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CALCULATIONS OF THE THREE-NUCLEON LOW ENERGY PARAMETERS*†

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ABSTRACT

Since exact calculations of the three-nucleon low energy parameters using separable models assume nuclear forces which do not reproduce experimental two-nucleon data, and variational calculations with "realistic" forces have not converged, comparison of either type of calculation with experiment is premature. However, the exact calculations can be used for parameter studies, and these studies shed some light on which features of the two-nucleon interaction must be known in order to make reliable calculations from first principles. After a review of the mathematical and physical structure of the non-relativistic quantum mechanical three-body problem to clarify what physical input is needed, and a review of whether and if so to what accuracy this information is available from two-nucleon experiments, existing calculations of the binding energy of the triton ϵ_t, and of the n-d doublet and quartet scattering lengths, a_2 and a_4, are reviewed from this point of view. So far as two-nucleon input goes, new

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experimental information has removed a serious source of difficulty by confirming the theoretical prediction from charge-independence that the n-p $^1S_0$ effective range r_s = 2.73 ± 0.03 F, but a comparable sensitivity to the percentage D state in the deuteron (where the evidence is now at least as good for 4-$\frac{1}{2}$% as for the conventional value of 7%) pointed out by Phillips will require extensive further investigation. Existing calculations of ϵ_t and a_2 do not discriminate clearly between effects due to differences in the models arising from differences in their fit to nucleon-nucleon S phases up to 300 MeV, differences due to different behavior of the phases at infinite momentum, differences (which are quite large between local and separable potentials) near but off the energy shell, and differences in asymptotic behavior far from the energy shell. However, they do show clearly that such differences either separately or in conjunction can lead to differences in the calculated value of ϵ_t of 1-4 MeV, and of a_2 of 1-2 F. Both uncertainties are one to two orders of magnitude larger than the experimental error in either quantity, and also larger than estimated effects due to three-body forces arising from meson exchange. They also make it likely that, unless further work is done on electromagnetic properties of the two-nucleon system with an eye on extracting off-shell (wave function) information, a discrepancy of a few MeV in ϵ_t between equally realistic models for the nuclear force will continue to exist even after variational and Faddeev calculations have converged to the same unique (mathematically speaking) answers for these models.

I. INTRODUCTION

Theoretical nuclear physics has developed an impressive body of phenomenology over the past three decades without knowing much about the nuclear force other than that it is strong, short-range and approximately charge-independent, aided by generous empirical input about specific systems. Yet one would like to believe that, given a knowledge of the nuclear force, one should be able to calculate all of these phenomenological models from first principles, just as one believes that all of atomic and molecular physics are in principle calculable knowing the charge and mass of the electron, Planck's constant, the velocity of light, and the nuclear masses (and for subtle effects their spins). Yet this confidence in the case of atomic systems came about because of quantitative success in calculating the three as well as the two particle systems; in particular, the success in understanding Helium gave confidence in the exclusion principle on which all of chemistry rests. Such a basis is still missing in nuclear physics.

Two places exist where quantitative tests of our understanding of the nuclear forces may be attempted with some hope of success during the next decade. One of these, surprizingly enough, is the calculation of the binding energy and density of infinite nuclear matter. But this assumes at the outset that all coulomb effects and surface effects have been exactly represented by the semi-empirical mass formula for finite nuclei, and that once these are turned off there is exact charge-independence of the nuclear force. Calculation of the three-nucleon system from first principles is harder, but has the advantage that there are

more parameters to be compared with experiment, and that quite subtle effects can be tested experimentally if there is theoretical reason to think them interesting[1,2]. Familiarity with the partial conservation law of charge-independence has tended to breed acceptance, and even imitation in the proliferating models for elementary particles. But it is a very peculiar thing to have a symmetry which is not directly tied to a symmetry of the system of description (in which case it would be exact) and hence must represent some feature of the interaction which clashes with other aspects. It is obviously useful to glean any information we can about how this works out in more complicated systems. It is also clearly important for nuclear physics to know at what point the three-nucleon problem ceases to be a three-body problem and begins to bring in measurable meson effects; until this is known, the limits of where nuclear physics can safely be treated as a non-relativistic problem (or even if there is ever such a situation), and where it becomes necessarily a part of elementary particle physics, cannot be set.

Unfortunately, the three-body problem is so complicated that until now attempts to answer these questions have led only to ambiguous answers. Fortunately by now, however, utterly different calculational and mathematical techniques (variational and Faddeev) have led to identical numerical answers for the simplest three-body models. Hence these models can be used with confidence, on the numerical side, to estimate the sensitivity of the problem to various uncertainties. These results are rather disturbing, at least if one is concerned with the physics of the three-nucleon problem and not just with solving a mathematical puzzle. We will therefore try to keep our focus on the physics of the situation,

and leave to other speakers the discussion of more technical mathematical aspects of the problem, and whether one or another approach will prove more practical. For our own part, we will hesitate to believe any complicated three-nucleon calculation until at least two independent calculations using different mathematical and numerical techniques have reproduced the same numerical answer for the same model; but that is only a prelude to the physics.

II. MATHEMATICAL AND PHYSICAL STRUCTURE OF THE THREE-PARTICLE SCHROEDINGER EQUATION

We consider three distinguishable, spinless particles of masses m_1, m_2, m_3 interacting only through forces between each pair. Initially we consider forces with depend only on the magnitude of the distances $\underline{\rho}_i = \underline{R}_j - \underline{R}_k$ between particle j at $\underline{R}_j$ and particle k at $\underline{R}_k$, and which can be derived from the three local potentials $V_i(\rho_i)$, but the generalization to non-local and/or velocity-dependent interactions becomes trivial at a later stage. If the total energy eigenvalue of the time-independent Schroedinger equation for this system is Z, and K is the usual kinetic energy operator, the equation we wish to solve is

$$(K + V_1 + V_2 + V_3)\Psi = Z\Psi \tag{II.1}$$

For three free particles in the final state, the general stationary state scattering wave function given by Goldberger and Watson[3] becomes

$$\Psi(\underline{R}_1,\underline{R}_2,\underline{R}_3) = \chi_I(\underline{R}_1,\underline{R}_2,\underline{R}_3) + \frac{1}{(2\pi)^{9/2}} \int d^3P_1 \int d^3P_2 \int d^3P_3 \delta^3(\underline{P}-\underline{P}_1-\underline{P}_2-\underline{P}_3) \frac{e^{i(\underline{P}_1\cdot\underline{R}_1+\underline{P}_2\cdot\underline{R}_2+\underline{P}_3\cdot\underline{R}_3)} < \underline{P}_1\underline{P}_2\underline{P}_3 | T(Z) | I >}{Z + i\epsilon - (P_1^2/2m_1 + P_2^2/2m_2 + P_3^2/2m_3)} \tag{II.2}$$

where I is any initial state and the limit $\epsilon \to 0^+$ is implied. For a normal laboratory experiment, we have to start with a bound state of two of the particles, so in that case χ_I will correspond to that bound state wave function, a plane wave for the motion of the center of mass of that state, and a plane wave for the third particle. In other words, it will be a stationary state solution of the simpler equation $(K + V_1)\chi_I = Z\chi_I$ if we choose the bound state to be of particles 2 and 3. Then, in addition to the term given in (II.2), we must include an outgoing wave corresponding to elastic scattering from this bound state, and if there are any other bound states between this or any other pair in the system, these must also be included. Formally, this can be accomplished by assuming that T contains additional terms with the appropriate δ-functions needed to insure the right kinematic relations for these states. We assume Z to be zero at three-particle breakup threshold, so if Z is negative, only these two-particle bound state plus single free particle terms will survive, and if Z is so negative that none of these are allowed, we look for bounded solutions of the homogeneous equation corresponding to three-particle bound states.

This complicated set of boundary conditions on (II.1) is not the easiest thing in the world to apply. If we insert (II.2) into (II.1) and go over to momentum space, thus obtaining the Lippmann-Schwinger equation for T, we find that the mathematical problem so posed is ambiguous. This difficult problem was solved by Faddeev[4,5], who found that the separation of the T matrix into channels in which one of the particles was asymptotically free, while the other two could continue to interact, after a sufficient number of iterations led to integral equations of the Fredholm

type. In configuration space, this amounts to replacing (II.1) by the three equations

$$\begin{aligned}(K + V_1 - Z)\Psi_1 &= - V_1(\Psi_2 + \Psi_3)\\ (K + V_2 - Z)\Psi_2 &= - V_2(\Psi_3 + \Psi_1)\\ (K + V_3 - Z)\Psi_3 &= - V_3(\Psi_1 + \Psi_2)\end{aligned} \qquad \text{(II.3)}$$

with $\Psi = \Psi_1 + \Psi_2 + \Psi_3$ and the boundary condition that the Ψ_i approach stationary solutions of the three separate equations obtained by setting the right hand sides of (II.3) equal to zero. Clearly this will work only if the right hand sides vanish asymptotically, which we now prove for short-range forces.

These equations in 9 variables can be reduced to two continuous variables and a set of discrete quantum numbers as follows. First take out the center of mass coordinate $(m_1\underline{R}_1 + m_2\underline{R}_2 + m_3\underline{R}_3)/M$ with conjugate momentum $\underline{P}$, and the corresponding energy by defining $z = Z - P^2/2M$. The remaining six coordinates can be represented by the vector between any two particles $\underline{\rho}$, and the vector $\underline{r}$ from the center of mass of that pair to the third particle, in three different ways; these vectors, and their conjugate momenta are

$$\underline{r}_i = \underline{R}_i - (m_j\underline{R}_j + m_k\underline{R}_k)/(m_j + m_k);\ \underline{p}_i = [(m_j + m_k)\underline{P}_i - m_i(\underline{P}_j + \underline{P}_k)]/M$$

$$\underline{\rho}_i = \underline{R}_j - \underline{R}_k;\ \underline{q}_i = (m_k\underline{P}_j - m_j\underline{P}_k)/(m_j + m_k) \qquad \text{(II.4)}$$

$$z = p_i^2/2M_i + q_i^2/2\mu_i \qquad M_i = \frac{(m_j + m_k)m_i}{M} \qquad \mu_i = \frac{m_j m_k}{m_j + m_k}$$

If $\underline{r}_i$ and $\underline{\rho}_i$ make angles $\Theta_i\Phi_i$ and $\theta_i\varphi_i$ respectively with space-fixed we can immediately reduce the left hand side of (II.3) to two radial variables

by introducing the orthonormal angular momentum functions $Y^M_{J\ell\lambda}$ as defined by Blatt and Weisskopf[6] and the expansion

$$\Psi^M_i = \sum_{J\ell\lambda} \frac{u^i_{\ell\lambda}}{r_i \rho_i} Y^M_{J\ell\lambda} (\Theta_i \Phi_i; \theta_i, \varphi_i) \qquad \text{(II.5)}$$

Since J and M are constants of the motion, we will omit them whenever possible in the following equations. The difficulty, of course, is that the right hand side of (II.3) is expressed in terms of the wrong variables.

In order to calculate the projection of the right hand side onto our space of two continuous variables by the operator $\int Y^{M*}_{J\ell\lambda} d\Omega_j$, we make a transformation of unit Jacobian to a set of four new angles. Following Omnes[7] we pick these to be the three Euler angles α, β, γ of a body-fixed axis lying in the plane of the triangle defined by the three particles, the angle ζ between $\underline{r}$ and $\underline{\rho}$, and a constant parameter ξ which gives the angle between this body-fixed axis and one of the vectors in the plane; for definiteness we take this to be the angle to $\underline{r}_1$. It is then a simple exercise in rotation matrices to show that the angular momentum functions in these coordinates become, in the notation of Edmonds[8],

$$Y^M_{J\ell\lambda}(\theta_1, \psi_1; \theta_1, \psi_1) = \sum_{M'} Y^{M'}_{J\ell\lambda}(\xi, 0; \xi+\zeta_1, 0) D^J_{M'M}(\alpha, \beta, \gamma) \qquad \text{(II.6)}$$

Since the angles α, β, γ are common to all three coordinate systems, three of the four integrations just give an orthogonality integral, and we are left with a single integral over ζ. Since all vectors now lie in a plane, a series of plane rotations are all that is needed to relate the angle ξ in one coordinate system to that in another, and all angular and radial

coordinates in one system to the other. To be explicit, we consider how to express r_2, ρ_1, ζ_2, and ζ_{12} (the angle between $\underline{r}_1$ and $\underline{r}_2$) in terms of r_1, ρ_1, and ζ_1. This is

$$\begin{aligned} r_2^2 &= a^2 r_1^2 + b^2 \rho_1^2 - 2abr_1\rho_1 \cos\zeta_1 \\ \rho_2^2 &= r_1^2 + c^2\rho_1^2 + 2cr_1\rho_1\cos\zeta_1 \\ r_2\rho_2\cos\zeta_2 &= ar_1^2 - bc\rho_1^2 + (ac-b)r_1\rho_1\cos\zeta_1 \\ r_1 r_2\cos\zeta_{12} &= ar_1^2 - br_1 r_2\cos\zeta_1 \end{aligned} \qquad \text{(II.7)}$$

$$a = m_1/(m_2+m_3) \qquad b = m_3 M/(m_2+m_1)(m_3+m_2) \qquad c = m_2/(m_2+m_3)$$

The corresponding expressions for r_3, ρ_3, ζ_3 and ζ_{12} are obtained by the cyclic permutation $2 \to 3$, $3 \to 1$, $1 \to 2$ in a, b, c and by $\zeta_1 \to -\zeta_1$.

The final result is the differentio-integral equation

$$\left[\frac{d^2}{dr_1^2} - \frac{\ell(\ell+1)}{r_1^2} + \frac{d^2}{d\rho_1^2} - \frac{\lambda(\lambda+1)}{\rho_1^2} - V_1(\rho_1) + E\right] u^1_{\ell\lambda}(r_1,\rho_1) =$$

$$+ V_1(\rho_1) \sum_{\ell'\lambda'} \sum_{s=2,3} \int_{-1}^{1} d(\cos\zeta_1) K^{1s}_{\ell\lambda;\ell'\lambda'}(r_1,\rho_1,\cos\zeta_1) \frac{r_1\rho_1 u^s_{\ell'\lambda'}(r_s,\rho_s)}{r_s\rho_s} \qquad \text{(II.8)}$$

and the corresponding equations for u^2 and u^3. Here r_s and ρ_s are to be expressed in terms of r_1, ρ_1 and $\cos\zeta_1$ by means of (II.7), and the kernel is given by

$$K^{12}_{\ell\lambda,\ell'\lambda'} = \frac{8\pi^2}{(2J+1)} \sum_{MM'} Y^{M*}_{J\ell\lambda}(\xi,0,\zeta_1+\xi,0)\, Y^{M'}_{J\ell'\lambda'}(\xi+\zeta_{12},0,\xi+\zeta_{12}+\zeta_2,0) \qquad \text{(II.9)}$$

or the obvious generalizations. Since our purpose at this point is to

investigate the structure of the source term, we do not attempt further simplification.

We see immediately from (II.8) that if V_1 vanishes asymptotically like, say, $e^{-\rho_1/R}$ that the source has the necessary compactness in this coordinate. This is to be contrasted with the situation which would have resulted if we had not made the Faddeev channel decomposition, since in that case the source term continues to exist along a diagonal strip in the ρ, r plane which runs to infinity; that fact frustrated earlier attempts to formulate three-particle boundary conditions in configuration space. It might be hoped that since particle 1 in this channel is asymptotically free, and is outside the range of force of the interacting pair for $r_1 > R$, that the source would also fall off like $e^{-r_1/R}$, but this is not the case. In fact, for $\rho_1 < R$ and r_1 large, we see from (II.7) that $r_2 \to a r_1$ and $\rho_2 \to r_1$. Hence, apart from angular factors of order unity, the source term approaches $\rho_1 V_1(\rho_1) u^s(a r_1, r_1)/a r_1$, and since u^s is also of order unity asymptotically, the source only falls off like $1/r_1$.

This long-range character of the source in the limit we are considering (ρ_1 less than the range of forces and r_1 large) has a ready physical interpretation which is illustrated in Figure 1. If all three particles were outside the range of forces, they would have to have their energy and momentum individually connected as they are for free particles, and if we are discussing the final state, no further scattering would be possible and the source would have to be zero (which is guaranteed by the vanishing of V_1). However, if particles 2 and 3 are still within the range of force, their energy and momentum are not necessarily connected as they are for free particles, and if they can pick up momentum from

somewhere, can still scatter into the final state. This momentum can be supplied from the outgoing spherical wave in one of the two other channels, as illustrated, and since the amplitude of this wave only falls off like $1/r_1$, the behavior of the source term has been explained. Were this not the case, we could readily write down the Green's function for the left hand side of (II.8), $(K + V_1 - z)^{-1}$, and applying this to the equations obtain integral equations for u^s of very similar structure to those for two-particle scattering, as has been discussed formally by Sasakawa[9]. We can, of course, do this but the resulting integrals extend over such a large region of r_1, that it looks unlikely that such an approach will provide a practical method for solving three-body problems even with short-range forces. We have presented the result in order to clarify the physical origin of the difficulty of solving these equations in configuration space, and also because the explicit formal structure in ρ and r may still have some advantages in clarifying asymptotic forms and phases in configuration space. But we agree with Faddeev that simple singularities in momentum space are easier to handle than $1/r$ source terms in coordinate space, and now turn to that approach.

If we specialize (II.2) by assuming that χ_I is a plane wave (i.e. solve for the scattering of three free particles to three free particles - from which we can construct any other case), take out the center-of-mass motion, and introduce momenta $\underline{p}$ and $\underline{q}$ conjugate to $\underline{r}$ and $\underline{\rho}$ respectively, the insertion of (II.2) into (II.1) and transformation to momentum space gives immediately the Lippmann-Schwinger equation

$$\langle \underline{p}'\underline{q}' \left| T(Z) \right| \underline{pq} \rangle = \langle \underline{p}'\underline{q}' \left| V \right| \underline{pq} \rangle$$

$$+ \int d^3p'' \int d^3q'' \langle \underline{p}'\underline{q}' \left| V \right| \underline{p}''\underline{q}'' \rangle \frac{1}{z - \frac{p''^2}{2M_i} - \frac{q''^2}{2M_i}} \langle \vec{p}''\vec{q}'' \left| T(Z) \right| pq \rangle \qquad \text{(II.10)}$$

or, in operator form

$$T(Z) = V + V\, G_0(Z) T(Z) \qquad \text{(II.11)}$$

As noted above, this as it stands does not pose a well-defined mathematical problem, but if we write $T = \Sigma_i T_i$, $V = \Sigma_i V_i$, and rearrange terms we see immediately that it is equivalent to the system of equations

$$\begin{aligned}
(1 - V_1 G_0(z)) T_1(z) &= V_1 + V_1 G_0(z)[T_2(z) + T_3(z)] \\
(1 - V_2 G_0(z)) T_2(z) &= V_2 + V_2 G_0(z)[T_3(z) + T_1(z)] \qquad \text{(II.12)} \\
(1 - V_3 G_0(z)) T_3(z) &= V_3 + V_3 G_0(z)[T_1(z) + T_2(z)]
\end{aligned}$$

We saw above in our configuration space treatment that if we knew the stationary state solutions of $(K + V_i - z)\Psi_i = 0$, we could immediately construct the integral equations for the three-body wave function. This amounts to solving the two-particle Lippmann-Schwinger equation in the three-particle Hilbert space, i.e. solving

$$(1 - V_i G_0(Z)) t_i(Z) = V_i \qquad \text{(II.13)}$$

or

$$\langle \underline{q}' \left| t_i(z - \frac{p^2}{2M_i}) \right| \underline{q} \rangle = \langle \underline{q}' \left| V_i \right| \underline{q} \rangle$$

$$+ \int d^3q'' \langle \underline{q}'' \left| V_i \right| \underline{q}'' \rangle \frac{1}{z - \frac{p^2}{2M_i} - \frac{q''^2}{2\mu_i}} \langle \underline{q}'' \left| V_i \right| \underline{q} \rangle \qquad \text{(II.14)}$$

where we have factored out $\delta^3(\underline{p}' - \underline{p})$ from both sides. This differs from the usual equation for two-particle elastic scattering in that (a) the energy is $z - p^2/2M_i$ rather than the on-shell value $q^2/2\mu_i$ and (b) that q' can differ from q in magnitude as well as in direction. This equation can be solved as we will discuss further below, so the operator $(1 - V_i G_0(z))$ has a well defined inverse. Substituting the expression for V_i given in (II.13) into (II.12) gives immediately

$$(1 - V_i G_0(z))T_i = (1 - V_i G_0(z))t_i[1 + G_0(z)(T_j + T_k)] \qquad \text{(II.15)}$$

so that by applying the inverse $(1 - V_i G_0)^{-1}$ to both sides we immediately obtain the Faddeev equations

$$T_i = t_i + t_i G_0 (T_j + T_k) \qquad \text{(II.16)}$$

or more explicitly

$$\begin{aligned} &\langle \underline{p}'\underline{q}' | T_i(z) | \underline{p}\underline{q} \rangle = \langle \underline{q}' | t_i(z - \frac{p^2}{2M_i}) | \underline{q} \rangle \delta^3(\underline{p}' - \underline{p}) \\ &+ \int d^3p'' \int d^2q'' \langle \underline{q} | t_i(z - \frac{p''^2}{2M_i}) | \underline{q} \rangle \delta^3(p' - p'') \frac{1}{z - \frac{p''^2}{2M_i} - \frac{q''^2}{2\mu_i}} \\ &\langle \underline{p}''\underline{q}'' | T_j(z) + T_k(z) | \underline{p}\underline{q} \rangle \end{aligned} \qquad \text{(II.17)}$$

Clearly we could now reduce these equations to two continuous variables by using the orthonormal angular momentum functions, which was in fact the route followed by Ahmadzadeh and Tjon[10]. An alternative route is to first follow Omnes[7] by introducing the three discrete quantum numbers J, M, λ corresponding to the total angular momentum and its projection on space- and body-fixed axes, and the three energies $\omega_i = P_i^2/2m_i$; Osborn and Noyes[11] showed that this system can then be reduced to two variables. The final

result as given by Osborn is[12]

$$F^{P(i)}_{\ell m}(E', e'_i) = \frac{(m_j + m_k)\hbar^2}{2\pi\sqrt{2m_i e'_i}} \left\{ \delta(\omega_i - e'_i) P^m_\ell(\cos\gamma_i) d^J_{\lambda m}(-\alpha_i) \right.$$

$$t^{(i)}_\ell(E - r_i e_i, E' - r_i e'_i, z - r_i e_i) - \sum_{s=j,k} \int_0^\infty de''_s \int_{r_s e''_s}^\infty dE'' \qquad \text{(II.18)}$$

$$\frac{t^{(i)}_\ell(E' - r_i e'_i, E'' - r_i e'_i; z - r_i e'_i)}{E'' - z} \sum_{\ell'' m''} K^{(i,s)}_{\ell m;\ell'' m''}(E', e'_i; E'', e''_s)$$

$$\left. F^{P(s)}_{\ell'' m''}(E'', e''_s) \right\}$$

where we have made the expansion

$$\langle \vec{\omega}' J \lambda' \left| T_i(z) \right| \vec{\omega} J \lambda \rangle = \sum_{\ell=0}^{\infty} \sum_{m=-J}^{J} (2\ell+1) \frac{(\ell - m)!}{(\ell + m)!} d^J_{\lambda' m}(+\alpha'_i) P^m_\ell(\cos\gamma'_i) \qquad \text{(II.19)}$$

$$F^{P(i)}_{\ell m}(E', e'_i)$$

with $e'_i = \omega'_i$,

and the kernel is

$$K^{(i,s)}_{\ell m;\ell'' m''}(E', e'_i; E'', e''_s) = (2\ell''+1) \frac{(\ell'' - m'')!}{(\ell'' + m'')!} P^m_\ell\left(\cos\gamma_i(E'', e''_s, \cos\Gamma''_s)\right)$$

$$P^{m''}_{\ell''}(\cos\Gamma''_s) d^J_{mm''}(\theta^{''}_{is}) \left\{ \Theta\left[E'' - r_s e''_s - \right.\right. \qquad \text{(II.20)}$$

$$\left. - \left(\sqrt{\frac{m_{s'} m_i}{m_{s'}}} e'_i - \sqrt{\frac{m_i m_s}{m_{s'}(m_{s'} + m_i)}} e''_s \right)^2 \right]$$

$$\left. - \Theta\left[E'' - r_s e''_s - \left(\sqrt{\frac{m_{s'} + m_i}{m_{s'}}} e'_i + \sqrt{\frac{m_i m_s e''_s}{m_{s'}(m_s + m_i)}} \right)^2 \right] \right\}$$

where cos Γ_s is given by

$$\cos \Gamma_s(e_i', E'', e_s'') = \pm \frac{\frac{m_{s'}+m_i}{m_{s'}} e_i' - (E'' - r_s e_s'') - \frac{m_i m_s e_s''}{m_{s'}(m_{s'}+m_i)}}{2\left(\frac{m_i m_s e_s''}{m_{s'}(m_{s'}+m_i)}\right)^{1/2} (E'' - r_s e'')^{1/2}} \qquad \text{(II.21)}$$

where the + applies for s = j and the - for s = k, and $r_i = M(/m_j + mk)$.

The argument of the d function θ_{is}'' must be determined in terms of E'', e_s''. This can be done by taking the ratio of $\sin^2\theta_{is}$ and $\sin^2\gamma_s$.

$$\frac{\sin^2\theta_{is}}{\sin^2\gamma_s} = \frac{\frac{4m_1m_2m_3}{(m_i+m_{s'})}\omega_s\left(\omega_i+\omega_s+\omega_{s'} - \left(1+\frac{m_s}{m_{s'}+m_k}\right)\omega_s\right)}{4m_i m_s \omega_i \omega_s} \qquad \text{(II.22)}$$

$$= \frac{m_{s'}}{m_i + m_{s'}} \frac{(E - r_s \omega_s)}{\omega_i}$$

Thus the argument to be used in $d^J_{mm''}(\theta_{is}'')$ is determined by

$$\sin^2\theta_{is}'' = \frac{m_{s'}}{m_i+m_{s'}} \sin^2\Gamma_s(e_i', E'', e_s'') \frac{(E'' - r_s e_s'')}{\omega_i(E'', e_s'', \Gamma_s)} \qquad \text{(II.23)}$$

A number of points need to be made about this system of equations. The first is that the sum over ℓ runs to infinity, so the reduction is of use only for interactions such that a small number of angular momentum states dominate the two-body interactions; fortunately this is true of the nuclear force at low energy. The second is that the driving term and the kernel of the equation require a knowledge of the off-shell two-body t-matrix over a range of variables inaccessible to two-body scattering experiments. If we have some model for the potential, or more generally

for the off-shell T-matrix, this is no problem, but if we have to rely on empirical input, this means we must know much more than the two-body phase shifts[1,2]. It is easy to show[13] that in fact what we need to know in addition to the phase shifts is the wave function inside the range of forces (or, equivalently $t_\ell(p,q;p^2/2\mu) = t_\ell(q,p;p^2/2\mu)$) from which the full off-shell function $t_\ell(p,q;z)$ can be constructed by a quadrature (the Low equation). The availability of this information for the two-nucleon system will be discussed in the next section. The third point to note is that, now we have introduced the explicit two-body angular momentum states, the generalization to non-local, and in particular separable, interactions is trivial. Of course, if we were to restrict ourselves to separable interactions from the start, a simpler derivation is possible, as was shown long ago by Mitra[14]; obviously, it is a useful check on the complicated algebra to follow both routes. More importantly, we see explicitly that the assumption of a separable interaction commits us to a specific assumption about the form of the wave function inside the range of nuclear forces. We are therefore required either to show that the results are insensitive to this assumption, or that it is compatible with other information about the two-nucleon system. Unfortunately, neither assumption is true.

III. THE NUCLEON-NUCLEON INTERACTION

We now know the nucleon-nucleon phase shifts to high accuracy over the entire elastic scattering range[15,16]. If the three-nucleon problem depends crucially on what happens above pion production threshold, the hope of constructing any reasonable non-relativistic model for nuclear physics is vain, until this hope is destroyed, we will go on the assumption that we now have complete information about the on-shell scattering amplitude, and that any reasonable phenomenology for the short-range high-momentum behavior will do. The obvious non-relativistic approach, given this data, is to construct a static local potential from the phase shifts. This problem is complicated by the nucleon spin, which requires us to specify 5 functions for p-p, 5 for n-n, and 12 for n-p scattering; even if we assume charge independence, we need to specify 10 functions (5 for each isospin state). But even so, it has been known for a long time that a simple static model wouldn't work. Thus Christian and Noyes[17] found that simple monatonic potentials plus tensor forces did not lead to a charge-independent description of the n-p and p-p high energy differential cross section data. Jastrow[18] showed that charge-independence could be approximately restored by introducing a hard core, but Gammel, Christian and Thaler[19] found that this type of model still was not quantitatively acceptable. Hence Gammel and Thaler[20] found they had to go outside the static potential framework and include first order momentum-dependent terms ($L \cdot S$), as had first been suggested by Case and Pais[21]. But as soon as the first phase shift analysis became available[22], and was made unique by taking the highest partial waves from one-pion-exchange[23],

it became clear that even this was insufficient. It was originally noted[24] that (for solution 2) it was impossible to reconcile the $^1S_2$ and $^1D_2$ phases at 310 MeV with any simple static potential model, and that it would take a region of non-locality in the interaction of the order of 0.7 F in radius in order to achieve agreement. After more extensive data became available, Noyes[25] showed that this was a general difficulty with the singlet state at all energies above 50 MeV, and an exhaustive investigation by Reid[26] confirmed this conclusion.

Once we have been forced to abandon the static potential assumption, the elastic scattering data do not in themselves allow us to determine what model to use. For example, the phenomenological Hamada-Johnston[27] and Yale[28] models introduce second-order momentum dependent terms in the form of quadratic spin-orbit interactions in order to take care of this effect. Reid[26] makes his model orbital/angular momentum dependent by fitting a separate "static" model to each partial wave. Bryan and Scott[29] avoid the problem by not applying their model to the S-waves. Feshbach, Lomon and Tubis[30] have a boundary condition parameter which is fitted separately to each state. Scotti and Wong[31] found that the momentum dependence arising from vector meson exchange automatically produces the desired result. And of course any separable model has a different interaction in each orbital angular momentum state, so has the necessary flexibility built in. Although there are large differences in the adequacy with which these different models agree with the elastic scattering data, especially if we concentrate attention on any particular angular momentum state, this does not mean that the elastic scattering data can be used to distinguish between them; often, all these differences reflect is the amount of care the

authors have devoted to that state in their fit, since usually the fit could be improved by adding more phenomenological parameters. If the three-nucleon problem depended only on the fit to the on-shell data, we could therefore pick any of these models which was most convenient to use, provided only we insured the adequacy of the fit of the model to the data. Even that has not been investigated in most cases, but even worse, we will see in the next section that the three nucleon parameters do depend on the off-shell characteristics of the model. Consequently, we must find some way to narrow down these possibilities if we are to get anywhere with the three-nucleon problem.

One way to narrow these possibilities is to rely more extensively on theory. The special theory of relativity and the uncertainty principle require that if there are other massive particles strongly coupled to nucleons, they will give rise to a short-range force. The discovery of the pion, and determination of its spin, parity and coupling constant therefore determine the longest range part of the nuclear force, and this OPE prediction has been quantitatively confirmed in a number of ways[32,33]. The $^1S_0$ scattering length and effective range then show an attractive force of intermediate range, confirmed by the $^1D_2$ and $^1G_4$ departures from OPE, and shown to be spin-independent by the central force component of the ^3P waves[2]. Whether this 0^+ boson exchange is an actual resonance (σ meson), a virtual state (ABC effect), a correlated J=0 state of two pions, or simply uncorrelated 2π exchange, is still debated by theorists; the effect on the nuclear force will be similar, and until the point is settled, the parameter fitting will have to remain phenomenological. The ^3P splitting, which switches from tensor to L·S signature at 210 MeV

is evidence for the exchange of a heavy vector meson, as is the short-range repulsion evidenced by the change in sign of the $^1S_0$ phase from attraction to repulsion near 250 MeV. The ω-meson fits neatly into this pattern[34], and if one takes into account n-p data, the ρ meson is also there in about the expected strength[35]. The η meson is not expected to be important[35], but careful analysis of the forward nucleon-nucleon dispersion relations reveals that 3π exchange does make a significant contribution[36]; some models mock this up (incorrectly) by η exchange. Hence the very complicated spin, isospin, and energy dependence of the nucleon-nucleon phase shifts is in complete accord with what we expect from the exchange of the lightest known bosons, insofar as these expectations can be made quantitative. For an excellent review of these one boson exchange[37] models, see Bryan's paper for the Gatlinburg Conference[35]. Unfortunately, simply knowing that the nucleon-nucleon interaction can be fitted by this combination of boson exchanges in a covariant S-matrix description (and even this can be done in a number of ways[35]), does not specify how to go from this to a non-relativistic model for the nuclear force. One procedure suggested by Wong[38] has been followed by Ingber[39] and leads to a non-local velocity-dependent interaction of which he keeps only the p^2 terms. It is perhaps significant that the resulting model is claimed[39] to give a better fit to nuclear matter than the Reid L-dependent potential, using several fewer parameters; this improvement results primarily from the indirect cause that the fit to the two-nucleon data allows a longer range central force (lower mass σ meson), and a higher ratio of central to tensor force in the triplet even states than do the "static" models.

But this approach requires a phenomenological cutoff at short distances or high momenta, and more or less ignores inelastic processes. A more phenomenological approach, which stems from a suggestion of Breit and Bouricius[40], expressed in current language, is to note that the large number of degrees of freedom corresponding to the many bosons and boson resonances which we now know to be strongly coupled to nucleons, will be excited once the two nucleons approach within about half a pion Compton wave length of each other. Hence essentially all knowledge of the elastic entrance channel will be lost inside this radius, and the wave function for two nucleons will fall rapidly to zero, the independence of the entrance channel being approximately phenomenologically by fixing the logarithmic derivative of the wave function to some energy-independent constant; if this is assumed, Hoenig and Lomon[41] have shown that the wave function must be zero inside this radius as a consequence of the Wigner causality condition on the phase shift, and this continues to be true even if there is a static, local interaction ("potential tail"), outside the radius. This model was revived by Saylor, Bryan and Marshak[42] and by Feshbach, Lomon and Tubis[30]. In its latest version[43], it uses OPE and a static two-pion-exchange potential with two phenomenological parameters to represent the theoretical controversies about the "derivation" of this term; since the boundary condition falls at 0.7 F, the vector meson exchanges do not make a major contribution, but are included, again as static potentials. This model leads to as good a fit to the two-nucleon data as does the covariant calculation of Scotti and Wong, and both models are superior in this respect to the Hamada-Johnston and Yale pehnomenological potentials[44]. Faced with two models "in agreement with theory and experiment",

we must look still farther to find ways to distinguishing between them. We know in advance that they will give different results in the three-nucleon problem since the boundary condition model has a "hole" in the wave function of about 0.7 F in radius, while vector meson repulsion (or phenomenological hard cores) only keep the two nucleons out of a region of about 0.5 F in radius, which is nearly three times smaller in volume.

One way in which the models differ is in electromagnetic properties. Non-relativistically, one expects to be able to compute the D-state admixture in the deuteron from the n, p, and d magnetic moments, and gets 4%. However, if one takes a model with OPE as the longest range force, integrates the wave function in assuming a hard core at about 0.5 F (which is required to fit the scattering at higher energy) and adjusts the intermediate part of the wave function to fit the binding energy, ϵ_d, scattering length, a_t, and quadrupole moment, Q, one finds about 7% D state. (This discrepancy is larger than has usually been "calculated" from meson currents, but no one has pushed this discrepancy to the point of a believable contradiction. Since the corresponding meson current effect in $n + p \rightarrow \gamma + d$ is 10% in cross section at threshold, and that discrepancy depends only on the low energy parameters and <u>not</u> on the model[45], it has usually been thought that meson currents are that large.) However, the Feshbach-Lomon model just described can fit the same data with only $4\frac{1}{2}$% D state. Evidence for 7% D state has often been claimed from photodisintegration of the deuteron, coherent π^o production from deuterium by electrons, etc., but examination of these calculations shows that usually most of the contribution comes from the overlap between

the deuteron wave function and the scattering wave function in the final state, the comparison being made with models that did not have OPE in this long-range region. Since the Feshbach-Lomon model agrees with conventional models outside a fermi or so, most of this "evidence" is likely to be irrelevant in the current context. However, Casper and Gross[46] have shown that low energy e-d scattering is sensitive to the difference between the Feshbach-Lomon model and conventional (7% D state) models. Since we will see below that the difference between 4% and 7% D state shifts both the calculated triton binding energy, ϵ_t, and the doublet n-d scattering length, a_2, by singificant amounts, this question obviously deserves much more careful study.

A second place to look for differences in off-shell behavior is in nucleon-nucleon bremsstrahlung. Unfortunately, the large discrepancies between theory and experiment originally reported were due to a neglected term in the calculation, and recent results are in reasonable agreement with experiment[47]; further, most of the effect can be computed from the on-shell nucleon-nucleon scattering matrix[48], so much more precise p-p experiments than those currently available would be required to spot significant off-shell differences between models. There is a possibility that low energy n-p bremsstrahlung might be more promising, according to McGuire and Cromer[49]. The quantity which is measured is $t_\ell(p,q;q^2/2\mu)$, or in Cromer's notation[50], the quasi-phase

$$q_\ell e^{-i\delta_\ell} t_\ell(p,q;q^2/2\mu) = \Delta_\ell(p,q) = (p/q)^\ell \sin\delta_\ell \qquad \text{(III.1)}$$

$$+ (q^2 - p^2)\int_0^\infty dr\, F_\ell(pr)[u_\ell(qr) - \cos\delta_\ell(q)F_\ell(qr) - \sin\delta_\ell(q)G_\ell(qr)]$$

where to first order in the off-shell parameter $(p^2 - q^2)$

$$\Delta_\ell(p,q) = \left(\frac{p}{q}\right)^\ell \sin \delta_\ell(q) \left[1 + \frac{(q^2 - p^2)\Lambda_\ell}{2(2\ell + 1)}\right] \qquad \text{(III.2)}$$

Their calculations show significant differences in n-p bremsstrahlung at low energy when Λ_ℓ varies from 0 to 2 F. Since for exponential, square or Yamaguchi interactions fitted to the same scattering length and effective range, they quote values of 1.20, 1.11 and 4.22 F respectively for Λ_ℓ, we see that even this first order off-shell variation is significantly different between local and non-local models. Clearly models which agree on-shell at least up to 300 MeV, and not just at low energy, should be compared, but the result looks promising.

A somewhat different difficulty with calculations of three-nucleon low energy parameters from two-nucleon input arises because of our uncertainty about the n-p and n-n $^1S_0$ effective range. Since Thomas showed long ago that a zero-range nuclear force leads to infinite binding for the three-nucleon system, we expect in advance that ϵ_t will be sensitive to this parameter, and as we will see shortly, this is indeed the case. However, until recently, there has been a discrepancy between the n-p effective range directly measured in low energy n-p scattering using the most accurate measurements, and the value predicted from charge independence (2.85 F, or, correcting for the $\pi^\pm$-π° mass difference, 2.73 F)[45]. Fortunately, a re-evaluation[51] of the experiment of Houk and Wilson reported last year[52] taken together with a new experiment by Koester also report last year[53] removes this difficulty, as we now show.

The route leading from p-p scattering data to the prediction that $r_s^{np} = 2.73 \pm 0.03$ F is a little involved, so we review it here. It is first necessary to show that the p-p data below 3 MeV can be analysed for the $^1S_0$ phase shift unambiguously. Taking account of vacuum polarization corrections is straightforward, but it is also necessary to understand the nuclear scattering in other angular momentum states. The coulomb-nuclear interference term in the differential cross section gives a direct measurement of the central-force combination $\delta_{1,0} + 3\delta_{1,1} + 5\delta_{1,2}$ (in the notation of Ref. 22), but the original analysis of the p-p data below 3 MeV[54] had in addition to assume that the tensor force combination of the P phases was given (to an accuracy of 50%) by one-pion-exchange (OPE), that the $^1D_2$ phase was also given by OPE to an accuracy of 50%, and that the L·S conbination of the P phases is less than 50% of the tensor force combination. The lowest energy at which the L·S effect is actually measured is about 25 MeV, where it is about a quarter of the tensor effect, and since all models of the nuclear force (especially models based on vector meson exchange), assume it to be short range, it should only be a few percent of the tensor force at 10 MeV and below. Under this assumption, Noyes and Lipinski[55] showed that it is possible to obtain a unique analysis of p-p scattering at 9.69 MeV, which analysis confirms the remaining assumptions to better than the requisite accuracy, and hence a fotiriori justifies them at 3 MeV and below. Having established the values of the S phase at single energies, a shape-dependent effective range analysis then confirms (to about the 30% experimental accuracy available in the shape parameter) the prediction[56,57] given by OPE. Since the OPE shape effect is also seen at 10[55] and 25[25] MeV, the

theoretical value can be adopted, and a_s^{pp} and r_s^{pp} extracted from the data. Since electromagnetic effects other than vacuum polarication and the e^2/r coulomb repulsion can be shown to be negligible, charge symmetry then allows the unambiguous prediction[45] that $a_s^{nn} = -16.96 \pm 1$ F and $r_s^{nn} = 2.846 \pm 0.02$ F, where the errors include a generous estimate of the model-dependence of the result. Since Bander[58] has shown that the analysis of the reaction $d(\pi^-,\gamma)2n$ for the n-n scattering length had a theoretical uncertainty of at most 1 F, the experimental result of Haddock, et al.[59] that $a_s^{nn} = 16.4 \pm 1.9$ F is in excellent agreement with the prediction; unfortunately there is no direct experimental evidence on the value of r_s^{nn}. Exact charge independence would predict the same values for a_s^{np} (and r_s^{np}), in disagreement with the experimental value of -23.7146 F, but as has been known for a generation, any slight charge-dependent correction (which we expect but do not know how to calculate) could account for this discrepancy. Independent of how we adjust the model phenomenologically to account for this discrepancy with the predicted scattering lengths, and including the known $\pi^{\pm}$-π^0 mass difference in the OPE part of the model, we showed[45] that the predicted value for the effective range is $r_s^{np} = 2.73 \pm 0.03$ F; estimates of other possible charge-dependent effects at shorter range[60] show that they could contribute a correction at most half as big. As we have emphasized repeatedly[2,25,45,61,62], there is less than 5% probability that the value of $r_s^{np} = 2.44 \pm 0.11$ F obtained from accepted values of ϵ_d, a_{nH}, $\sigma_{np}(0)$ and the two n-p total cross sections measured by Engelke, Benenson, Melkonian and Lebowitz[63] at 0.4926 and 3.205 MeV is compatible with this prediction. That the discrepancy might rise instead from one or more of the other ingredients of the analysis was

emphasized by Breit, Friedmann, and Seamon[64]. The new measurement of a_{nH} by Koester[53] and the preliminary result of Houk and Wilson as reported at Gainsville[52] did not significantly alter this situation.

Recent analysis of the data of Houk and Wilson has shown[51] that the published value should be replaced by $\sigma_{np}(0) = 20.442 \pm 0.023$ b. Taken together with Koester's result[53] that $a_{nH} = -3.719 \pm 0.002$ F, this gives immediately that the triplet scattering length is $a_t = 5.4255 \pm 0.0043$ F and the singlet scattering length $a_s = -23.7146 \pm 0.0127$ F with error correlation $\langle \delta a_t \delta a_s \rangle = -0.9566 \, |\delta a_t| \, |\delta a_s|$. At this level of accuracy it can be seen immediately from our previous error analysis[65] that, in calculating r_s, the uncertainty arising from the quoted errors in ϵ_d, a_t, and a_s is at most a few percent of that coming from the error in published n-p total cross section measurements in the 0.5-5.0 MeV range. This same analysis showed that, in the same energy range, the contribution from higher partial waves is almost negligible, and that the single and triplet shape corrections nearly cancel. The effective range is then given by

$$r_s = 2\left(\sqrt{\frac{\pi}{\sigma_s} - k^2} + \frac{1}{a_s}\right)/k^2 \qquad \text{(III.3)}$$

where

$$\sigma_s = \sigma_{np} - \sigma_t - \sigma_{\ell \neq 0}$$

$$\frac{3\pi}{\sigma_t} = \left(-\frac{1}{a_t} + \frac{(\gamma a_t - 1)k^2}{\gamma^2 a_t} \right)^2 + k^2 \qquad \text{(III.4)}$$

$$\gamma^2 = \mu \epsilon_d / \hbar^2$$

The error analysis is clearly trivial. Results for the individual cross

section measurements, previously selected by Hafner[65] on an experimental basis, are given in Figure 2, and the weighted average for r_s using three different data selections, both with and without the restriction $r_s = 2.73 \pm 0.03$ F, in Table I. Clearly, complete agreement between the theoretical prediction and the data has now been achieved.

We now examine the sensitivity of the result to the assumptions. Our reason for believing the OPE shape correction has been documented above, but all that is in fact involved in this n-p analysis is the assumption that the shape correction is approximately the same in the singlet and triplet S-waves, since the opposite sign of a_s and a_t then insures a cancellation; if we omitted the triplet shape correction, the weighted average of r_s would rise by only 0.028 F, and if we omitted the singlet shape correction, it would fall by the same amount. Note that if we included accurate data above 5 MeV, this would no longer be true, as the maximum triplet shape correction occurs at about 5 MeV.

The sensitivity to the data selection is more serious. If we use only the two published Columbia measurements[63], r_s falls by 0.11 F, while if we use only the older measurements, it rises by 0.13 F. Both sets of data are still separately consistent with the prediction, but barely consistent with each other, as can be seen from Figure 1 and Table I. Further, as has been pointed out to Houk and Wilson[52], Koester's value for a_{nH} requires the change of the previously accepted value for a_C/a_H by five standard deviations. If the old ratio were used, r_s would fall by 0.15 F, and the Columbia measurements would still imply a failure of charge-independence. Hence, continued attention to this problem is still needed.

According to Houk[66], if we use only the two published Columbia cross sections, plus two unpublished measurements by Lebowitz at 3.204 and 5.874 MeV, and treat the singlet shape parameter as free, $r_s = 2.704 \pm 0.095$ F, again in agreement with the prediction. At this higher energy, and particularly in order to include the accurate unpublished measurement by Lebowitz at 7.942 MeV, careful attention to the shape correction and to the contribution from higher partial waves will be needed in order to exploit the accuracy now potentially available. Careful analysis of the experiments of Houk and Wilson and of Lebowitz is now in progress and will be reported elsewhere[66].

Additional evidence for the expectation that r_s is closer to 2.7 than 2.5 F is provided by a recent analysis of data up to 350 MeV by Breit, Friedman, Holt and Seamon[67]. So far as the test of charge independence goes, Breit[33] has also shown that this could be carried out to high precision in the 20-30 MeV region, where coulomb corrections are smaller than at low energy, high precision p-p data already exist which overdetermine the I=1 phase shifts, and the necessary spin-dependent experiments are easier to perform. We hope, however, that we will not soon again be confronted with basic numbers shifting by several standard deviations, and that this reconfirmation of charge independence presented here will stand for some time at the current level of accuracy.

IV. CALCULATIONS OF ϵ_t, a_2, AND a_4

Although n-d cross section measurements at low energy allow two alternative sets of doublet and quartet scattering lengths, by measuring the transmission of neutrons through a polarized deuterated target, Alfimenkov, Lusichikov, Nikolenko, Taran, and Shapiro[68] demonstrated that the set with $a_4 > a_2$ is correct. The latest experiments and analysis by van Oers and Seagrave[69] yields the values $a_2 = 0.11 \pm 0.07$ F and $a_4 = 6.14 \pm 0.06$. In the quartet state, the exclusion principle keeps the neutron from approaching too close to the deuteron, and it is reasonably easy to obtain the correct value for a_4 from any model that has the right low energy behavior for on-shell nucleon-nucleon scattering; hence the separable model calculations reproduce a_4 quite well and need not be discussed further. Also k ctn δ_4 has the usual type of energy dependence we expect for elastic scattering with no bound state, and since the quartet scattering dominates the low energy region, fairly simple calculations reproduce the low energy behavior quite well, as Amado will discuss this afternoon.

The behavior of k ctn δ_2 at low energy is quite another matter, as can be seen from the curve given by van Oers and Seagrave[69] which we reproduce at Figure 3. The fitted curve of the form k ctn $\delta_2 = - A + Bk^2 - C/(1 + Dk^2)$ looks very strange at first sight to those used to the two-nucleon effective range expansion, but the occurence of a pole in k ctn δ_2 very close to threshold in fact has a simple explanation. Delves[70] gives a plausibility argument for this pole, which can now be considerably sharpened. Aaron, Amado, and Yam[71] and Bander[72] both found

that three identical bosons interacting via separable potentials of the Yamaguchi or square-root Yamaguchi type and of approximately the right range and strength to fit the average of the singlet and triplet nucleon-nucleon S-wave interactions not only give a bound state somewhat more tightly bound than the triton (as we will discuss below), but also a very weakly bound first excited state. Bander noted that the residue of the pole in T corresponding to this state has the opposite sign to that of the residue of the triton pole, which in a two-body system would imply a "ghost" state of imaginary probability; this "wrong" sign for the residue is not so obviously a difficulty in a three-body system. The situation has been clarified by Osborn[73], who finds that local Yukawa and exponential interactions also lead to this weakly bound first excited state, and that it has a perfectly reasonably wave function. In fact, as one might guess, the wave function for this state corresponds to a very diffuse "orbit" about a tightly bound core consisting of the other two particles, in contrast to the much more compact and nodeless ground state Although 90% of the actual triton wave function corresponds to the fully symmetric state which would be obtained for three identical bosons interacting via central forces, the actual spin dependence of the two-nucleon interaction brings in admixtures of other states, and apparently is sufficient to move the first excited state above the threshold for n-d scattering (i.e. force it up to the point where it is "virtual", or in S-matrix language, move the pole onto the second sheet of the Riemann surface.) However, between the triton pole in T and this virtual pole, the T matrix must go through zero, and if this happens below threshold (where k ctn δ - ik = T^{-1} is real) this will guarantee a pole in k ctn δ.

That this actually happens is shown by the fit in Figure 3. Clearly, now that this situation is understood, three-nucleon calculations should be aimed at getting this zero in T right, as well as a_2, or better still, the four constants A, B, C, D of the van Oers-Seagrave fit. We also see that the precise value of a_2 obtained by a calculation can be expected to be quite sensitive to the details of the nuclear force which is assumed.

Since Thomas[74] showed that zero-range nuclear forces would give infinite binding energy to the triton, we must also expect calculations of ϵ_t to be quite sensitive to the range of the force and to its radial variation, in marked contrast to the shape-independent approximation for low energy nucleon-nucleon parameters. The variational calculations of various authors[75] more than a decade ago showed that purely attractive central forces considerably overbind the triton, and that the result is sensitive to the radial form assumed. This result has been reconfirmed by Noyes ans Osborn[76] by direct solution of the Faddeev equations for the bound state of three identical bosons of nucleonic mass interacting via the sum of two potentials fitted to the parameters a_s = - 23.688 F, r_s = 2.7251 F, a_t = 5.4039 F, γ = 0.231608 F^{-1}. For two Yukawa potentials, these give a binding energy of 12.76 MeV, for two exponential potentials 10.50 MeV, and for two Yamaguchi potentials, 11.24 MeV. Before commenting on this result, we will first attempt to establish the accuracy of the numbers.

The first question to ask is about numerical accuracy. The code used gaussian quadrature for the double integrals, and many checks showed the results good to at least $\frac{1}{2}\%$. More convincing is the exact agreement (to better than this accuracy) with a calculation by Ball and Wong[77] which

used an expansion in Stürmian functions which Wong will discuss in the next talk. The code also (as a two-variable code) agreed with the single-variable calculations of the Yamaguchi interaction, which can be done to much higher accuracy; the latter calculations agreed exactly with published results of Sitenko and Kharchenco when their parameters were used. The second question is whether the neglect of higher angular momentum states in the two-nucleon interaction is justifiable. For gaussiam interactions, the exact calculation of Baker, Gammel, Hill and Wills[78] indicates a contribution of about 3% of the binding energy from $\ell > 0$ states, but is is not clear that the calculation is completely reliable on this point. Ball and Wong[77] in fact find only $\frac{1}{2}$% correction from $\ell = 2$ states in their calculation using Yukawa potentials. Further, a new calculation by Humberston, Hall and Osborn[79] using variational techniques which give both upper and lower bounds which are very close together, and which include all angular momentum states, bracket Osborn's curves[73] very closely in the region of interest. Incidentally the last mentioned calculation shows that the apparent collapse of the gound state found by Osborn[73] at higher coupling constants was spurious, and that the linear variation of $\epsilon^{\frac{1}{2}}$ with potential strength continues into this region. We conclude that the binding energy of three identical bosons interacting via purely attractive local potentials can be accurately and unambiguously calculated. Unfortunately, this has still not been demonstrated for local potentials with short-range repulsion, or I would be talking about that instead of separable potentials. In particular, there is no evidence that the upper and lower bounds obtained by the variational methods will lie nearly as close together in the case of hard-core potentials, and good reason to expect the opposite.

The implication of these numerical results (which, as already noted are equivalent in the case of the local potentials to variational results nearly two decades old) are quite distrubing. To put the 2.26 MeV discrepancy between the exponential and the Yukawa models in context, we note that Loiseau and Nogami[80] have estimated that the three-body force due to meson exchanges will contribute about 1.5 MeV (attraction) to ϵ_t. Of course, this is not a very reliable number, but does show that we will have to calculate ϵ_t to considerably better than an MeV in order to get new physics out of a discrepancy between theory and experiment. But the local potential result shows immediately that simply fitting the effective range parameters is not enough to determine the binding energy to the requisite accuracy - even for local central forces the high energy behavior (or radial variation of the interaction) is significant. The result for the non-local Yamaguchi interaction is even more disturbing. As we will see below, the on-shell behavior of this model is closer to the experimental S phases than a purely attractive local potential would give. If we construct a local potential to fit the Yamaguchi singlet phase at <u>all</u> energies (which can be readily done using the procedure given by Newton[81] for constructing a generalized Bargmann potential), this local potential is slightly repulsive at short distance, so would be expected to give even less binding than the exponential potential (see below). Yet the non-local Yamaguchi potential, as we have just seen, gives 0.74 MeV more binding than the exponential potential, indicating that the difference of off-shell behavior between local and non-local models is likely to be more than an MeV. Hence reliable models for the off-shell as well as for the on-shell two-nucleon interaction will have

to be developed before we can trust calculations of ϵ_t. As discussed in the last section, theory is not a very reliable guide for this, and we will have to extract as much as we can from electromagnetic experiments.

This suspicion about the importance of off-shell behavior in triton binding energy calculations has been dramatically confirmed by preliminary results we have just received from van Wageningen's group. Concentrating still on the bound state of three identical particles bound by an average nucleon potential, if the two-nucleon force gives rise to a single bound state, there are two ways to define a local potential equivalent to the Yamaguchi interaction. One is to require complete agreement with the phase shift at all energies from zero to infinity, which gives the generalized Bargmann potential mentioned above. The second is to require complete agreement with the bound state wave function, in which case the local potential is the Hulthen potential. In all three cases the scattering length and binding energy of the two-body bound state are the same. L. P. Kok[82] finds that for a Yamaguchi interaction which binds the triton with 12.263 MeV, the equivalent Hulthen potential binds it with 14.352 MeV, while G. Erens[83] finds that the equivalent generalized Bargmann potential binds it with 10.689 MeV. The two local potential results are variational, but comparison with the exact results of Osborn and of Ball and Wong for similar potentials shows that the variational calculations should be good to at least 1%. Hence whether we ask for the local equivalent of the Yamaguchi potential on the basis of the two-nucleon wave function, or on the basis of the on-shell scattering, leads to nearly 4 MeV difference in the calculated value of ϵ_t. This destroys at one

blow many commonly expressed assumptions, such as that all that is needed is to fit the scattering length and binding energy, or to get the right two-nucleon bound state wave function. It also shows immediately that just how we take the two-nucleon scattering matrix off shell in the three-body problem has more effect on ϵ_t than the anticipated effect of three-body forces! Hence we must get the physics of this off-shell extension right, if we are to draw any fundamental physical conclusions from these calculations.

Since we have just seen that central force Yamaguchi models give results quite comparable to purely attractive local potentials (i.e. overbind the triton), the pioneering calculations[84,85,86,87] which established this fact were mainly important in demonstrating the ease with which the three-body problem could be solved in the separable approximation, and we turn to the more recent and more ambitious calculations which have attempted to include tensor forces and/or short range repulsion in the model used for the nuclear force. Unfortunately, here we have no local potential calculations for comparison. After herculean efforts, the variational calculations of ϵ_t using the Hamada-Johnston potential had improved the 2-3 MeV binding obtained by Blatt and Delves[88] to about 5.7 MeV, and as of last summer Delves[89] estimated that by 1970 they might converge to about 7 MeV leaving the 1.5 MeV estimate of attractive three-body forces by Loiseau and Nogami[80] to take up the slack. We will have a progress report on this program from Delves this afternoon. In contrast, simple separable models overbind the triton, and the hope has been that by making the models more "realistic", the binding energy would be reduced.

The simplest way to present the latest results is to compare them on a very instructive plot given by Phillips[90] which compares simultaneously the value for a_2 and for ϵ_t, and is reproduced in Figure 4. At that time, as we discussed in the last section, it was still an open question whether to use the n-p experimental value of 2.44 MeV for the singlet effective range, the value of 2.73 MeV predicted[45] by charge independence from p-p scattering, or the n-n value of 2.85 MeV predicted by charge-symmetry from p-p scattering. The seriousness of this uncertainty in the three-nucleon problem is obvious at a glance, and we are happy that we could report above that is it probably no longer a worry. However, the sensitivity of the results to the D-state probability remains, and we emphasize again the importance of re-analyzing e-d scattering, photodisintegration, etc. to see if any more light can be shed on this problem.

The model used by Phillips for this calculation was

$$V_S(p,q) = -\frac{\lambda s}{M}\frac{1}{(p^2+\beta_s^2)}\frac{1}{(q^2+\beta_S^2)}$$

$$V_t(p,q) = -\frac{\lambda t}{M}\left(\frac{1}{p^2+\beta_t^2)} - \frac{tp^2 S_{12}(\hat{p})}{(p^2+\bar{\beta}_t^2)^2}\right)\left(\frac{1}{q^2+\beta_t^2} - \frac{tq^2 S_{12}(\hat{q})}{(q^2+\bar{\beta}_t^2)^2}\right)$$

with the singlet parameters adjusted as indicated, and the triplet parameters adjusted to change the percentage D state while maintaining a fit to a_t = 5.397 F, ϵ_d = 2.2245 MeV, and Q = 0.282 F^2. A still more extensive parameter study on the effect of varying the singlet effective range (but not the percentage D state) was carried out earlier by Sitenko, Kharchenco, and Petrov[91], and further details published by Kharchenco, Petrov and Storozhenco[92] more recently. In order not to confuse the plot,

we give only two representative points from these calculations, labeled 1_t and 3_t. The difference between the 1_t point and the corresponding value by Phillips is due solely to the difference in a_s, as can be seen from the study of the dependence on a_s contained in those papers, or from the earlier work of Mitra and collaborators[85]. Of course this insensitivity ot a_s is due to the fact that a change of only a percent or so in potential strength is all that is needed to shift a_s from - 17 to - 23 F. Corresponding points for differing values of r_s would be simply shifted over from the values given by Phillips by approximately the same amount.

A different effect studied by the same authors is indicated by the point labeled 3_t which replaced the Yamaguchi central force form factor in both the singlet and the triplet interaction by $(p^2+\beta^2)^{-3}$, while maintaining the same fit to the two-nucleon parameters. Clearly, this has as much effect as changing the percentage D state by $1\frac{1}{2}\%$. In order to try to see why, we have computed the change in the on-shell singlet phase shift for the still more extreme case $(p^2+\beta^2)^{-8}$ which requires an increase in λ from 0.15×10^0 to 2.96×10^{21} and β from 1.18 to 5.22. Surprizingly, there is remarkably little change in the on-shell phase shift, as can be seen from Figure 5.

A still more ambitious calculation has been made by Schrenk and Mitra[93], and the results are also presented in Figure 4. Here the triplet force model is the same as above for the points labeled $(C + T)_Y$, but uses the Naqvi[94] parameters for $(C + T)_N$; the second triplet force is less realistic in that the original fit was made including an L·S term, which has simply been dropped here. The main change is that the singlet

potential is of the form

$$V_S(p,q) = -\frac{\lambda_S}{M}\left(\frac{1}{(p^2+\beta_S^2)}\frac{1}{(q^2+\beta_S^2)} - \frac{t'p^2q^2}{(p^2+\bar{\beta}_S^2)^2(q^2+\bar{\beta}_S^2)^2}\right)$$

where the second (replusive) term causes the singlet phase to change sign at high energy. The point labeled N is the fit due to Naqvi[95], which is somewhat unrealistic because the effective range is much too short (2.33 F). The points labeled G are several fits due to Gupta[96] which are still unpublished; we compare the one which is claimed to be best with experimental values of the $^1S_0$ phase in Figure 5 and agree that the fit is reasonably good. Over the same range (50-330 MeV), published phases for the Naqvi model are very close to the Gupta curve. Unfortunately the calculation is quite lengthy, so we did not have time to compute values for the other Gupta parameters to see how they agree on-shell. There seems to be surprizingly little change in ϵ_t and a_2 in going from 2.33 to 2.7 F in the singlet effective range, a result we do not know how to interpret. Clearly, however, the on-shell changes between the models can produce differences as important as those found by Sitenko, et al. in changing the form, or by Phillips in changing the percentage D state. Whether introducing repulsion into the triplet model as well would finally produce "agreement with experiment" is anybody's guess at this point.

Two calculations which bear on this point have been made by Tabakin[97], and by Borysowicz and Dabrowski[98]. Unfortunately, neither includes tensor forces, but Tabakin has taken the care to make sure that the models he compares agree on-shell to 0.01 radians in δ_0 over the entire elastic

scattering range. He uses two second rank separable potentials, one of a smooth type similar to that used by Naqvi and Gupta, and the second a "hard shell" potential of the type invented by Puff[99]. The latter keeps most (but because of the non-locality of the separable model, not all) of the wave function outside the core radius, and hence produces different short-range correlations, and a phase shift which goes to infinity like $-kr_c$, rather than going to zero at infinite momentum. He finds that the hard-shell model givs 0.4 MeV less binding to the system, showing that even on-shell agreement to 330 MeV and similar (separable) off-shell behavior, will not lead to the same value of ϵ_t to that accuracy, if the infinite momentum behavior of the two models differs.

The effect could be considerably larger than the estimate from Tabakin's calculation, as has been shown by Borysowicz and Dabrowski[98]. Since Tabakin fitted an average between the singlet and the triplet S phases, his hard-shell radius came out to be only 0.2 F, which is considerably smaller than one would expect from hard-core local models. Borysowicz and Dabrowski find a decrease of binding energy of 2.7 MeV when going from a no-core model to a Puff model with r_c = 0.4 F, as compared with the 0.93 MeV Tabakin finds in a similar comparison. Unfortunately, the Borysowicz-Dabrowski version of the Puff model, has, as it stands, too much short-range repulsion, as can be seen from the comparison with $^1S_0$ phase shifts given in Figure 5. It also does not attempt to answer the question raised by Tabakin about short-range correlations (or different asymptotic behavior for δ), since the two models were <u>not</u> fitted to the same on-shell scattering, as Tabakin was careful to do.

And the absolute value of the binding energy obtained (8.3 MeV) has been questioned by Jaffe and Reiner[100], who obtain 8.7 MeV for what is supposed to be an identical calculation.

Another interesting aspect of the Borysowicz-Dabrowski calculation is that it includes the effect of splitting the singlet parameters in a charge-dependent way (from $a_s = -23.69$, $r_s = 2.5$ to the pair $a_s^{np} = -23.69$, $r_s^{np} = 2.5$ plus $a_s^{nn} = -16.6$, $r_s^{nn} = 2.8$); this drops the binding energy from 11.6 MeV to 10.7 MeV for no-core, and from 8.9 to 8.3 MeV with a core. This would seem to indicate a smaller sensitivity to r_s in the presence of a core than without it, as was indicated by the calculation of Schrenk and Mitra discussed above. Unfortunately, they do not indicate how much isospin 3/2 admixture this charge-dependent effect introduces, but they do find a rather small change in the percentage of S' state due to this cause. The percentage of S' state changes from 1.1 to 2.0 when the core is added, or from 1.4 to 2.4 when the core is added in the charge-dependent model.

V. WAVE FUNCTIONS OF H^3 AND He^3

An extensive review of the work on the ground state properties of the three-nucleon system up until 1967 has been given by L. M. Delves[89] in his lectures on "The Nuclear Three-Body Problem" given at the Symposium on Light Nuclei at Brela. We shall only give a short summary of his conclusions and report somewhat more fully the work which has been done since.

V.1 The Coulomb Energy of He^3

If the nucleon-nucleon interaction is charge symmetric the Coulomb energy of ^3He, which is experimentally found to be

$$\Delta E_c = 0.764 \text{ MeV}$$

must be wholly ascribed to electromagnetic effects. Unfortunately, although much work has been done, the question if this is really true has not yet been settled. Direct variational calculations[101] of ΔE_c, assuming charge symmetry, lead to an estimate

$$\Delta E_c \sim 0.6 \text{ MeV}$$

which is likely to be low, since the trial wave functions used underbind the triton. A number of authors[102] have followed a different approach. They calculated the Coulomb energy with wave functions which have been fitted to the experimental electron scattering form factors. The conclusion as reported by Delves has been that it is very difficult if not impossible to obtain a Coulomb energy of 0.76 MeV. The result of very careful calculations by Okamato and Lucas[103] has been a discrepancy of 0.13 MeV, on the low side, in the Coulomb energy. Calculations of other authors[104] using local potentials have been more or less consistent with this. If this is true we can

conclude that the n-n interaction is stronger than the p-p interaction, or that charge asymmetric three-body forces exist. Interpreting the result as a charge asymmetry of the two nucleon potential, the discrepancy of 0.13 MeV corresponds to about 1% in the strength of the n-n and p-p interaction, according to a rough calculation of Okamoto[105].

There are counter arguments against this interpretation. One is that no charge asymmetry has been found in the Coulomb energy of the heavier mirror nuclei. This argument is somewhat weakened by the difficulty of interpreting small quantitative effects in complex nuclei. The second argument came from the work of Mitra and his group[106]. They found for those separable potentials, which approximately fit the triton binding energy, a Coulomb energy which was too high rather than too low. Since repulsive core effects had not been included in their potentials they expected agreement with experiment after inclusion of such effects.

Since then more attempts to settle this question have been made with separable potentials. Some authors were mostly concerned with the problem of treating the Coulomb interaction in the framework of separable potentials. Adya[107] is concerned with testing the validity of two different approximations to the Coulomb Green's function in a simplified model triton. The best approximation is just a little bit better than the straightforward perburbation calculation with a triton wave function. This best value for ΔE_c, which is 1.3 MeV, is expected to be much too high because of the approximations involved. An exact treatment of the Coulomb interaction in ^{3}He combined with separable two-nucleon potentials of rank one has been given by Alessandrini, et al.[108]. To do this they employed the formalism of Alt, Grassberger and Sandhas[109] (AGS). They found that in the case that

there is a non-separable interaction between only one pair of particles, the AGS equations can be solved exactly. This allowed them to take the Coulomb interaction into account without any approximations. However, no repulsive core effects were included. The contribution of the tensor forces was estimated by using the Sitenko and Kharchenko[110] parameters in the Yamaguchi potential. This overbinds the triton. Using only the central part of the $(C_N + S_N)$[106] Naqvi potential, which underbinds it, they respectively obtained 0.68 and 0.84 MeV for ΔE_c. Although the average value is in reasonable agreement with the experimental value, hard core effects are expected to lower it by 15%.

This procedure was also used by Gupta and Mitra[106] and was recently criticized by Okamoto and Lucas[111]. They reject the subtraction of the tensor part, since it produces an incomplete potential. The parameters of the Yamaguchi potential were also criticized because they were fitted to a singlet effective range of 2.15 fm instead of $r_s = 2.820 \pm 0.044$ F, the "experimental" value[112]. Because the two protons are believed to be predominantly in the singlet state the use of the correct value of r_s is probably important for the calculation of ΔE_c.

To investigate this point Okamoto and Lucas make variational calculations with local exponential potentials fitted to the same low energy scattering data as the Yamaguchi potential used by Gupta and Mitra, and found nearly the same Coulomb energy $\Delta E_c = 0.85$ MeV. Next they repeated the calculation with an exponential potential fitted to $r_s = 2.83$ F and with various hard core radii. They then find $\Delta E_c = 0.618$ MeV for the hard core radius of 0.486 fm and conclude that Gupta and Mitra's calculation probably contains

a large error due to the wrong choice of r_s and the neglect of the hard core.

A separable calculation of ΔE_c with a rank-two separable central potential of the Puff type, which takes hard core effects into account, was recently reported by Jaffe and Reiner[100]. They fitted Puff potentials not only to the n-n interactions, but also to the p-p interaction, a procedure which does not do full justice to the Coulomb potential. On variational grounds the Coulomb energy should be bounded by its expectation value as calculated with trition and He^3 wave functions. Jaffe and Reiner found values of 0.71 and 0.73 MeV respectively. This confirms the importance of the repulsive core of the Coulomb energy, but still does not allow a definite conclusion concerning the charge dependence of nuclear forces in view of the approximations involved.

Simonov and Badalyan[113] also reported a calculation of the Coulomb energy of ^{3}He using a promising method of partial wave expansion of the wave function in six dimensions, which allows an exact solution of the three-nucleon bound state problem. Although agreement was obtained with the experimental data to 5-10%, the central potential used was extremely unrealistic, being a square well without a repulsive core.

Clearly the question of the charge dependence of nuclear forces has not yet been settled, although the gap between the calculations with separable and with local potentials has been somewhat narrowed.

V.2 The Charge Form Factors and Radii of ^{3}H and ^{3}He

The elastic scattering of electrons from ^{3}H and ^{3}He can be expressed in terms of two functions of the momentum transfer q, the charge and magnetic form factors of the target particle. Rather accurate experiments have

been carried out on e - ^{3}H and e - ^{3}He scattering, and analysed[114,115]. The resulting form factors[114] are identified with the Fourier transform of the spatial distributions of the electric charge and magnetic moment of the bound states of ^{3}H and ^{3}He.[114] Since charge exchange effects are practically absent and the magnetic exchange effects likely to be large, the charge form factors provide the most reliable data on the properties of the three-nucleon bound state wave function. Although the wave functions of ^{3}H and ^{3}He are not expected to differ much, since the Coulomb potential and other charge dependent effects are relatively weak, the ^{3}H and ^{3}He charge form factors are different. For small q^2 we can write

$$F_{ch} = 1 - \frac{1}{6} r_{ch}^2 q^2$$

where r_{ch} is the root mean square radius of the charge distribution of the nucleus. From experiment[114]

$$r_{ch}(He^3) = 1.87 \pm 0.05 \text{ F}$$

$$r_{ch}(H^3) = 1.70 \pm 0.05 \text{ F}$$

At higher energies the ^{3}He charge form factor continues to fall off more rapidly with q^2 than the ^{3}H form factor. However, this is not necessarily a contradiction since the charge form factor measures the charge distribution rather than the mass distribution. Schiff[114] and Dalitz and Thacker[116] have pointed out that this is likely to be true, since the interaction between the like particles, which are predominantly in the singlet state, is weaker that that between unlike particles. This causes the like particles to spread out over a bigger region in space than the unlike particles.

This effect goes in the right direction to explain the difference in form factors even if the wave functions of ^{3}H and ^{3}He are substantially the same.

Delves[89] has given a careful discussion of the difference between the ^{3}H and ^{3}He form factors in terms of the classification of the triton wave function. A difference between the ^{3}H and ^{3}He form factors is expected as soon as other states than the principal S totally symmetric state are present in the wave functions. If the ^{3}He and ^{3}H wave functions are nearly identical, the most important admixtures are expected to be the S' state of mixed symmetry, and the D-states. The ^{3}He wave function will contain an additional T = 3/2 mixed symmetry S state, induced by the Coulomb potential. The form factors can be analyzed in terms of these admixtures, leading to estimates for the probability of the various states. The first calculation by Schiff[115] assumed that only the S' state contributed, leading to an unlikely high probability P(S') of 4%. The inclusion of the D-states by Gibson[117] failed to remove the discrepancy. However, the necessary S' probability depends quite strongly on the assumed shape of the S' wave function, which is clearly demonstrated by the fact that the Dalitz-Thacker wave function[116] corresponds to an S' probability of about 1.5%. To investigate the role of the induced T = 3/2 state in ^{3}He, Gibson[118] included it in his calculations and found that it was possible to reach agreement with experiment with $P_{S'} = 2\%$ and $P_{3/2} \simeq 0.25\%$. However, $P_{S'} = 2\%$ is still uncomfortably large compared with the variational result[101] $P_{S'} \simeq 1.2\%$ and the result from inelastic e - ^{3}H scattering[115]. This can be reduced if the neutron form factor is not taken to be identically zero[106,119]. With a

positive form factor for the neutron as given by experiment[120] it is possible to obtain a $P_{S'} \simeq 1\%$. The reliability of these conclusions is somewhat diminished by the strong wave function dependence of $P_{S'}$, especially on the asymptotic part of the wave function[116]. They hold only for a given assumed shape of the wave function.

Recently a new calculation of $P_{S'}$, with rank-two separable potentials of the Puff type, including repulsive core effects has been reported by Borysowicz and Dabrowski[98] who find $P_{S'} = 2.0 - 2.4\%$ respectively for charge independent and charge dependent forces, which is more than twice the value of $P_{S'} = 0.81 - 0.96\%$ found by Bhakar and Mitra[121]. This seems to show that inclusion of the hard core increases the discrepancy again. Although Phillips[122] favors a $P_{S'} = 1.2\%$ with $r_s = 2.7$ fm and $P_D = 7\%$, with a difference in the charge radii of 0.12 to 0.16 fm, he does not include repulsive core effects. He does also point out that the variation of $P_{S'}$ with r_s is fairly large.

In a recent article Ohmura[123] discusses the differences in the electromagnetic form factors due to the Coulomb potential, assuming local central potentials without a hard core, having no spin and isobaric spin dependence. This means that the ^3H wave function is exactly given by the principal S-state. It is then found that the Coulomb repulsion alone is clearly insufficient to explain the observed difference in the electromagnetic form factors. The author also comes to the conclusion that the charge radii are not determined mainly by the asymptotic region, but that the wave function in the inside region is quite important, in disagreement with Dalitz and Thacker[112]. A very small probability for the $T = 3/2$ state of $0.001 - 0.006\%$ is found in the He^3 nucleus. The Schroedinger

equation for the three-body bound state is solved approximately by an expansion of the wave function with Legendre polynomials omitting the higher order terms. A negligible admixture of T = 3/2 states is also predicted in the previously discussed work with separable potentials by Alessandrini, et al.[108].

From the previous discussion we see that recent estimates of P_S, range from 2.4% - 0.8% and the T = 3/2 probability from 0% - 0.25%. It has clearly not yet been demonstrated that the difference between the charge radii of ^3H and ^3He can be satisfactorily explained, without coming into conflict with other experimental evidence concerning the three-nucleon systems. In his review article, Delves[89] points out that the absolute magnitude of the form factors is rather easier to fit than the ^3H - ^3He difference. This is not surprising since we have the initial condition F(0) = 1 and the initial slope as determined by the charge radius. Hence any wave function giving the correct charge radius will fit the small momentum transfer part of the form factor. Only at momentum transfers $q^2 \gtrsim 7\text{-}8$ fm^{-2} some structure appears in the form factors. They then drop below the straight line predicted for a Gaussian charge distribution, which can be explained by the presence of a repulsive core in the nucleon-nucleon potential. It is found that even wave functions which are not particularly good can give quite reasonable fits to the form factors[115]. On the other hand, it is not enough to fit the triton binding energy, since different wave functions yielding nearly the same binding energy give different form factors[89,119].

V.3 The Beta Decay of the Triton

In his review article[89] Delves discusses the information on the ^3H

wave function which may be contained in the beta-decay of ^3H into ^3He. In particular, Blin-Stoyle[124,125] has suggested that a comparison of this decay with neutron decay should lead to a test of the PCAC theory of weak interactions and give information on the S'- and D-states probabilities. It can be shown that the ft value for an allowed beta-decay is given by

$$(ft)^{-1} = \frac{m^5 c^4}{2\pi^3 \hbar^7 \ell n 2} [G_V^2 |M_V|^2 + G_A^2 |M_A|^2]$$

Where G_V and G_A are the polar and the axial vector coupling constants and M_V and M_A the matrix elements of the corresponding weak Hamiltonians. An analysis of 0^+ to 0^+ decays[126,127] combined with the measured value of (ft) for neutrons[128] leads to values of G_V and G_A using $|M_V|^2 = 1$, $|M_A|^2 = 3$ for the neutron. The measured ft value for ^3H is[125]

$$(ft)_{^3H} = 1137 \pm 20 \text{ secs.}$$

If we assume that the ^3H and ^3He wave functions are nearly equal, it follows that

$$|M_V|^2 = 1,$$

which is independent of the ^3H wave function.

For the axial vector matrix we find according to Blatt[130]

$$|M_A|^2 = 3[P_S - \tfrac{1}{3} P_{S'} + \tfrac{1}{3} P_D]$$

where he neglected any T = 3/2 states. Inserting variational values Delves[89] finds

$$|M_A|^2 \sim 2.70$$

In any case $|M_A|^2 < 3$, independent of the accuracy of the variational calculations. Even that value leads to contradiction with the ft value of the neutron[124,131]. Blin-Stoyle and Tint[125] tried to remove the discrepancy by taking exchange effects into account, using CVC and PCAC theory. Unfortunately, they obtained a reduction instead of an enhancement in the value of $|M_A|^2$, increasing the discrepancy. Delves[89] points out that there are several possible reasons for this discrepancy:

1. neglect of $T = 3/2$ and other charge dependent effects,
2. assumption of CVC and PCAC theories,
3. the experimental results for the reaction $p + p \rightarrow d + \pi$, used by Blin-Stoyle and Tint[125], are in error, and
4. the measured ft values of the neutron and/or of ^3H are in error.

The last mentioned reason seems to be the most likely. Some independent reasons for doubting the ft value of the neutron are given by Blin-Stoyle[125]. Until new experiments have been performed no additional information on the three-nucleon bound state wave function can be deduced from the beta decay of ^3H.

VI. CONCLUSIONS

Two points emerge clearly from this review. The first is that it will be necessary to understand the deuteron well enough to tell the difference between $4\text{-}\frac{1}{2}$ and 7% D state, if we are to avoid a basic uncertainty of around 1 MeV in calculations of ϵ_t and of around 1 fermi in calculations of a_2. In order to do this we will have to understand the wave function inside distances of 1 F, and in particular be able to discriminate between a hole in the wave function 0.5 F in radius as suggested by hard core models and vector meson repulsion, or 0.7 F in radius as suggested by the boundary condition model and the large number of elementary particle degrees of freedom which can be excited inside this radius. Fortunately, the comparable uncertainty due to uncertainty in the correct value to use for r_s^{np} and r_s^{nn} has probably been removed.

The second is related in that we have also seen that it will be important to understand the off-shell behavior of the two-nucleon T matrix better than we do now, and the size of the hole in the wave function has an important bearing on this question. This is pretty well defined if we stick to local potentials, but differs in one way for the non-local boundary condition model, and in another for the non-local separable models. Work on momentum dependent models is still virgin territory in the three-nucleon problem, but the work of Ingber on the closely related nuclear matter problem indicates this may also be crucial, at least indirectly, because of the difference it allows in the two-nucleon central force model. Again, current uncertainties on these points frustrate meaningful comparison with experiment at the 1-4 MeV level in ϵ_t and the 1-2 F level in a_2 calculations.

Other conclusions are more controversial. It seems likely that separable models can be brought into good agreement with nucleon-nucleon phase shifts, that a rather low cutoff in the number of two-nucleon angular momentum states can be used without much error, and less certainly, that when the small effects are included, such models will give more binding to the triton than variational calculations with local hard-core potentials. It will, therefore, be necessary to also claculate such hard-core models by other methods (Faddeev, or methods giving close bounds on ϵ_t from both sides) in order to be sure the variational calculations have converged. It is our guess that when this has been carried out, there will still be a 2 MeV or so discrepancy between separable and local models which give the same fit to two-nucleon elastic scattering data up to 300 MeV. Hence, it is expected that we will have to extract off-shell information froma variety of two-nucleon electromagnetic experiments before we can say whether or not there are three-body forces in the three-nucleon system.

The point of view taken here is to treat the three-nucleon system as a problem in physics rather than in mathematics or in nuclear phenomenology. We mean in no way to belittle the real accomplishment made by the separable models in showing that very simple interactions and equations can give a reasonable description of both the two- and the three-nucleon systems at the 10-15% level. But we hope that these real successes will not blind us to the fact that a much harder task remains before a convincing connection between nuclear and elementary particle physics can be forged out of these studies.

We are indebted to Houk for informing us of his new result, which is so crucial for our subject, and for permission to quote it and his preliminary analysis prior to his publication. We are most grateful to Delves for sending us his Brela lectures prior to publication, and we have taken the liberty of relying heavily on his review, particularly in Chapter V. We are also most grateful to the many workers in this field who have kept us informed of their work through preprints and private communications, in particular to Amado, Erens, Kok, Mitra, Osborn, Phillips and Wong. Most of all, we wish to express our gratitute to Nuttall for organizing this interesting conference and making possible this fruitful discussion and interchange of ideas.

FOOTNOTES AND REFERENCES

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Table I. Values of the $^1S_0$ n-p Effective Range for Various Data Selections

	Present Analysis				Previous Analysis[a]			
	$\sigma_{np}(0) = 20.442 \pm 0.023$ b $a_{nH} = -3.719 \pm 0.002$ F				$\sigma_{np}(0) = 20.35 \pm .05$ B $a_{nH} = -3.741 \pm .011$ F			
	$a_t = 5.4255$ F $a_s = -23.7146$ F							
	$\epsilon_d = 2.22452$ MeV							
	(errors negligible in determination of r_s)				(a_t and a_s codetermined with r_s)[a]			
Data Selection	r_s	D.F.	χ^2	% Prob	r_s	D.F.	χ^2	% Prob
ALL[b]	2.758 ± 0.053	7	5.942	54.8	2.52 ± 0.10	6	4.45	62.1
EBML[c]	2.646 ± 0.072	1	0.060	80.8	2.44 ± 0.11	1	0.15	68.5
ALL - EBML	2.891 ± 0.078	5	2.449	78.4	2.64 ± 0.13	4	1.27	86.5
Adding constraint $r_s = 2.73 \pm 0.03$ F								
ALL	2.736 ± 0.026	8	5.993	64.8	2.71 ± 0.03	7	8.40	33.8
EBML	2.718 ± 0.028	2	0.232	89.1	2.71 ± 0.03	2	6.31	4.25
ALL - EBML	2.750 ± 0.028	6	2.750	81.7	2.73 ± 0.03	5	1.77	87.8

[a] H. P. Noyes, Nucl. Phys. 74, 508 (1965), Table 2.

[b] H. P. Noyes, Phys. Rev. 130, 2025 (1963), Table 1. The attentive reader will note that the degrees of freedom do not add up for the previous analysis. On checking the computer output I find that for some reason which now escapes me, the cross section at 2.54 MeV (Table 1, No. 6) was omitted; if this is done in the current analysis, the value for r_s for the remaining seven points becomes 2.734 in embarrassing good agreement with the prediction.

[c] C. E. Engelke, R. E. Benenson, E. Melkonian and J. M. Lebowitz, Phys. Rev. 129, 324 (1963).

FIGURE CAPTIONS

Figure 1. Possibility that the outgoing wave from another channel (channel 3 in the illustration) can always scatter the two particles in the direct channel (2 and 3 in the illustration) if they are inside the range of forces, which gives a source term in channel 1 which only vanishes like ρ_1/r_1 for r_1 large.

Figure 2. Values of r_s^{np} computed from n-p total cross section measurements below 5 MeV compared with the prediction from charge-independence, corrected for the $\pi^{\pm}$ - π^{o} mass difference in the OPE term, that $r_s^{np} = 2.73 \pm 0.03$ F.

Figure 3. Fit to the low energy n-d doublet S phase shift of the form $k \operatorname{ctn} \delta_2 = - A + Bk^2 - C/(1 + Dk^2)$ as given by van Oers and Seagrave, Phys. Letters 24B, 562 (1967).

Figure 4. Comparison of calculations of ϵ_t and a_2 as given by A. C. Phillips, Nuclear Physics A107, 209 (1968), V. F. Kharchenko, N. M. Petrov and S. A. Storozhenko, Nuclear Physics A106, 464 (1968), and G. L. Shrenk and A. N. Mitra, preprint and Brela Symposium. The much more extensive results obtained by the second group in the reference cited and in the earlier publication by A. G. Sitenko, V. F. Kharchenko and N. M. Petrov, Physics Letters 21, 54 (1966) have mostly been omitted in order not to confuse the plot. They use only 4% D state, and are slightly shifted from Phillips' results (open circle, solid

circle, open triangle) because of the slightly different value for a_s, as is illustrated for r_s = 2.7 F by the open square labeled 1_t. Values for other values of r_s also agree with Phillips if shifted by about the same amount. The value labeled 3_t is obtained by these authors by cubing the Yamaguchi form factor in the central but not the tensor parts of the interaction. Results from Shrenk and Mitra are not directly comparable, since they include a second rank singlet potential fitted by Naqvi and Gupta. The $(C+T)_Y$ points use the same (Yamaguchi) triplet interaction as Phillips 4% D state points. The $(C+T)_N$ points use the Naqvi triplet parameters, omitting the L·S term. The designation of the singlet model used $(N,G_1,G_1',G_1'',G_2',G_2,G_3)$ refers to parameters taken from Naqvi and Gupta by Shrenk and Mitra, and occurs in the same order along both dotted curves; for clarity the points are labeled only along the $(C+T)_N$ curve.

Figure 5. Comparison of various separable singlet models with experiment between 50 and 330 MeV. The dotted experimental curve is from the n-p Yale IV fit (R. E. Seamon, K. A. Frideman, G. Breit, R. D. Haracz, J. M. Holt and A. Prakash, Phys. Rev. 165, 1579 (1968)) and the experimental points the latest Livermore p-p results (M. H. MacGregor, R. A. Arndt and R. M. Wright, "Determination of the Nucleon-Nucleon Scattering Matrix VII. (p,p) Analysis from 0.400 MeV" UCRL 70075 (Part VII) and Phys. Rev. (in press)) corrected by the n-p to p-p difference used

by the Yale group; the Livermore energy-dependent fit is essentially identical in spite of the very different fitting procedures and different criteria for data selection used by the two groups. The shape-independent curve is the same as the separable model using the square root of the Yamaguchi form factor (Bander). n=1 is the Yamaguchi form, and n=8 those form factors raised to the 8th power. The Puff potential result is computed from the parameters given by Borysowicz and Dabrowski. The Gupta result has been computed by us from the parameters for the G_1 model as given by Shrenk and Mitra; surprisingly, the Naqvi parameters give a curve between 50 and 330 MeV which lies almost on top of G_1, according to the phase shifts published by Naqvi, although it will differ significantly at low energy due to the different effective range.

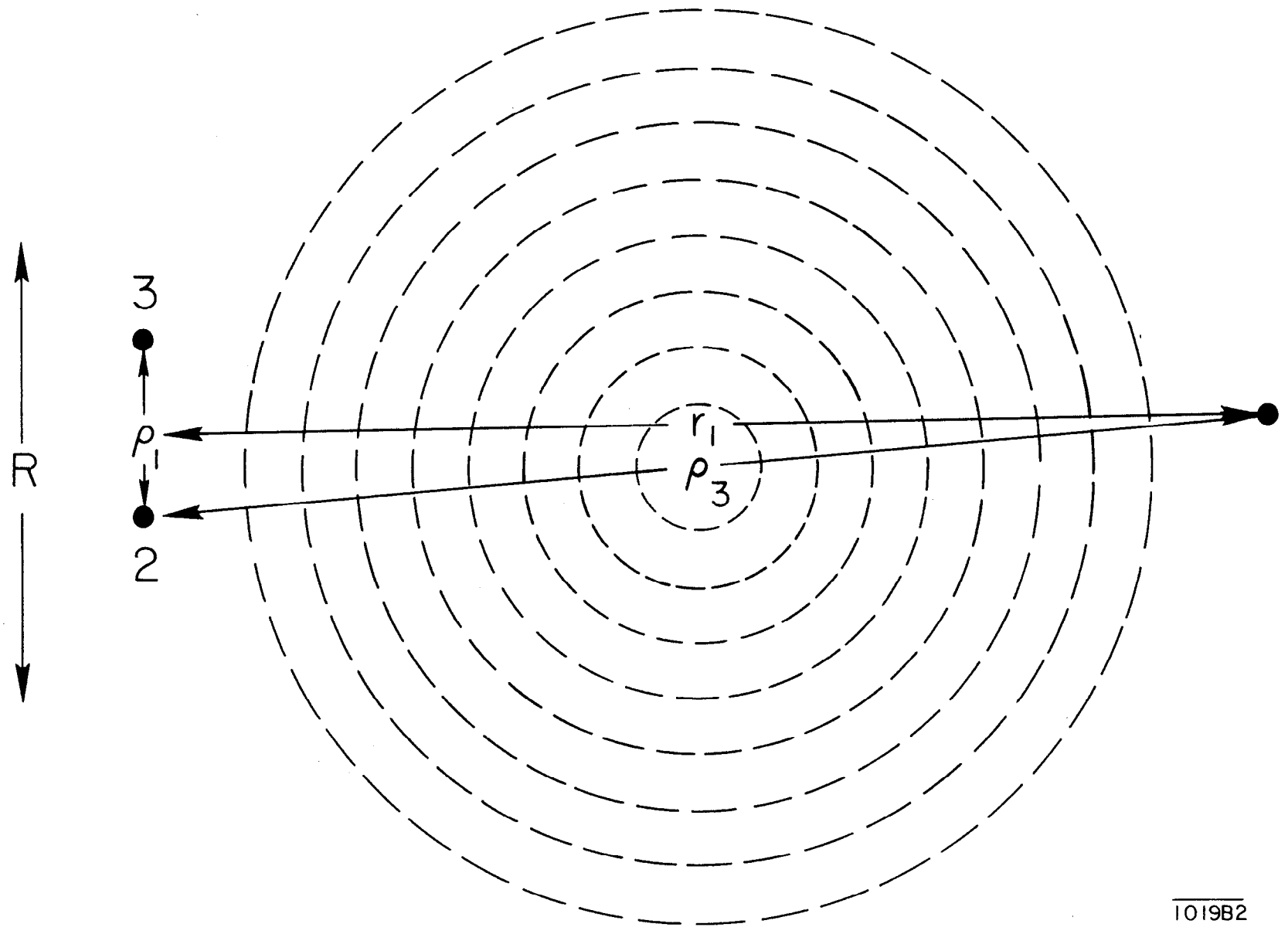

Fig. 1

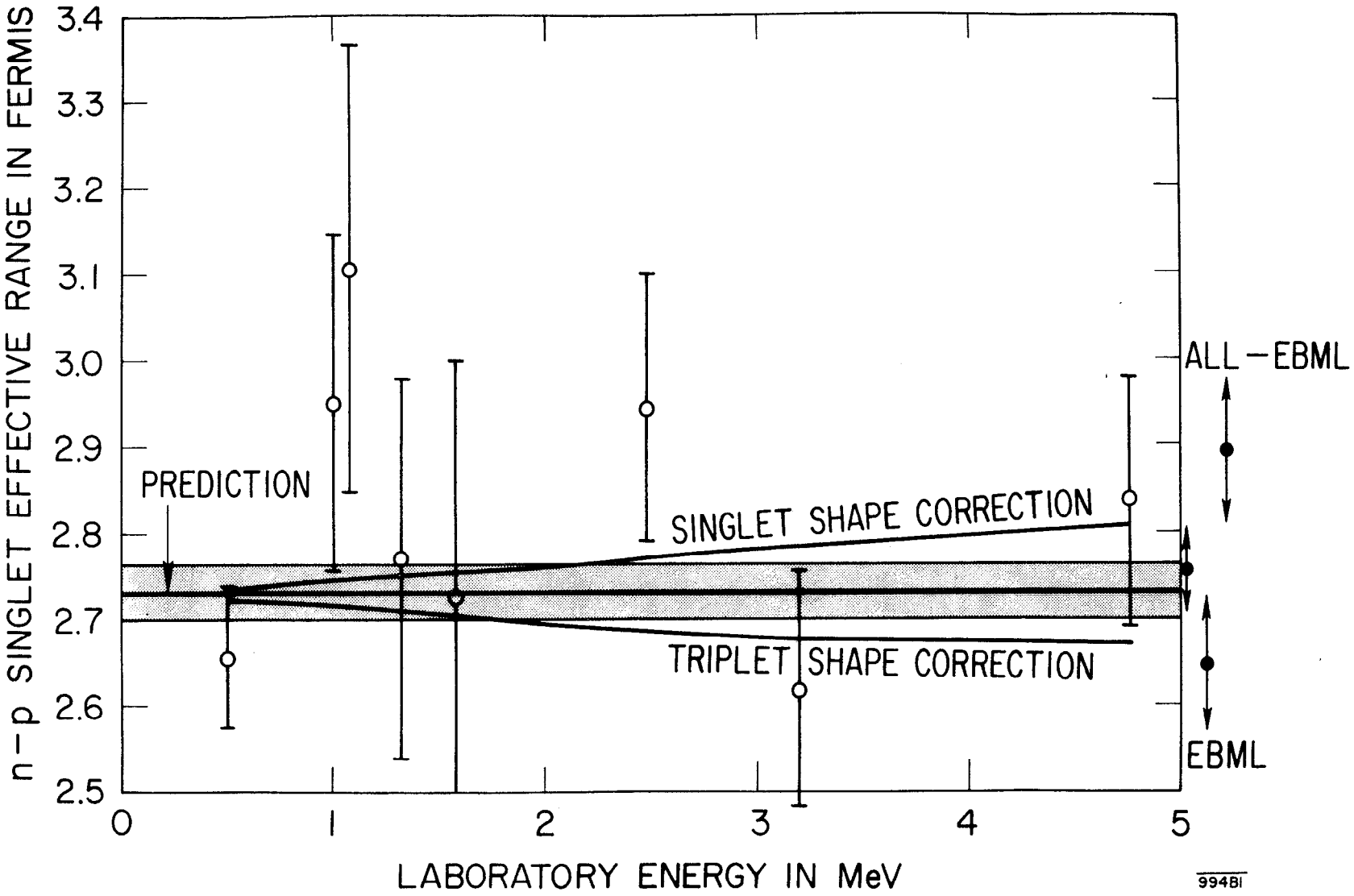
n-p SINGLET EFFECTIVE RANGE IN FERMIS
3.4
3.3
3.2
3.1
3.0
2.9
2.8
2.7
2.6
2.5
0
1
2
3
4
5
LABORATORY ENERGY IN MeV
PREDICTION
SINGLET SHAPE CORRECTION
TRIPLET SHAPE CORRECTION
ALL-EBML
EBML
994BI

Fig. 2

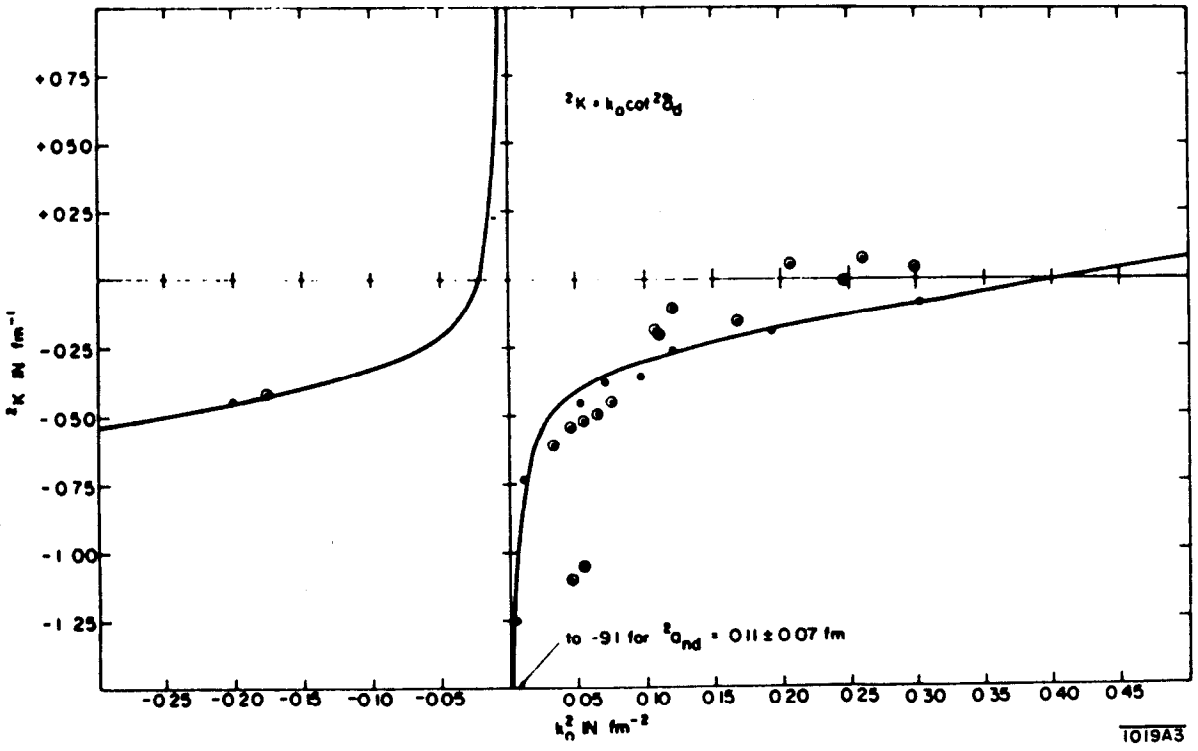
+0.75
+0.50
+0.25
-0.25
-0.50
-0.75
-1.00
-1.25
-0.25 -0.20 -0.15 -0.10 -0.05
0.05 0.10 0.15 0.20 0.25 0.30 0.35 0.40 0.45
^2K IN fm^{-1}
k_n^2 IN fm^{-2}
$^2K = k_0 \cot {}^2\delta_d$
to -9.1 for $^2a_{nd}$ = 0.11 ± 0.07 fm
1019A3

Fig. 3

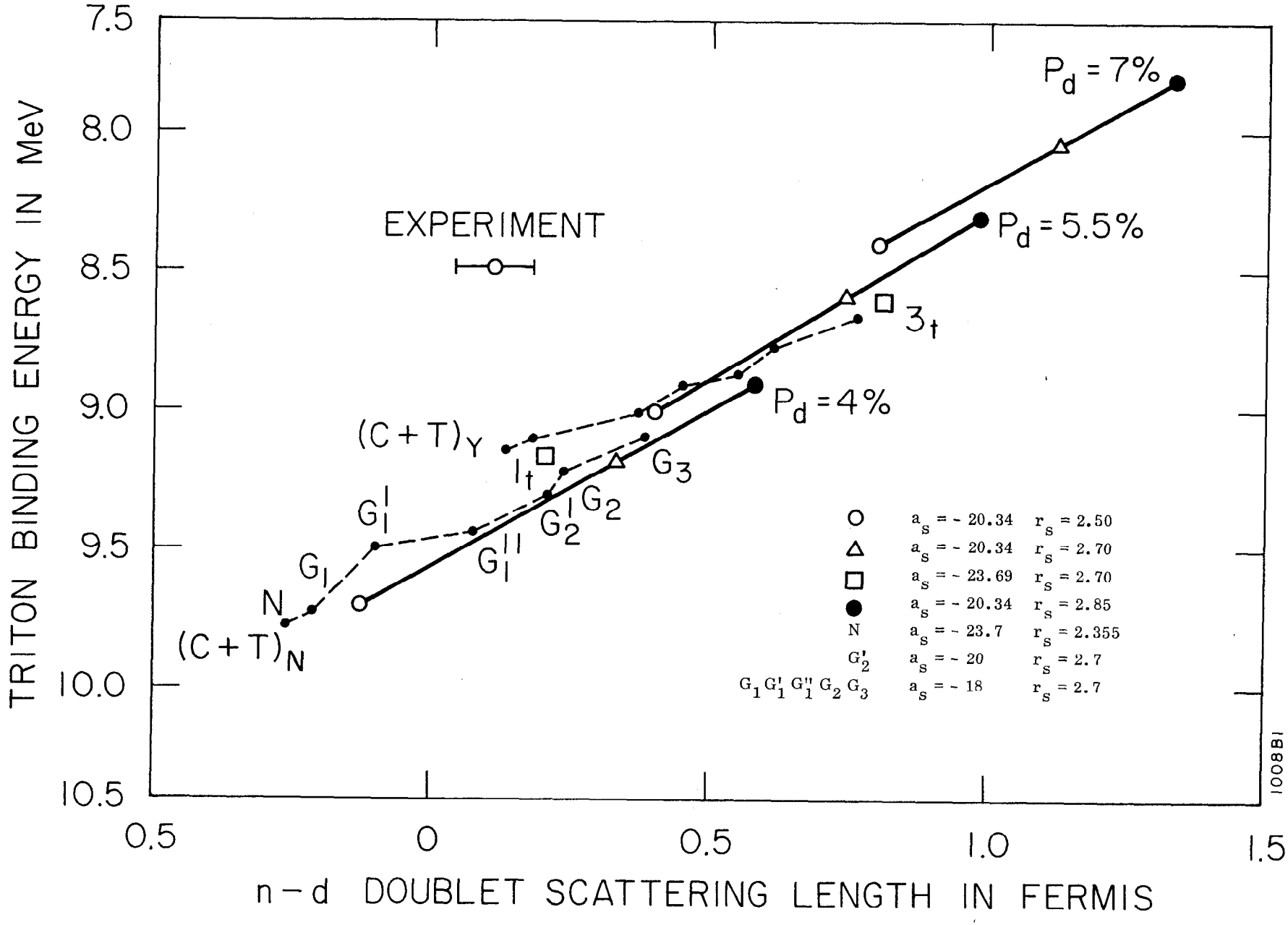
TRITON BINDING ENERGY IN MeV
7.5
8.0
8.5
9.0
9.5
10.0
10.5
0.5
0
0.5
1.0
1.5
n – d DOUBLET SCATTERING LENGTH IN FERMIS
EXPERIMENT
P_d = 7%
P_d = 5.5%
P_d = 4%
^3_t
$(C+T)_Y$
$(C+T)_N$
N
G_1
G_1'
G_1''
G_2
G_2'
G_3
I_t
○ a_s = - 20.34 r_s = 2.50
△ a_s = - 20.34 r_s = 2.70
□ a_s = - 23.69 r_s = 2.70
● a_s = - 20.34 r_s = 2.85
N a_s = - 23.7 r_s = 2.355
G_2' a_s = - 20 r_s = 2.7
$G_1 G_1' G_1'' G_2 G_3$ a_s = - 18 r_s = 2.7
1008B1

Fig. 4

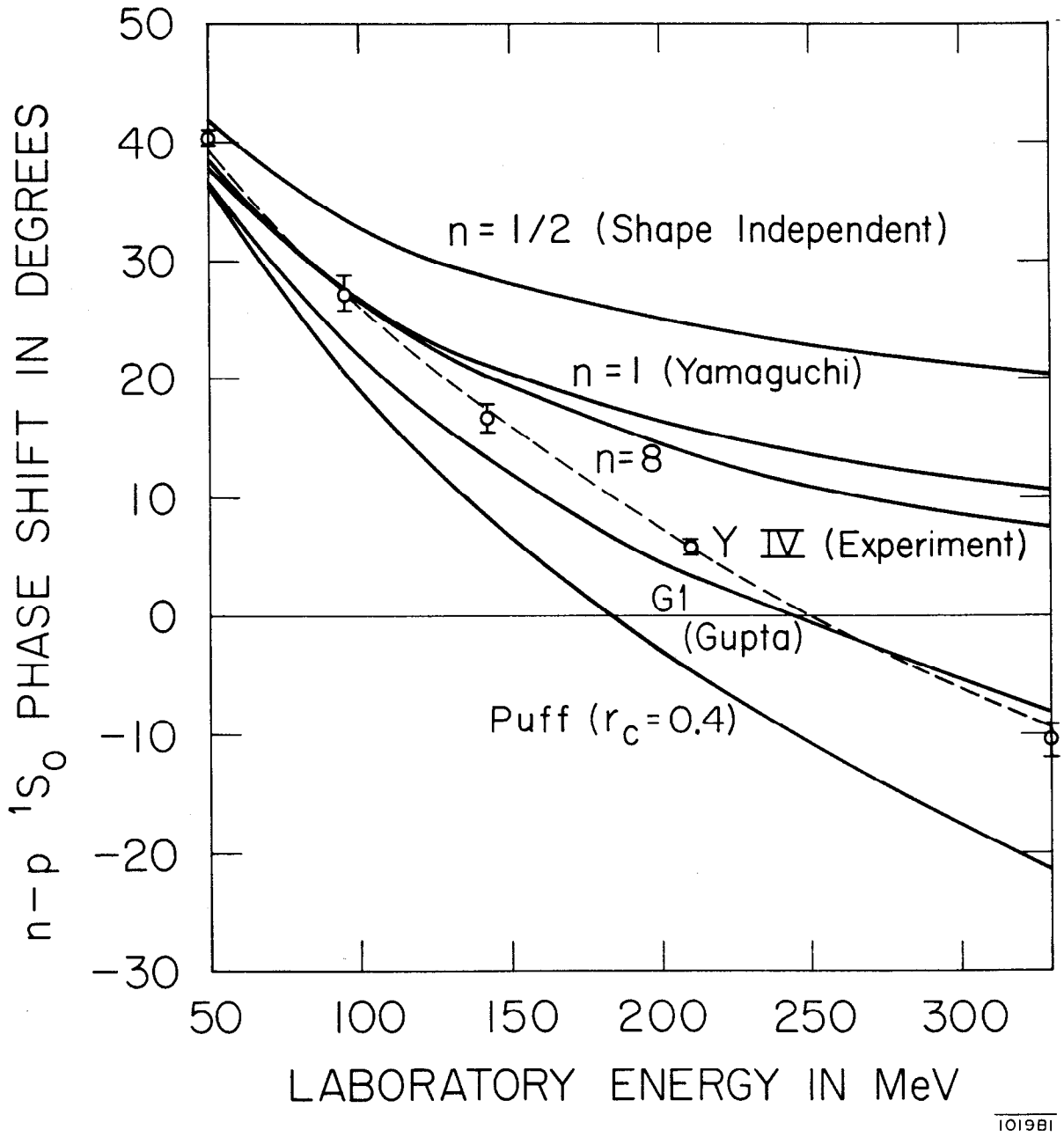
50
40
30
20
10
0
−10
−20
−30
n − p ¹S₀ PHASE SHIFT IN DEGREES
n = 1/2 (Shape Independent)
n = 1 (Yamaguchi)
n=8
Y IV (Experiment)
G1
(Gupta)
Puff (r_c = 0.4)
50
100
150
200
250
300
LABORATORY ENERGY IN MeV
101981

Fig. 5