 2 \phi\left\{\begin{array}{l}
+0.03  \tag{III.25}\\
-0.06 \\
-0.21
\end{array}\right\} \quad \begin{aligned}
& \theta=60^{\circ} \\
& \theta=90^{\circ} \\
& \theta=120^{\circ}
\end{aligned}
$$

Since $\kappa \approx 0.5$ and with $|\sin 2 \phi| \approx 1$, this experiment to determine Im $B_{7}$ should be feasible yielding an accuracy of less than $25 \%$ around $\theta=120^{\circ}$. According to (III. 19), (III. 22) and (III, 21)

$$
\begin{equation*}
\operatorname{Im}\left(A_{5}-B_{6}\right)=18 \operatorname{Im} E_{1+} M_{1+}^{*} \tag{III.26}
\end{equation*}
$$

and $\operatorname{Im} \mathrm{C}_{5}$ are independent quantities to determine $\mathrm{E}_{1+}$, but $\operatorname{Im}\left(\mathrm{A}_{5}-\mathrm{B}_{6}\right)$ is less accessible.
3. Again from (III. 19) and (III. 22) one obtains

$$
\begin{equation*}
\operatorname{Im}\left(2 A_{5}+B_{6}\right)=9 \operatorname{Im} M_{1-}\left(M_{1+}-E_{1+}\right)^{*} \tag{III.27}
\end{equation*}
$$

which is a quantity similar to (II. 3). Together with (III. 18) it fixes completely the complex number $\mathrm{M}_{1-}\left(\mathrm{M}_{1+}-\mathrm{E}_{1_{+}}\right)$, which would serve to determine $\mathrm{M}_{1-}$.

Finally we should like to mention that measurements of the quantities (III.12) and (III. 20) yield the real and imaginary part of $\mathrm{E}_{0+}\left(\mathrm{E}_{1+}-\mathrm{M}_{1+}\right)^{*}$, which would determine $\mathrm{E}_{0+}$ completely.
D. Final Remarks

The measurement of the cross section for simultaneously plane-polarized $\gamma^{\prime}$ s and a polarized target yields the additional information on $\operatorname{Im} L_{i}$ for $i=8,9,10$. In the approximation that all multipoles with $\ell \geq 2$ can be neglected, one obtains ${ }^{41}$

$$
\begin{align*}
& -\operatorname{Im} \mathrm{L}_{9}=+\operatorname{Im} \mathrm{L}_{10}=\mathrm{P}_{2}(\mathrm{E}, \cos \theta)_{\text {unpol. }}  \tag{III.28}\\
& {\operatorname{Im~} B_{8}}=\operatorname{Im} C_{8}=0 \tag{III.29}
\end{align*}
$$

Because of these equations and (III. 19) for $\operatorname{Im} \mathrm{A}_{8}$, no new information can be obtained from these kind of experiments as long as the analysis is restricted to multipoles with $\ell \leq 2$.

From the preceding discussion we have learned for some cases how one has to choose the experimental arrangements to improve the present accuracy for our experimental determination of the multipoles with $\ell=0$ and 1 . It turned out that measurements of the recoil polarization for photoproduction with planepolarized $\gamma^{\text {is }}$ and a polarized target are suitable for this purpose。 But utmost care should be taken in choosing the right kind of experimental arrangements. It is now the challenge to the experimentalist to decide which of the useful experiments are feasible. The results of these experiments can serve as an improved check for the approximation scheme in solving the partial amplitude dispersion relations. This scheme is presently based mainly on the isobar approximation as discussed in Section I. Furthermore these kinds of experiments performed with sufficient accuracy in $\pi^{ \pm}$- and $\pi^{0}$-photoproduction would yield, ultimately a check of the Watson theorem, which is an implication of unitarity and time reversal invariance.

## APPENDIX

We give here the relation of the parity conserving helicity amplitudes $H_{\lambda}^{\mathrm{J} \pm}(\mathrm{W})$ used in Section $I$ - to the more familiar multipole amplitudes $F_{\ell \pm}^{I}, M_{\ell \pm}^{I}{ }^{3}$ Let for $J=\ell+1 / 2$

$$
\begin{align*}
& \sqrt{2} \quad \mathrm{~F}_{3 / 2}^{\mathrm{J}-}(\mathrm{W})=[\ell(\ell+2)]^{1 / 2}\left[\mathrm{E}_{\ell+}(\mathrm{W})-\mathrm{M}_{\ell+}(\mathrm{W})\right]  \tag{A,1a}\\
& \sqrt{2} \quad \mathrm{~F}_{3 / 2}^{\mathrm{J}+}(\mathrm{W})=-[\ell(\ell+2)]^{1 / 2}\left[\mathrm{M}_{(\ell+1)-}(\mathrm{W})+\mathrm{E}_{(\ell+1)-}(\mathrm{W})\right]  \tag{A.1b}\\
& \sqrt{2} \quad \mathrm{~F}_{1 / 2}^{\mathrm{J}-}(\mathrm{W})=(\ell+2) \mathrm{E}_{\ell+}(\mathrm{W})+\ell \mathrm{M}_{\ell+}(\mathrm{W})  \tag{A.1c}\\
& \sqrt{2}  \tag{A.1d}\\
& \sqrt{2} \\
& \mathrm{~F}_{1 / 2}^{\mathrm{J}+}(\mathrm{W})=-(\ell+2) \mathrm{M}_{(\ell+1)-}(\mathrm{W})+\ell \mathrm{E}_{(\ell+1)-}(\mathrm{W})
\end{align*}
$$

with the inversion (again $J=\ell+1 / 2$ )

$$
\begin{array}{rlr}
\mathrm{E}_{\ell+} & =\frac{1}{\sqrt{2}} \frac{1}{\ell+1}\left(\mathrm{~F}_{1 / 2}^{\mathrm{J}-}+\left(\frac{\ell}{\ell+2}\right)^{1 / 2} \mathrm{~F}_{3 / 2}^{\mathrm{J}-}\right) \\
\mathrm{M}_{\ell+} & =\frac{1}{\sqrt{2}} \frac{1}{\ell+1}\left(\mathrm{~F}_{1 / 2}^{\mathrm{J}-}-\left(\frac{\ell+2}{\ell}\right)^{1 / 2} \mathrm{~F}_{3 / 2}^{\mathrm{J}-}\right) & \ell>0 \\
\mathrm{E}_{(\ell+1)-} & =\frac{1}{\sqrt{2}} \frac{1}{\ell+1}\left(\mathrm{~F}_{1 / 2}^{\mathrm{J}+}-\left(\frac{\ell+2}{\ell}\right)^{1 / 2}\right. & \left.\mathrm{F}_{3 / 2}^{\mathrm{J}+}\right) \\
\mathrm{M}_{(\ell+1)-} & =\frac{-1}{\sqrt{2}} \frac{1}{\ell+1}\left(\mathrm{~F}_{1 / 2}^{\mathrm{J}+}+\left(\frac{\ell}{\ell+2}\right)^{1 / 2}\right. & \left.\mathrm{F}_{3 / 2}^{\mathrm{J}+}\right) \tag{A.2d}
\end{array}
$$

Then we define

$$
\begin{equation*}
H_{\lambda}^{J \pm, I}(W)=F_{\lambda}^{J \pm, I}(W) \frac{1}{u_{\lambda}^{J \pm}(W)} \tag{A.3}
\end{equation*}
$$

with

$$
\begin{equation*}
u_{\lambda}^{J \pm}(W)=\mp \frac{C(\mp W)(q k)}{k^{\lambda-1 / 2}} \tag{A.4}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{C}(\mathrm{~W})=\frac{\mathrm{s}-\mathrm{M}^{2}}{16 \pi \mathrm{~s}}\left\{(\mathrm{~W}+\mathrm{M})^{2}-1\right\}^{1 / 2} \tag{A.5}
\end{equation*}
$$

Note that

$$
\begin{equation*}
C(-W)=-\frac{q(W)}{E_{2}(W)+M} \quad C(W) \tag{A.6}
\end{equation*}
$$

The McDowell symmetry relation has for the $H_{\lambda}^{\mathrm{J} \pm, \mathrm{I}}$ the very convenient form

$$
\begin{equation*}
H_{\lambda}^{\mathrm{J} \pm, \mathrm{I}}(-\mathrm{W})=\mathrm{H}_{\lambda}^{\overline{\mathrm{J}+}, \mathrm{I}}(\mathrm{~W}) \tag{A.7}
\end{equation*}
$$

which follows from (A.3) and the corresponding relation (2.10) in Ref. 20 for the $F^{p} s$

$$
\begin{equation*}
F_{\lambda}^{J \pm, I}(-W)=(-1)^{\lambda+1 / 2} F_{\lambda}^{J \bar{J}}(-W) \tag{A.8}
\end{equation*}
$$

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TABLE 1. $T_{\ell \pm}^{\mathrm{I}}(E)$ for the $J=1 / 2,3 / 2$ multipoles $M_{\ell \pm}^{\mathrm{I}}$ at different energies $E$

| E ( MeV ) | $\mathrm{E}_{0+}^{\mathrm{I}}$ |  |  | $\mathrm{M}_{1-}^{\mathrm{I}}$ |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 152 | 0.351 | 0.148 | 0.249 | 0.007 | 0.006 | 0.007 |
| 200 | 0.447 | 0.200 | 0.323 | 0.051 | 0.042 | 0.047 |
| 300 | 0.644 | 0.324 | 0.484 | 0.148 | 0.125 | 0.137 |
| 400 | 0.868 | 0.492 | 0.680 | 0.298 | 0.259 | 0.278 |
|  | $\mathrm{E}_{1+}^{\mathrm{I}}$ |  |  | $\mathrm{M}_{1+}^{\mathrm{I}}$ |  |  |
| 152 | 0.001 | 0.000 | 0.001 | 0.004 | 0.002 | 0.003 |
| 200 | 0.010 | 0.005 | 0.007 | 0.031 | 0.013 | 0.022 |
| 300 | 0.049 | 0.031 | 0.040 | 0.097 | 0.054 | 0.075 |
| 400 | 0.142 | 0.103 | 0.122 | 0.213 | 0.142 | 0.177 |
|  | $\mathrm{E}_{2}^{\mathrm{I}}$ |  |  | $\mathrm{M}_{2-}^{\mathrm{I}}$ |  |  |
| 152 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| 200 | 0.007 | 0.003 | 0.005 | 0.002 | 0.001 | 0.001 |
| 300 | 0.040 | 0.023 | 0.031 | 0.018 | 0.015 | 0.017 |
| 400 | 0.121 | 0.084 | 0.103 | 0.075 | 0.068 | 0.071 |

TABLE 2. $R\left(M_{\ell \pm}^{I}, M_{\ell \pm}^{I \prime}, W_{0}, W\right)$ for the $J=1 / 2, J^{\prime}=1 / 2,3 / 2$ multipoles for

$$
W_{0}=9.15\left(\mathrm{E}_{0}=400 \mathrm{MeV}\right) \text { and } \mathrm{W}=8.61(\mathrm{E}=300 \mathrm{MeV})
$$

| $\mathrm{M}_{\ell^{\prime}}^{\mathrm{I}} \mathrm{M}_{\ell^{\prime} \pm}^{\mathrm{I}^{\prime}}$ | $\mathrm{E}_{0+}^{1 / 2}$ | $\mathrm{E}_{0+}^{3 / 2}$ | $\mathrm{M}_{1-}^{1 / 2}$ | $\mathrm{M}_{1-}^{3 / 2}$ | $\mathrm{E}_{1+}^{1 / 2}$ | $\mathrm{E}_{1+}^{3 / 2}$ | $\mathrm{M}_{1+}^{1 / 2}$ | $\mathrm{M}_{1+}^{3 / 2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\mathrm{E}_{0+}^{1 / 2}$ | - | 0.02 | 0.01 | 0.01 | 0.74 | 0.42 | 0.01 | 0.01 |
| $\mathrm{E}_{0+}^{3 / 2}$ | 0.04 | - | 0.03 | 0.02 | 0.84 | 0.32 | 0.01 | 0.01 |
| $\mathrm{M}_{1-}^{1 / 2}$ | 0.00 | 0.00 | - | 0.00 | 0.01 | 0.04 | 0.01 | 0.01 |
| $\mathrm{M}_{1-}^{3 / 2}$ | 0.01 | 0.00 | 0.01 | - | 0.07 | 0.03 | 0.02 | 0.01 |


| $\mathrm{M}_{\ell \pm}^{I} \mathrm{M}_{\ell \pm}^{\mathrm{I}^{\prime}}$ | $\mathrm{E}_{0+}^{(0)}$ | $\mathrm{M}_{1-}^{(0)}$ | $\mathrm{E}_{1+}^{(0)}$ | $\mathrm{M}_{1+}^{(0)}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\mathrm{E}_{0+}^{(0)}$ | - | 0.03 | 0.10 | 0.01 |
| $\mathrm{M}_{1-}^{(0)}$ | 0.00 | - | 0.06 | 0.02 |

TABLE 3. $\widetilde{R}\left(M_{\ell \pm}^{I}, W_{0}, W\right)$ for the $J=1 / 2$ multipoles. $\mathrm{W}_{0}, \mathrm{~W}$ as in Table 2.

| $\mathrm{E}_{0+}^{(0)}$ | $\mathrm{E}_{0+}^{1 / 2}$ | $\mathrm{E}_{0+}^{3 / 2}$ | $\mathrm{M}_{1-}^{0}$ | $\mathrm{M}_{1-}^{1 / 2}$ | $\mathrm{M}_{1-}^{3 / 2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 0.06 | 0.07 | 0.06 | 0.03 | 0.03 | 0.03 |

TABLE 4. Comparison of results from (1.9)(first line) and from Ref. 10 (second line) $\mathrm{m}_{\pi}=\mathrm{m}_{\pi+}$ at $\mathrm{E}=200 \mathrm{MeV}$.

| $\operatorname{Re~M}_{\ell+}^{\mathrm{I}}$ | $\mathrm{E}_{0+}^{3 / 2}$ | $\mathrm{E}_{0+}^{1 / 2}$ | $\mathrm{E}_{0+}^{0}$ | $\mathrm{M}_{1-}^{3 / 2}$ | $\mathrm{M}_{1-}^{1 / 2}$ | $\mathrm{M}_{1-}^{0}$ | $\mathrm{E}_{1+}^{1 / 2}$ | $\mathrm{E}_{1+}^{0}$ | $\mathrm{M}_{1+}^{1 / 2}$ | $\mathrm{M}_{1+}^{0}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | -1.50 | 3.40 | -0.14 | -0.53 | 0.42 | -0.08 | 0.36 | 0.00 | -0.60 | 0.03 |
|  | -1.71 | 3.55 | -0.13 | -0.56 | 0.40 | -0.07 | 0.37 | 0.00 | -0.58 | 0.03 |
| $\operatorname{Re~M} \mathrm{M}_{2+}^{\mathrm{I}}$ | $\mathrm{E}_{2-}^{3 / 2}$ | $\mathrm{E}_{2-}^{1 / 2}$ | $\mathrm{E}_{2-}^{0}$ | $\mathrm{M}_{2-}^{3 / 2}$ | $\mathrm{M}_{2-}^{1 / 2}$ | $\mathrm{M}_{2-}^{0}$ |  |  |  |  |
|  | -0.23 | 0.41 | -0.03 | 0.03 | 0.06 | 0.00 |  |  |  |  |
|  | -0.24 | 0.43 | -0.03 | 0.03 | 0.06 | 0.00 |  |  |  |  |

TABLES 5, 6, 7. Predictions for the coefficients $A_{i}, B_{i}, C_{i}$ in the isobar approximation (sol. 5).

TABLE 5

| E <br> $[\mathrm{MeV}]$ | $\operatorname{Re~} \mathrm{A}_{2}$ | $\operatorname{Re} \mathrm{C}_{2}$ | $\operatorname{Re} \mathrm{~B}_{9}$ | $\operatorname{Re} \mathrm{~B}_{5}$ | $\operatorname{Re} \mathrm{~A}_{7}$ | $\mathrm{~B}[\mu \mathrm{~b} / \mathrm{st}]$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 280 | -0.6 | 0.6 | -1.2 | -0.5 | 1.3 | -2.7 |
| 340 | 2.8 | -3.7 | -4.1 | 7.2 | 8.0 | 3.0 |
| 400 | 3.8 | -3.2 | -2.9 | 7.1 | 7.6 | 4.7 |

TABLE 6

| E <br> $[\mathrm{MeV}]$ | $\operatorname{Re~} \mathrm{B}_{2}$ | $\operatorname{Re~} \mathrm{~A}_{5}$ | $\operatorname{Re} \mathrm{D}_{2}$ | $\operatorname{Re~} \mathrm{C}_{9}$ | $\operatorname{Re} \mathrm{~B}_{7}$ | $\operatorname{Re~} \mathrm{C}_{5}$ | A | $\mathrm{C}[\mu \mathrm{b} / \mathrm{st}]$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 280 | 26.1 | -6.3 | -34.2 | -33.6 | 31.1 | 31.7 | 23.9 | -15.2 |
| 340 | 36.7 | -15.1 | -43.9 | -43.4 | 38.9 | 39.4 | 33.2 | -26.2 |
| 400 | 17.8 | -10.4 | -21.5 | -21.0 | 18.3 | 18.8 | 17.7 | -15.1 |

TABLE 7

| E <br> $\mathrm{MeV}]$ | $\operatorname{Im} \mathrm{A}_{5}$ | $\operatorname{Im} \mathrm{~A}_{8}$ | $\operatorname{Im} \mathrm{C}_{5}$ | $\operatorname{Im} \mathrm{~B}_{7}$ | $\operatorname{Im} \mathrm{~B}_{6}$ | $\operatorname{Im} \mathrm{~A}_{7}$ | $\operatorname{Im} \mathrm{~B}_{5}$ | $\operatorname{Im} \mathrm{~B}_{8}[\mu \mathrm{~b} / \mathrm{st}]$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 280 | 3.3 | -2.9 | 6.4 | 6.2 | -6.0 | -5.5 | -6.7 | 2.2 |
| 340 | 5.4 | -4.9 | 8.9 | 8.3 | -4.1 | -2.9 | -5.1 | 4.2 |
| 400 | 3.0 | -2.3 | 4.5 | 4.1 | 0.4 | 1.2 | -0.3 | 2.6 |

## FIGURE CAPTIONS

1. Real parts of $J=1 / 2,3 / 2$ multipoles for the three isospin combinations $\mathrm{I}=0,1 / 2,3 / 2$. Pole term approximation: dashed line, isobar approximation: full line with $\operatorname{Im} \mathrm{E}=0$, dash-point-line with $\operatorname{Im} \mathrm{E}_{0+} \neq 0$. For $\mathrm{J}=\mathrm{I}=3 / 2$, $P=+1$, solution 5 (Section In) has been taken. In some indicated cases results have been multiplied by a factor of 10 or 100 .
2. Real parts of $\mathrm{J}=1 / 2$ multipoles following from subtracted dispersion relations in the isobar approximation (solution 2). Dash-point-line unsubtracted and full line subtracted dispersion relations; dashed line error limits arising from the subtraction constant.
(a) $\pi^{0}$-production
(b) $\pi^{+}$-production
3. Present range of uncertainty for $\operatorname{Im} M_{1+}^{3 / 2}\left(W_{f}\right)$ and $\operatorname{Im} E_{1+}^{3 / 2}\left(W_{f}\right)$ with $W_{f}$ $=9.016$, i.e., $\mathrm{E}_{\mathrm{f}}=320 \mathrm{MeV}$. The numbers of 1 to 6 correspond to the solutions used in this paper $\mathrm{m}_{\pi}=\mathrm{m}_{\pi \mathrm{o}}$.
4. The coefficients $\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}$ and $\mathrm{A}-\mathrm{I}_{0}$ in $\pi^{0}$-production for solutions $1,2,3$ and 5 over the photon energy E. Experimental results for A: $\Phi_{1} 22,25,26, \not{ }^{23}$,
 interpolated values for $A$ used to calculate ( $\mathrm{A}-\mathrm{I}_{0}$ ).
5. (a) The ratio $\mathrm{I}_{0} / \mathrm{C}$ and $\mathrm{E}_{1+}^{3 / 2} / \mathrm{M}_{1+}^{3 / 2}$ for the solutions 1, 2, 3 and 5. $\mathrm{I}_{0} / \mathrm{A}$ from $\Phi^{28}, \mathbb{I}^{29}$, $\mathbb{I}^{30}$, $\Psi^{31} ;$--- Multipole analysis ${ }^{21}$; A/C, see Fig. 5 b.
(b) The experimental ratio C/A with an eye-fit $\Phi^{25}, \Phi^{22}, \Phi^{23}$.
6. The coefficients $a$ and $b$ for the polarization of the recoil proton in $\pi^{\circ}$ production, old data $\Phi^{33}$, new data $\Phi^{34}$.
7. $\pi^{0}$-production in backward direction for solutions $1,2,3$ and 5 ; $\mathbb{I}^{25}$, $\Phi^{22}$, $\Phi^{15 \mathrm{~b}}$, $\Phi^{35}$ with magnet spectrometer; 古 $^{35}$ with telescope spectrometer.
8. $\pi^{+}$excitation curves at $\theta=0^{\circ}$ and $90^{\circ}$ for $\frac{\mathrm{d} \sigma}{\mathrm{d} \Omega}(\theta, E)$ unpol. and for $\frac{\mathrm{d} \sigma}{\mathrm{d} \Omega}(\theta, E)_{\text {unpol. }}-\frac{\mathrm{d} \sigma}{\mathrm{d} \Omega}\left(\theta, \mathrm{E}, \phi=90^{\circ}\right)$ at $\theta=90^{\circ}$. Experimental results from $36,37 \mathrm{a}$ data and the compilation. 37 b .
9. The difference $S_{\exp }-S_{\text {theor. }}$ (see (II.5)) and $\Delta \mathrm{S}$ (see text) for solutions $2\left(\begin{array}{l}\Phi \\ (1)\end{array}\right.$ and $5\binom{\mathrm{I}}{\mathrm{I}}$. Experimental results, Refs. 28, 29 and 30.
10. Angular distribution for $\frac{d \sigma}{d \Omega}(E, \theta)$ unpol. at $E=350 \mathrm{MeV}$ and for $\frac{\mathrm{d} \sigma}{\mathrm{d} \Omega}\left(\mathrm{E}, \theta, \phi=90^{\circ}\right.$ ) at $\mathrm{E}=280 \mathrm{MeV}$. Dashed line prediction according to Part I. At 280 MeV the experimental results Ref. 36 at 350 MeV from Rcf. 37a, b.
11. $\pi^{+}$excitation curve for the asymmetry $A_{R}(E, \theta)=\frac{d \sigma_{1}-d \sigma_{11}}{d \sigma_{1}+d \sigma_{11}}$ at Ref. 36.
12. Coordinate system for photoproduction.


95ci
FIG. I


FIG. 2a


FIG. 2b


FIG. 3
$\overline{102 \mathrm{BA}^{2}}$


FIG. 40
$\overline{1028 \mathrm{CB}}$


FIG. $4 b$



FIG. Sa


FIG. 5b


FIG. 6




FIG. 9


FIG. 10 a


FIG. 10b
$\overline{1028 C 11}$



FIG. I2

$$
\overline{1028 \mathrm{Al3}}
$$

