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DIFFERENCE OF NUCLEON AND PION ELECTROMAGNETIC RADII*

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ABSTRACT

The nucleon isovector Pauli radius, $< R^2_{2V}>$, is calculated using sidewise dispersion relations to be significantly larger than the predictions of ρ dominance, in accord with observation. It is also predicted that $< R^2_{2V}>$ is significantly larger than the pion charge radius, $< R^2_{\pi}>$. Elastic scattering of pions from electrons at very high energies (viz. Serpukhov) will give a clear confrontation with this prediction.

In this note we report on the application of sidewise dispersion relations to the calculation of the electromagnetic structure of nucleons. This formulation of the dispersion relation of the electromagnetic vertex as a function of the nucleon mass was developed first by Bincer[1] and expresses the appropriate form factor as

$$F(\ell^2) = \frac{1}{\pi} \int_{(M+\mu)^2}^{\infty} \frac{dW^2}{W^2 - M^2} \quad \mathrm{Im}\, F(W^2, \ell^2) \quad . \tag{1}$$

Im F is the amplitude for a virtual photon of mass $\ell^2 \equiv t$ to be absorbed by a nucleon and form a real intermediate state of total mass W which then couples to an off-

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shell nucleon of the same mass as in Fig. 1. The threshold of the dispersion integral lies in the physical region, and to the extent that the absorptive amplitude is dominated by its low mass contributions, $W \sim M$, we can approximate it by the threshold electropion production amplitude times the pion-nucleon coupling strength. For real photons the exact low energy behavior of the photopion production amplitude is known and is given by the Kroll-Ruderman theorem. For virtual photons the low energy limit of electropion production is constructed using PCAC and current algebra.

An earlier application of this idea to the calculation of the nucleon g-2 value led to encouraging results.[2] Both the isovector character of the nucleon moment and its approximate numerical value were reproduced fairly well when only the contribution to the absorptive amplitude between $M \leq W \leq 1.5M$ was retained, and the threshold theorems were used. The usual grief which befalls the perturbation calculations was found to be in the high mass contributions $1.5M < W < \infty$, which the perturbation approximation severely distorts. This threshold dominance view also reproduces second and fourth order electron g-2 values and has made a definite prediction of the α^3 contribution, recently refined by more detailed studies of Parsons.[3]

Our motivations in undertaking this study were twofold:

1. The familiar dispersion theory studies of the nucleon electromagnetic form factors are based on a continuation in the photon mass. The vector dominance model which has emerged from these analyses has been successful for processes involving real ρ^o mesons, for example, as well as relating these to processes involving real photons interacting with hadrons[4]--viz. $\gamma + p \rightarrow p + \rho^o$ or $\pi^o \rightarrow \gamma + \gamma$. However, it is inadequate to describe the observed rapid decrease of the nucleon form factors for large momentum transfers $|t| \gtrsim 10(\mathrm{GeV/c})^2$. For processes in which there appear virtual photons of such high masses a more elaborate analysis with

several vector mesons or resonances in the $J^{PG} = 1^{-+}$ channel joining the photon to hadrons is needed. This is not surprising, because the dispersion integral

$$F(t) = \int_{4\mu^2}^{\infty} \frac{\rho(\sigma^2)\, d\sigma^2}{\sigma^2 - t} \tag{2}$$

converges only very slowly, with all contributions to the spectral function $\rho(\sigma^2)$ being weighted essentially equally up to large $\sigma^2 = -t$. It is certainly an extravagant optimism to assume that only one octet (or nonet) of low-lying resonance contributions dominates in Eq. (2) in this case. On the other hand, the photon mass is only a parameter in the present calculation, and the denominator in Eq. (1) is unaltered if we extend these studies to finite ℓ^2 values.

2. A mean square radius of a nucleon is given by $< R^2 > \equiv 6 \int_{4\mu^2}^{\infty} d\sigma^2 \rho(\sigma^2)/\sigma^4$ in the usual approach, Eq. (2), and heavily weights the low mass contributions to $\rho(\sigma^2)$. By the same token, however, it puts great emphasis on contributions to the spectral amplitude from the two body $\pi\pi$ continuum of mass $\sigma^2 \sim 4\mu^2$ in addition to the ρ^0 resonance, and the evaluation of this contribution to $\rho(\sigma^2)$ requires knowledge of the analytic continuation of the amplitude for $\pi\pi \rightarrow N\overline{N}$ below its physical threshold of $4M^2$. Using the sidewise approach of Eq. (1), no such analytic continuation is required, and the amplitude in the threshold region emphasized in this approach is constructed with the help of PCAC and current algebra. The ρ dominance model predicts a radius of $< R^2 > = 6/M_\rho^2 = 0.4\, f^2$ for the isovector form factor, whereas experimentally[5] for the isovector part of the Pauli magnetic moment form factor a much larger result is obtained

$$< R_{2V}^2 > = 0.7\, f^2 \tag{3}$$

and our goal in this calculation is to achieve a better account of this difference.

We denote by $\overline{u}(p)\,\Gamma_\mu(p, p+\ell)$ the electromagnetic vertex for an off-shell nucleon of $(\text{mass})^2 = (P+\ell)^2 = W^2$ to emit a virtual photon of $(\text{mass})^2 = \ell^2$ and become a real nucleon on the mass shell $p^2 = M^2$. The general form of this vertex and the technique for projecting out the Pauli magnetic moment form factor $F_2(\ell^2)$ have already been discussed and we need only quote the results here.[1,2] The Ward identity expressing differential electromagnetic current conservation assures us that the proton Dirac charge form factor $F_1^p(\ell^2)$ obeys a subtracted dispersion relation in W^2. Whether this subtraction term is the constant 1 or a function of ℓ^2 is a matter of assumption, as is the decision whether the subtraction is to be made at $W^2 = M^2$, say, or at $W^2 = \infty$. We thus choose to confine ourselves to the $F_2(\ell^2)$ dispersion relation,[6] and in order to avoid complications of anomalous thresholds we stick to the scattering region $\ell^2 < 0$.

With the threshold dominance assumption we retain only the two-body πN intermediate state in computing the absorptive part:

$$\mathrm{Abs}\left\{\bar{u}(p,s)\Gamma_\mu(p,p+\ell)\right\} = \frac{|\mathbf{q}|}{4\pi W} M \sum_{s',c} \int \frac{d\Omega_{\hat{q}}}{4\pi}\left[-i\bar{u}(p,s)J_\mu^c u(k,s')\right]\bar{u}(k,s')g\Gamma_5\tau_c \tag{4}$$

where $\left[\bar{u}(p,s)\,J_\mu^c\,u(k,s)\right]$ is the amplitude for electropion absorption, $\frac{|\mathbf{q}|}{W}\int d\,\Omega_{\hat{q}}$ is the two-body phase space for the intermediate real pion of momentum $\underline{q}$ and nucleon with $\underline{k} = -\underline{q}$ and spin s' in the center of mass system, and $\bar{u}(k,s')\,g\Gamma_5\,\tau_c$ is the general expression for the π - N vertex for the nucleon emerging off the mass shell with $k + q = p + \ell$ and $(k+q)^2 = W^2$ (see Fig. 1).

For the electropion absorption amplitude at threshold $W^2 \approx M^2$ or $W = M + \mu$, we time reverse the low-energy theorem for electropion production as derived by Adler and Gilman[7] assuming PCAC and the current algebra for axial and vector currents. Very simply, this amplitude is, in the massless pion limit, the three electroproduction pole terms, the usual non-pole term for gauge invariance, and in addition the PCAC and current algebra prediction for the threshold s-wave contributions of the dispersion theoretic continuum. We extrapolate this result to the threshold region for a pion of physical mass $\mu \neq 0$ to obtain in the above indicated order

$$\bar{u}(p,s)\, J^c_\mu\, u(k,s') = ig\bar{u}(p,s)\Bigg[V_\mu \frac{\not{p}+\not{\ell}+M}{(p+\ell)^2 - M^2}\gamma_5 \tau_c + \gamma_5 \tau_c \frac{\not{p}-\not{q}+M}{(p-q)^2 - M^2} V_\mu$$

$$- \frac{(2q-\ell)_\mu}{(q-\ell)^2-\mu^2} F_\pi \gamma_5 \frac{[\tau_c, \tau_3]}{2} - \gamma_5 \frac{\ell_\mu}{\ell^2}(F_\pi - 2F_1^V)\frac{[\tau_c, \tau_3]}{2} \tag{5}$$

$$- \gamma_5\, i \frac{\sigma_{\mu\nu}\ell^\nu}{2M^2}(\tau_c F_2^S + \delta_{c3} F_2^V)$$

$$+ \frac{\gamma_5}{2M}\left(\gamma_\mu - \not{\ell}\frac{\ell_\mu}{\ell^2}\right)(G_A - 2F_1^V)\frac{[\tau_c, \tau_3]}{2}\Bigg] u(k,s')$$

where $V_\mu = \gamma_\mu (F_1^S + \tau_3 F_1^V) - (i\sigma_{\mu\nu}\ell^\nu/2M)(F_2^S + \tau_3 F_2^V)$, F_1 and F_2 denote the usual Dirac and Pauli nucleon form factors, G_A is the axial form factor, and S and V denote isotopic scalar and vector parts respectively.

For the pion-nucleon vertex we have the general form

$$\bar{u}(k,s') g\Gamma_5 \tau_c = g\bar{u}(k,s')\gamma_5 \left[K_+(W^2)\frac{\not{p}+\not{\ell}+M}{2M} + K_-(W^2)\frac{M-\not{p}-\not{\ell}}{2M}\right]\tau_c \tag{6}$$

where the form factors $K_\pm(W^2)$ are themselves determined from sidewise dispersion relations as exhibited by Bincer[1] with $K_+(M^2) = 1$ defining the Watson-Lepore coupling constant. To be consistent with neglecting the resonance variations in electropion production, we also neglect them in $K_\pm(W^2)$ and obtain the pseudoscalar meson perturbation theory result $K_\pm(W^2) = 1$ and $\Gamma_5 = \gamma_5$.[8]

To illustrate most simply the separate terms contributing to the absorptive part of F_2^V, we compute the coefficient of the two-body phase space factor $\frac{|\underline{q}|}{W}$ in Eq. (4) at threshold $\underline{q} = \underline{k} = 0$, $W = M + \mu$. This simplifies all angular dependence since the electropion production is pure S-wave.[9] Using threshold kinematics

$q = \frac{\mu}{M+\mu}\,(p + \ell) = \frac{\mu}{M}k$, we can rewrite

$$\bar{u}(p,s)\, J_\mu^c\, u(k,s') = \frac{ig}{M}\,\bar{u}(p,s)\left[\gamma_\mu E_1^c - i\frac{\sigma_{\mu\nu}\ell^\nu}{2M} E_2^c + \ell_\mu E_3^c\right]\gamma_5\, u(k,s') \qquad (7)$$

where the E_i^c are scalar functions of ℓ^2 only. Introducing Eqs. (6) and (7) into Eq. (4), performing the spin sum and projecting out the absorptive part of interest at threshold, we find

$$\mathrm{Im}\, F_2^+(W^2 = (M+\mu)^2, \ell^2) \simeq \frac{g^2}{4\pi}\frac{|\underline{q}|}{W}\,\underline{E}_2 \cdot \underline{\tau}\,\frac{\mu}{M}\,\left[1 + 0\left(\frac{\mu}{M}\right)\right] \qquad (8)$$

The factor $\frac{\mu}{M}$ arises from the fact that at threshold the πN system of mass $W \simeq M + \mu$ is produced in an S-state of odd parity. This in turn projects into a positive parity nucleon of physical mass M only by an amount proportional, in lowest order, to its distance from the mass shell--i.e., $W - M \sim \mu$. We must now identify in Eq. (5) the leading order contribution to E_2^c and cancel the factor $\frac{\mu}{M}$ if the threshold contribution is to be important and hopefully dominant. The last three non-pole terms of Eq. (5) are finite as $\frac{\mu}{M} \to 0$ and thus do not contribute to Eq. (8) in leading order. The first two s channel and u channel nucleon pole terms also contribute finite parts to E_2^c as $\frac{\mu}{M} \to 0$. The only singular part in $\frac{1}{\mu}$ comes from the pion pole term near $\ell^2 = 0$, and we find

$$\mathrm{Im}\, F_2^+\,[(M+\mu)^2, \ell^2] \approx \frac{g^2}{4\pi}\,\frac{|\underline{q}|}{W}\, 2\,\tau_3\, F_\pi(\ell^2)\left(\frac{1}{1 - \ell^2/2\mu^2}\right). \qquad (9)$$

With $\ell^2 = 0$, Eq. (9) gives the same magnetic moment found earlier.[2] The slope of $\mathrm{Im}\, F_2^+$ as $\ell^2 \to 0$ gives the absorptive part at threshold for the nucleon's radius

$$\frac{d}{d\ell^2}\,\mathrm{Im}\, F_2^+\,[(M+\mu)^2, \ell^2]\Big|_{\ell^2=0} \approx \frac{g^2}{4\pi}\,\frac{|\underline{q}|}{W}\, 2\,\tau_3\left[\langle R_\pi^2\rangle + \frac{1}{2\mu^2}\right]. \qquad (10)$$

From a more careful evaluation using Eq. (5) throughout the threshold region $(M+\mu)^2 \leq W^2 \leq \Lambda^2(M+\mu)^2$, with $\Lambda^2 \approx 2$ chosen to give the magnetic moment $\mu_V = 1.85$, we find

$$< R_{2V}^2 > \; = \; < R_\pi^2 > + \frac{g^2/4\pi}{4M\mu} \frac{1}{\mu_V} \left[1 - \frac{12}{\pi}\frac{\mu}{M}\left(\ln\left(\frac{M}{\mu}\right) + \ln\left(\frac{\Lambda^2-1}{\Lambda^2}\right) - 1\right)\right] \quad (11)$$

$$\approx \; < R_\pi^2 > + 0.5\,f^2.$$

We do not propose to take seriously any quantitative predictions of this calculation in view of the approximate nature of the method.[10] However, there are important qualitative conclusions to be drawn from Eq. (11) with interesting experimental implications:

1) The nucleon isovector (Pauli) radius is significantly larger than both the pion charge radius and the predictions of ρ dominance.[11] This is due to important and known threshold contributions to the absorptive parts of Eq. (1). In the words of the uncertainty principle the size of the pion current distribution about the nucleon extends out as far as $\Delta x \sim c\Delta t \sim c\hbar/\mu c^2 \sim \hbar/\mu c$. For the π meson structure, however, the selection rule of conservation of G parity dictates that $\pi \rightarrow \pi + \rho$--i.e., the range of the pion current about a pion is restricted because of the requirement to make the ρ meson rest mass in the intermediate state and $\Delta x \sim c\hbar/M_\rho c^2 \sim \hbar/M_\rho c$. Translated to more formal terms, the linear divergence with $\frac{1}{\mu}$ as $\mu \rightarrow 0$ appears in Eq. (11) because the nucleon experiences no change in mass in the intermediate state $N \rightarrow N + \pi$ and for the pion case it is not present. This suggests that the ρ dominant prediction of a radius $6/M_\rho^2 = 0.4\,f^2$ should be a better approximation for the pion size than for the nucleon. A quantitative measurement of $< R_\pi^2 >$ and of its difference from $< R_{2V}^2 >$ is eagerly anticipated. To avoid theoretical uncertainties[12] and complications in the interpretation of $e\pi$ production and of $\pi^\pm - \alpha$ scattering results,

it will be necessary to do elastic scattering of pions from target atomic electrons at the momentum transfers of > 180 MeV/c first available[13] at Serpukhov so that $1/3|\ell^2| < R_\pi^2 > \gtrsim 10\%$.

2) There is no $\frac{1}{\mu}$ singularity with vanishing μ for the isoscalar Pauli form factor so that, to leading order, $< R_{2p}^2 > \approx < R_{2n}^2 >$.

Finally, turning to the large ℓ^2 limit we find it impossible to cancel the $\frac{\mu}{M}$ factor in Eq. (8), and thus threshold dominance does not appear to be a valid, or at least defensible approximation for large ℓ^2. In this limit, the pion current term in Eq. (9) is unimportant, being $\sim \mu^2/\ell^2$. The combined effects of s and u channel pole terms lead to

$$\mathrm{Im}\, F_2^+ \sim \frac{|q|}{W}\left[\frac{\mu}{M} F_2 + \frac{\mu}{M}\frac{M^2}{\ell^2} F_1\right]$$

or

$$F_2(\ell^2) \sim \frac{\mu}{M}\frac{M^2}{\ell^2} F_1$$

and for large ℓ^2

$$G_M(\ell^2) = G_E(\ell^2)(1 + 0(\frac{\mu}{M})).$$

Although suggestive of the "scaling law" this misses the "theoretically popular" limit of $G_M^p/G_E^p \approx 2.79$. Moreover, the main contributions to the spectral weight functions are probably not included by retaining only the threshold region[14] in the dispersion integral.[15]

FOOTNOTES AND REFERENCES

1. A. M. Bincer, Phys. Rev. 118, 855 (1960).

2. S. D. Drell and H. R. Pagels, Phys. Rev. 140, B397 (1965).

3. R. G. Parsons, Stanford Linear Accelerator Center preprint No. 354 (1967), Phys. Rev. (in press).

4. For a general review, see the reports to the International Symposium on Electron and Photon Interactions at High Energies, SLAC, September 1967.

5. L. H. Chan, K. W. Chen, J. R. Dunning, Jr., N. F. Ramsey, J. K. Walker, and Richard Wilson, Phys. Rev. 141, 1298 (1966). Our conventions are $F_2^V = \frac{1}{2}(F_2^p - F_2^n)$, $F_2^p(0) = 1.79$, and $F_2^n(0) = -1.91$. The experimental Pauli radius is determined from $\langle R_{2V}^2 \rangle = 6\, F_2^V(0)^{-1} (d\, F_2^V / d\ell^2)\Big|_{\ell^2 = 0}$ and $F_2^V = (G_M^V - G_E^V)/(1 - \ell^2/4M^2)$.

6. In the reduction to the Low equation for the electromagnetic vertex there is an additive constant, 1, in the amplitude for $F_1(\ell^2)$ coming from the equal time commutator term and the assumption of canonical commutation relations. Short of denying these relations, the 1 must be cancelled by important contributions from high mass intermediate states, and thus a threshold dominance approximation is probably not valid for F_1.

7. S. L. Adler and F. J. Gilman, Phys. Rev. 152, 1460 (1966).

8. One can disperse K_- using elastic unitarity and observed parameters for the P_{11} and S_{11} resonances, obtaining $K_-((M + \mu)^2) \approx 1$. H. Suura and L. M. Simmons, Jr., Phys. Rev. 148, 1579 (1966) calculate $K_-(M^2)$ using observed phase shifts and obtain $K_-(M^2) \approx 1$.

9. The complete evaluation of the low energy theorem contribution to the absorptive part, using the multipole analysis of Bincer, will be published separately.

10. The leading terms that diverge linearly and logarithmically as $\mu \to 0$ also appear in the perturbation theory results, as we would expect, since they come from the same graphs. See P. Federbush, M. L. Goldberger, and S. B. Treiman, Phys. Rev. 112, 642 (1958).

11. This conclusion is also arrived at in the analysis of J. S. Ball and D. Y. Wong, Phys. Rev. 130, 2112 (1963); N. G. Antoniou and J. E. Bowcock, Phys. Rev. 159, 1257 (1967); and P. Signell and J. W. Durso, Phys. Rev. Letters 18, 185 (1967) using the normal dispersion approach of the t channel as in Eq. (2).

In addition to the ρ^{0} they keep the non-resonant contributions in calculating the absorptive part. In particular, the N exchange contribution to the $\pi\pi$ - $N\bar{N}$ amplitude enhances the spectral amplitude near the threshold $4\mu^2$ in analogy with our result.

12. The most complete results on electropion production, $e + p \rightarrow e' + n + \pi^{+}$, attempting to measure $< R_{\pi}^{2} >$, are in C. W. Akerlof, W. W. Ash, K. Berkelman, C. A. Lichtenstein, A. Ramanauskas, and R. H. Siemann, Phys. Rev. 163, 1482 (1967). These data and the theory on which their analysis is based are not accurate enough to conclude whether or not there is a quantitative difference between the pion and nucleon radii (see in particular the slopes of the form factors in Fig. 10 on p. 1493). Uncertainties in the theory of the pion radius from the observed differences in π^{+} - α and π^{-} - α elastic scattering have also been discussed and remain to be fully understood. We refer to M. Ericson, Nuovo Cimento 47A, 49 (1967); G. B. West, Phys. Rev. 162, 1677 (1967); M. M. Block, Phys. Letters 25B, 604 (1967); and K. M. Crowe, private communication.

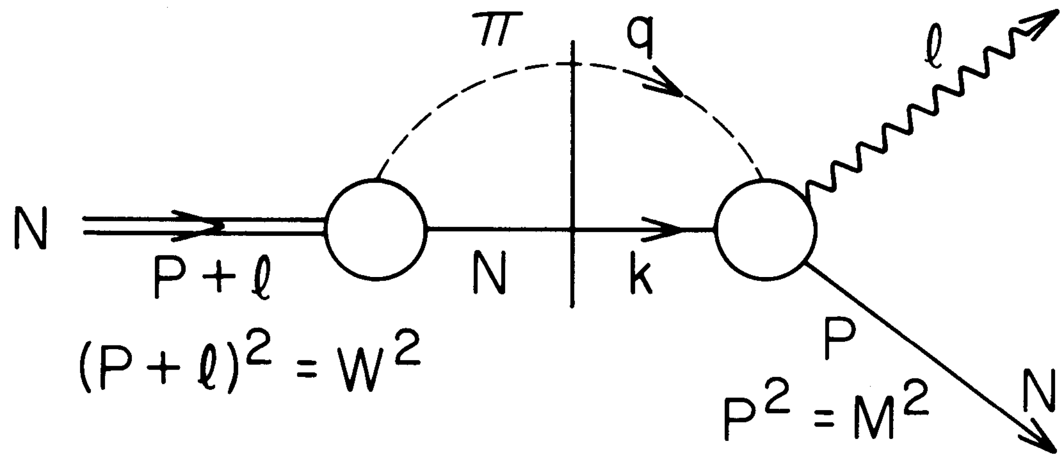
13. For a preliminary analysis of this type of experiment at lower ℓ^2 values, see D. G. Cassel, Ph.D. thesis, Princeton University, 1965.

14. For an earlier discussion of this in a less negative tone, see p. 10 of the proceedings of the International Symposium on Electron and Photon Interactions at High Energies, SLAC, September 1967.

15. We note that an analysis of the nucleon form factors from sidewise dispersion relations has also been carried out by D. U. L. Yu and L. Grünbaum, Bull. Am. Phys. Soc. 13, 24 (1968).

FIGURE CAPTION

Pion-nucleon intermediate state contribution to the absorptive part of the nucleon current.



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