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ASYMPTOTIC SUM RULES*

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ABSTRACT

New sum rules from current algebra valid in the limit of large negative four-momentum squared of the current are obtained; some of these depend upon commutators of space-components of the current densities. They contain convergence factors which make their validity plausible on most models of high-energy behavior, with the Regge-pole model being one of them. With one exception, their utility consists of placing constraints upon models of hadrons designed to saturate the current algebra scheme.

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I. Introduction

Sum rules derived from current commutation relations have thus far been restricted to relations involving the maximum helicity flip part of the current-particle scattering amplitude, and thus, based on Regge-pole arguments, the most convergent amplitudes.[1] Analogous sum rules naively derived for other amplitudes are not expected to hold[2] due to the interlocking morass of Schwinger terms, disconnected diagrams and subtractions in dispersion relations. We may however suspect more relations at large spacelike masses, Q^2, of the virtual particles associated with the currents. A heuristic argument for this hope may be obtained from analogy with nuclear physics.[3] If we consider the hadrons to be made up of other "simpler" constituents then at large current masses Q^2 or equivalently small wavelengths we sample the constituents incoherently and expect multiparticle correlations to vanish. For finite Q^2, it is these correlations that prevent us from obtaining new sum rules.

With one exception all the relations derived have no immediate possibility of experimental verification, as at large negative Q^2 no simple pole dominance is expected to be valid. The one exception is a sum rule devised previously which bounds backward high-energy electron scattering. A possible utility of these rules is that they place constraints on possible hadron models which may be constructed to saturate the current algebra scheme.

In Section II the kinematics, crossing properties, dispersion relations, and various asymptotic relations for the current-particle scattering amplitude are presented. In Section III the sum rules are derived in the $Q_0 \rightarrow i\infty$[5] limit and these results are discussed in Section IV. Some details of the calculations are listed in the Appendix.

II. Kinematics, Crossing Properties, Dispersion Relations and Asymptotics

The commutator of the currents will for simplicity be sandwiched between spin zero hadron states. In the forward direction the same analysis applies to spin averaged amplitudes. We then consider the scattering of a current of momentum q_1 on a state of momentum p_1 to a current of momentum q_2 and a state of momentum p_2:

$$p_1 + q_1 \rightarrow p_2 + q_2 \tag{2.1}$$

and introduce the variables

$$\begin{aligned} P &= p_1 + p_2 \\ \Delta &= p_2 - p_1 = q_1 - q_2 \\ Q &= q_1 + q_2 \end{aligned} \tag{2.2}$$

$$\nu = P \cdot Q, \qquad t = \Delta^2 , \qquad \delta = \Delta \cdot Q = q_1^2 - q_2^2$$

We now study the covariant amplitude

$$M_{\mu\nu}^{\alpha\beta}(P, Q, \Delta) = -(2\pi)^3 \; i \; \sqrt{4\omega_1\omega_2} \int d^4x \; e^{iq_1x} \; \theta(x_0) \langle p_2 \left| \left[j_\mu^\alpha(x) \; j_\nu^\beta(0) \right] \right| p_1 \rangle$$

$$+\text{possible polynomials in } (P_i, q_j) \; . \tag{2.3}$$

In the above the j's are Gell-Mann's[6] currents with the superscript refering to $U(3) \otimes U(3)$ indices; $\omega_i = \sqrt{\underline{p}_i^2 + m_i^2}$, the energy of the state. The difference between the covariant amplitude $M_{\mu\nu}^{\alpha\beta}$ (defined as the response of the S matrix to the variation of external sources coupled to the currents in question[7]) and the retarded product is an operator localized at $x_\mu = 0$, and hence its Fourier transform is a polynomial.[5] Hereafter we shall assume that the polynomial is no worse than a constant already encountered in the vacuum expectation value. The nature of this constant and consistency of this assumption will appear subsequently. The covariant expansion of $M_{\mu\nu}^{\alpha\beta}$ is, for the case of states and currents of the same parity:

$$M_{\mu\nu}^{\alpha\beta} = P_\mu P_\nu\, A_1^{\alpha\beta} + \left(P_\mu Q_\nu + P_\nu Q_\mu\right) A_2^{\alpha\beta} + \left(P_\mu Q_\nu - P_\nu Q_\mu\right) A_3^{\alpha\beta}$$
$$+ \left(P_\mu \Delta_\nu + P_\nu \Delta_\mu\right) A_4^{\alpha\beta} + \left(P_\mu \Delta_\nu - P_\nu \Delta_\mu\right) A_5^{\alpha\beta} + \left(Q_\mu \Delta_\nu + Q_\nu \Delta_\mu\right) A_6^{\alpha\beta}$$
$$+ \left(Q_\mu \Delta_\nu - Q_\nu \Delta_\mu\right) A_7^{\alpha\beta} + Q_\mu Q_\nu\, A_8^{\alpha\beta} + \Delta_\mu \Delta_\nu\, A_9^{\alpha\beta} + g_{\mu\nu}\, A_{10}^{\alpha\beta} \tag{2.4}$$

Crossing symmetry tells us

$$A_i^{\alpha\beta}(\nu, t, Q^2, \delta) = \eta_i\, A^{\beta\alpha}(-\nu, t, Q^2, -\delta) \tag{2.5}$$

with

$$\eta_i = \begin{cases} +1, & i = 1,3,4,7,8,9,10 \\ -1, & i = 2,5,6 \end{cases}$$

We likewise assume that for large spacelike Q^2 and fixed t, δ, the A_i's satisfy usual dispersion relations in ν and their asymptotic behavior in this variable is governed by the Regge-pole hypothesis for the absorptive parts of the A_i's.[8] Thus A_1, A_2 and A_5 satisfy unsubtracted dispersion relations. All others have one subtraction except for A_6, which has two.

$$A_i^{\alpha\beta} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{Im}\, A_i(\nu', t, Q^2, \delta)}{\nu' - \nu}\, d\nu' \quad ; \quad i = 1,2,5$$

$$A_i^{\alpha\beta} = A_i(0, t, Q^2, \delta) + \frac{\nu}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{Im}\, A_i(\nu', t, Q^2, \delta)}{\nu'(\nu' - \nu)}\, d\nu \quad ; \quad i = 3,4,7,8,9,10$$

$$A_6^{\alpha\beta} = A_6^{\alpha\beta}(0, t, Q^2, \delta) + \nu\, \frac{\partial A_6^{\alpha\beta}(0, t, Q^2, \delta)}{\partial \nu} + \frac{\nu^2}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{Im}\, A_6^{\alpha\beta}(\nu', t, Q^2, \delta)}{\nu'^2 (\nu' - \nu)}\, d\nu' \tag{2.6}$$

For spacelike Q^2, the absorptive parts are nonvanishing for $|\nu| \geq \frac{1}{2}\left(\frac{M^2}{4} - P^2 - Q^2\right)$ with M^2 being the lowest value of s or u leading to physical states.

In order to obtain asymptotic sum rules, we shall be interested in the limit $Q_0 \to i\infty$ with $\underset{\sim}{Q}$ fixed. In this limit $Q^2 \to -\infty$; ν and δ will likewise tend to ∞, though linearly rather than quadratically. In the dispersion relation for the A_i's we may thus neglect ν with respect to the lower limit of integration for ν'. The above conjecture is true barring pathological behavior of Im $A_i^{\alpha\beta}$. Thus all terms of the form of a dispersion integral behave as

$$\lim_{Q_0 \to i\infty} \frac{\nu^n}{\pi} \int \frac{\text{Im } A_i^{\alpha\beta}(\nu', t, Q^2, \delta)}{\nu'^n (\nu' - \nu)} d\nu' = \lim_{Q_0 \to i\infty} \frac{(P_0 Q_0)^n}{\pi} \int \frac{\text{Im } A_i^{\alpha\beta}(\nu', t, Q^2, \delta)}{(\nu')^{n+1}} d\nu' , \tag{2.7}$$

and asymptotically we are left with a polynomial in Q_0. A heuristic argument for this comes from the structure of the Jost-Lehmann-Dyson[9] representation, which we assume to be unsubtracted

$$A_i^{\alpha\beta} = \int d^4u \int d\, s \; \frac{\rho_i^{\alpha\beta}(P, \Delta, u, s)}{(Q-u)^2 - s} , \tag{2.8}$$

with the u integration going over a finite region. In the $Q_0 \to i\infty$ limit we may expand as follows

$$A_i^{\alpha\beta} \to \int d^4u \;\; ds \;\; \rho_i^{\alpha\beta} \left[\frac{1}{Q^2 - s} + \frac{2Q \cdot u}{(Q^2 - s)^2} + \cdots\cdots \right] \tag{2.9}$$

If the asymptotic behavior in s is reasonable then this leads to the answer below. With all the dangers inherent in using any representations for determining asymptotic behavior the above is at best a plausibility argument. Thus the limiting relations are

$$A_i^{\alpha\beta}(\nu, t, Q^2, \delta) \to A_i^{\alpha\beta}(0, t, Q^2 = -\infty, 0) + \nu \frac{\partial A_i^{\alpha\beta}(0, t, Q^2 = -\infty, 0)}{\partial \nu} + \delta \frac{\partial A_i^{\alpha\beta}(0, t, Q^2, 0)}{\partial \delta} \tag{2.10}$$

for i $\neq$ 6 and for i = 6 we likewise add the next order terms

$$A_6^{\alpha\beta}(\nu, t, Q^2\delta) \rightarrow A_6^{\alpha\beta}(0) + \nu \frac{\partial A_6^{\alpha\beta}}{\partial\nu} + \delta \frac{\partial A_6^{\alpha\beta}}{\partial\delta} + \frac{\nu^2}{2} \frac{\partial^2 A_6^{\alpha\beta}}{\partial\nu^2}$$

$$+ \frac{\delta^2}{2} \frac{\partial^2 A_6^{\alpha\beta}}{\partial\delta^2} + \nu\delta \frac{\partial^2 A_6^{\alpha\beta}}{\partial\nu\,\partial\delta} \qquad (2.11)$$

III. Derivation of Sum Rules

The idea of the derivation is the same as in Ref. 4. We write, for fixed $\underset{\sim}{Q}$ a dispersion relation in Q_0. This is just the Low equation

$$M_{\mu\nu}^{\alpha\beta} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dQ_0' \, \mathrm{Im}\, M_{\mu\nu}^{\alpha\beta}}{Q_0' - Q_0} \text{ (+ Polynomial)} \qquad (3.1)$$

As $Q_0 \rightarrow i\infty$

$$M_{\mu\nu}^{\alpha\beta} \rightarrow - \frac{1}{\pi Q_0} \int \mathrm{Im}\, M_{\mu\nu}^{\alpha\beta} \; dQ_0' \text{ (+ Polynomial)} \qquad (3.2)$$

The numerator is, however, the equal-time commutator of the currents, so that

$$M_{\mu\nu}^{\alpha\beta} \rightarrow \frac{2(2\pi)^3 \sqrt{4\omega_2\omega_1} \int \langle p_2 \left| \left[j_\mu^\alpha(\underset{\sim}{x}, 0), \; j_\nu^\beta(0) \right] \right| p_1 \rangle \, d^3x \, e^{-i\underset{\sim}{q}_1 \cdot \underset{\sim}{x}}}{Q_0} \qquad (3.3)$$

We assume the following commutation relations for the currents; they contradict no commonly used models (σ-model,[10] quark model,[11] and field algebra[12])

$$\left[j_0^\alpha(\underset{\sim}{x}, 0), \; j_0^\beta(0) \right] = i f^{\alpha\beta\gamma} \; j_0^\gamma(0) \, \delta^3(\underset{\sim}{x})$$

$$\left[j_0^\alpha(\underset{\sim}{x}, 0), \; j_k^\beta(0) \right] = -i S^{\alpha\beta}(0) \; \nabla_k \, \delta^3(\underset{\sim}{x}) + i f^{\alpha\beta\gamma} \; j_k^\gamma(0) \, \delta^3(\underset{\sim}{x}) \qquad (3.4)$$

$$\left[j_i^\alpha(\underset{\sim}{x}, 0), \; j_j^\beta(0) \right] = ic \, \delta_{ij} \, f^{\alpha\beta\gamma} \, j_0^\gamma(0) \, \delta^3(\underset{\sim}{x}) + \text{odd parity term}$$

The matrix elements of j_μ^γ and $S^{\alpha\beta}$ are given by

$$(2\pi)^3 \sqrt{4\omega_2\omega_1}\ \langle p_2 | j_\mu^\gamma(0) | p_1 \rangle = P_\mu F_1^\gamma(t) + \Delta_\mu F_2^\gamma(t)$$

$$(2\pi)^3 \sqrt{4\omega_2\omega_1}\ \langle p_2 | S^{\alpha\beta}(0) | p_1 \rangle = s^{\alpha\beta}(t) \tag{3.5}$$

The number c depends on the model for the commutation relation. If the fundamental carriers of the currents are spin one-half particles then $c = 1$. If these are spin zero objects or we deal with field algebra[12], then $c = 0$. The Schwinger term $S^{\alpha\beta}$ is likewise model dependent and may be a c number, as in field algebra. We do assume that it is symmetric in α, β.

To obtain the sum rules we compare the covariant expansion (2. 4) with that given in (3.3), (3.4) and (3.5). We start with the 0-0 components and find

$$P_0^2 A_1^{\alpha\beta}(Q^2, t, \nu, \delta) + 2P_0Q_0 A_2^{\alpha\beta} + 2P_0\Delta_0 A_4^{\alpha\beta} + 2Q_0\Delta_0 A_6^{\alpha\beta} + Q_0^2 A_8^{\alpha\beta}$$
$$+ \Delta_0^2 A_9^{\alpha\beta} + A_{10}^{\alpha\beta} \xrightarrow[Q_0 \to i\infty]{} \frac{2if^{\alpha\beta\gamma}}{Q_0}\left[F_1^\gamma(t)\, P_0 + F_2^\gamma(t)\, \Delta_0\right] \tag{3.6}$$

We have assumed that M_{00} vanishes as $Q_0 \to i\infty$ because for $\mathbf{q}_1$ or $\mathbf{q}_2 \to 0$, M_{00} involves a retarded product of a total charge operator. This operator is conserved or partially conserved; it does not couple to high-mass states. Another way of saying this is that we assume that $Q_\mu M_{\mu\nu}$ is bounded by a constant as $Q_0 \to i\infty$, as implied by the commonly accepted divergence conditions of vector and axial vector operators.[7, 13] We shall make the same assumption on M_{0i}, namely, it tends to zero as $Q_0 \to i\infty$. As discussed in Section II, the $A_i^{\alpha\beta}$ are assumed to converge rapidly as power-series expansions in ν and δ in the infinite-Q_0 limit, so that (except for $A_6^{\alpha\beta}$) we need keep only up to linear terms. Writing

$$A_i^{\alpha\beta} \to A_i^{\alpha\beta}(Q^2, t, 0, 0) + \nu \frac{\partial A_i^{\alpha\beta}}{\partial \nu}(Q^2, t, 0, 0) + \delta \frac{\partial A_i^{\alpha\beta}}{\partial \delta}(Q^2, t, 0, 0) \qquad i \neq 6 \tag{3.7}$$

(and a similar expansion for $A_6^{\alpha\beta}$) we then replace ν and δ by

$$\begin{aligned} \nu &\rightarrow Q_0 P_0 \\ \delta &\rightarrow Q_0 \Delta_0 \end{aligned} \tag{3.8}$$

and identify the coefficients of the various powers of P_0 and Δ_0 on the right and left hand sides of (3.6). From the coefficients of P_0, for example, we find

$$\lim_{Q_0 \rightarrow i\infty} \quad 2Q^2 A_2^{\alpha\beta}(Q^2,t,0,0) + Q^4 \frac{\partial A_8^{\alpha\beta}}{\partial \nu}(Q^2,t,0,0) + Q^2 \frac{\partial A_{10}^{\alpha\beta}}{\partial \nu}(Q^2,t,0,0)$$

$$= 2\,if^{\alpha\beta\gamma} \; F_1^{\gamma}(t) \tag{3.9}$$

A large number of additional relations are obtained in the same way and are catalogued in the Appendix.

The same method can then be applied to M_{0i}, whereupon another large number of asymptotic relations are generated. These are also catalogued in the Appendix. At this point, the equations can, for the most part, be broken down into conditions on the individual $A_i(Q^2,t,0,0)$ and their derivatives. One interesting result is that $A_{10}(Q^2,t,0,0) \rightarrow s(t)$ as $Q_0 \rightarrow i\infty$, with the consequence that M_{ij} cannot vanish as $1/Q_0$ if there exist operator Schwinger terms $S(t)$. We assume in this case that $M_{ij} \rightarrow$ (const) $\delta_{ij} + 0(1/Q_0)$, a behavior consistent with lowest-order perturbation theory of spin zero bosons, although perhaps not in higher orders, where space-space equal-time commutators have a propensity for not existing.[14]

However, upon assuming the space-space equal-time commutators exist, the same procedure we have applied to M_{00} and M_{0i} when also applied to M_{ij} completely fixes the asymptotic behaviors of the $A_i^{\alpha\beta}(Q^2,t,0,0)$ and their derivatives. We record the results here, and some intermediate steps in the Appendix.

Nonvanishing Limits

$$Q^2 A_2(Q^2,t,0,0) \longrightarrow 2if^{\alpha\beta\gamma} F_1^\gamma(t) \qquad Q^4 \frac{\partial A_8}{\partial\nu} \longrightarrow (c-1)\ 2if^{\alpha\beta\gamma}\ F_1^\gamma(t)$$

$$Q^2 A_6(Q^2,t,0,0) \longrightarrow 2if^{\alpha\beta\gamma} F_2(t) \qquad Q^4 \frac{\partial A_8}{\partial\delta} \longrightarrow (c-1)\ 2if^{\alpha\beta\gamma}\ F_2^\gamma(t)$$

$$Q^2 A_7(Q^2,t,0,0) \longrightarrow s(t) \qquad Q^2 \frac{\partial A_{10}}{\partial\nu} \longrightarrow -\ 2cif^{\alpha\beta\gamma}\ F_1(t)$$

$$Q^2 A_8(Q^2,t,0,0) \longrightarrow s(t) \qquad Q^2 \frac{\partial A_{10}}{\partial\delta} \longrightarrow -\ 2cif^{\alpha\beta\gamma}\ F_2^\gamma(t)$$

$$A_{10}(Q^2,t,0,0) \longrightarrow -s(t) \tag{3.10a}$$

Vanishing Limits

$$Q_0 A_i(Q^2,t,0,0) \longrightarrow 0 \quad i = 1,4,5,9 \qquad Q^2 \frac{\partial A_i}{\partial\delta} \longrightarrow 0 \quad i = 1,4,5,9$$

$$Q^2 A_3(Q^2,t,0,0) \longrightarrow 0 \qquad Q_0^3 \frac{\partial A_i}{\partial\delta} \longrightarrow 0 \quad i = 2,3,6,7$$

$$Q^2 \frac{\partial A_i}{\partial\nu} \longrightarrow 0 \quad i = 1,4,5,9 \qquad Q^4 \frac{\partial^2 A_6}{\partial\nu^2} \longrightarrow 0 \qquad Q^4 \frac{\partial^2 A_6}{\partial\delta^2} \longrightarrow 0$$

$$Q_0^3 \frac{\partial A_i}{\partial\nu} \longrightarrow 0 \quad i = 2,3,6,7 \qquad Q^4 \frac{\partial^2 A_6}{\partial\nu\partial\delta} \longrightarrow 0 \tag{3.10b}$$

Given the assumptions of Regge asymptotics,[8] some of these relations can be translated into asymptotic sum rules. These we also record.

$$Q_0 \int_{-\infty}^{\infty} \frac{d\nu'}{\nu'}\ \text{Im}\ A_1^{\alpha\beta}(\nu',Q^2,t,0) \longrightarrow 0 \qquad Q_0 \int_{-\infty}^{\infty} \frac{d\nu'}{\nu'}\ \text{Im}\ A_5^{\alpha\beta} \longrightarrow 0$$

$$Q^2 \int_{-\infty}^{\infty} \frac{d\nu'}{\nu'^2}\ \text{Im}\ A_1^{\alpha\beta} \longrightarrow 0 \qquad Q^2 \int_{-\infty}^{\infty} \frac{d\nu'}{\nu'^2}\ \text{Im}\ A_5^{\alpha\beta} \longrightarrow 0$$

$$\frac{Q^2}{\pi} \int_{-\infty}^{\infty} \frac{d\nu'}{\nu'}\ \text{Im}\ A_2^{\alpha\beta}(\nu',Q^2,t,0) \longrightarrow 2if^{\alpha\beta\gamma} F_1^\gamma(t) \qquad Q^4 \int_{-\infty}^{\infty} \frac{d\nu'}{\nu'^3}\ \text{Im}\ A_6^{\alpha\beta} \longrightarrow 0$$

$$Q_0^3 \int_{-\infty}^{\infty} \frac{d\nu'}{\nu'^2}\ \text{Im}\ A_2^{\alpha\beta} \longrightarrow 0 \qquad Q_0^3 \int_{-\infty}^{\infty} \frac{d\nu'}{\nu'^2}\ \text{Im}\ A_7^{\alpha\beta} \longrightarrow 0$$

$$Q_0^3\int_{-\infty}^{\infty}\frac{d\nu'}{\nu'^2}\,\mathrm{Im}\,A_3^{\alpha\beta}\longrightarrow 0 \qquad \frac{Q^4}{\pi}\int_{-\infty}^{\infty}\frac{d\nu'}{\nu'^2}\,\mathrm{Im}\,A_8^{\alpha\beta}(\nu',Q^2,t,0)\longrightarrow (c-1)\,2if^{\alpha\beta\gamma}F_1^{\gamma}(t)$$

$$Q^2\int_{-\infty}^{\infty}\frac{d\nu'}{\nu'^2}\,\mathrm{Im}\,A_4^{\alpha\beta}\longrightarrow 0 \qquad Q^2\int_{-\infty}^{\infty}\frac{d\nu'}{\nu'^2}\,\mathrm{Im}\,A_9^{\alpha\beta}\longrightarrow 0$$

$$\frac{Q^2}{\pi}\int_{-\infty}^{\infty}\frac{d\nu'}{\nu'^2}\,\mathrm{Im}\,A_{10}^{\alpha\beta}(\nu',Q^2,t,0)\longrightarrow -2cif^{\alpha\beta\gamma}F_1^{\gamma}(t) \tag{3.11}$$

The asymptotic form of the Fubini-Gell-Mann-Dashen[15] sum rule

$$\frac{1}{\pi}\int_{-\infty}^{\infty}d\nu'\,\mathrm{Im}\,A_1^{\alpha\beta}\;(Q^2,\nu',t,\delta) = 2if^{\alpha\beta\gamma}F_1^{\gamma}(t) \tag{3.12}$$

is not present among this myriad of sum rules. However, if one has a conserved (or partially conserved) current, Im A_2 can, for large Q^2, be related to ν Im A_1, and the sum rule (3.12) is recovered from (3.11). The "backward" sum rules for neutrino processes[4] can be obtained from the sum rule on the transverse amplitude A_{10}. Beyond that result, no practical application seems to be in sight. The generalization of these relations to other parity choices and cases with spin is left to the courageous reader.

IV. Conclusions

The main results of this work are contained in (3.10a), (3.10b) and (3.11). These sum rules are a direct consequence of locality and (perhaps optimistic) smoothness assumptions on the covariant amplitude describing scattering of a current from a hadron. The results appear to have no new direct applications. However, since the mechanism for saturating all local sum rules at high Q^2 is obscure (assuming indeed that the sum rules are correct!), it is hoped that these results might provide additional clues to what the physics looks like in these asymptotic regions.

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<u>Appendix</u>

The relations obtained from the asymptotic behavior of M_{00}, as described in Section III are as follows:

$$(1) \qquad Q_0\, A_{10}(0) + Q_0^3\, A_8(0) \longrightarrow 0$$

$$(P_0) \qquad 2Q^2\, A_2^{\alpha\beta}(0) + Q^4 \frac{\partial A_8}{\partial \nu} + Q^2 \frac{\partial A_{10}}{\partial \nu} \longrightarrow 2if^{\alpha\beta\gamma}\, F_1^{\gamma}(t)$$

$$(P_0^2) \qquad Q_0\, A_1(0) + 2Q_0^3 \frac{\partial A_2}{\partial \nu} \longrightarrow 0$$

$$(P_0^3) \qquad Q^2 \frac{\partial A_1}{\partial \nu} \longrightarrow 0$$

$$(\Delta_0) \qquad 2Q^2\, A_6(0) + Q^4 \frac{\partial A_8}{\partial \delta} + Q^2 \frac{\partial A_{10}}{\partial \delta} \longrightarrow 2if^{\alpha\beta\gamma}\, F_2^{\gamma}(t)$$

$$(\Delta_0^2) \qquad 2Q_0^3 \frac{\partial A_6}{\partial \delta} + Q_0\, A_9(0) \longrightarrow 0$$

$$(\Delta_0^3) \qquad Q^4 \frac{\partial^2 A_6}{\partial \delta^2} + Q^2 \frac{\partial A_9}{\partial \delta} \longrightarrow 0$$

$$(P_0\Delta_0) \qquad 2Q_0^3 \frac{\partial A_2}{\partial \delta} + 2Q_0\, A_4(0) + 2Q_0^3 \frac{\partial A_6}{\partial \nu} \longrightarrow 0$$

$$(P_0\Delta_0^2) \qquad 2Q^2 \frac{\partial A_4}{\partial \delta} + 2Q^4 \frac{\partial^2 A_6}{\partial \nu \partial \delta} + Q^2 \frac{\partial A_9}{\partial \nu} \longrightarrow 0$$

$$(P_0^2\Delta_0) \qquad Q^2 \frac{\partial A_1}{\partial \delta} + 2Q^2 \frac{\partial A_4}{\partial \nu} + Q^4 \frac{\partial^2 A_6}{\partial \nu^2} \longrightarrow 0 \qquad (A.1)$$

The factor in parentheses indicates the coefficient of the particular term . We suppress superscripts $\alpha\beta$.

From a similar calculation for $M_{0i}^{\alpha\beta}$ and $M_{i0}^{\alpha\beta}$ we obtain

$$(P_i) \qquad Q^2(A_2 \mp A_3) \longrightarrow 2if^{\alpha\beta\gamma}\, F_1^{\gamma}(t)$$

(P_iP_0) $\quad Q_0\left(A_1 + Q^2\,\frac{\partial A_2}{\partial\nu} \mp Q^2\,\frac{\partial A_3}{\partial\nu}\right) \longrightarrow 0$

$(P_iP_0^2)$ $\quad Q^2\,\frac{\partial A_1}{\partial\nu} \longrightarrow 0$

$(P_iP_0\Delta_0)$ $\quad Q^2\,\frac{\partial A_1}{\partial\delta} + Q^2\,\frac{\partial A_4}{\partial\nu} \mp Q^2\,\frac{\partial A_5}{\partial\nu} \longrightarrow 0$

$(P_i\Delta_0)$ $\quad Q_0\left[Q^2\,\frac{\partial A_2}{\partial\delta} \mp Q^2\,\frac{\partial A_3}{\partial\delta} + A_4 \mp A_5\right] \longrightarrow 0$

$(P_i\Delta_0^2)$ $\quad Q^2\left(\frac{\partial A_4}{\partial\delta} \mp \frac{\partial A_5}{\partial\delta}\right) \longrightarrow 0$

(Q_i) $\quad Q^2\,A_8 \longrightarrow s(t)$

(P_0Q_i) $\quad Q_0\left(A_2 \pm A_3 + Q^2\,\frac{\partial A_8}{\partial\nu}\right) \longrightarrow 0$

$(P_0^2Q_i)$ $\quad Q^2\left(\frac{\partial A_2}{\partial\nu} \pm \frac{\partial A_3}{\partial\nu}\right) \longrightarrow 0$

$(P_0\Delta_0Q_i)$ $\quad Q^2\left(\frac{\partial A_2}{\partial\delta} \pm \frac{\partial A_3}{\partial\delta} + \frac{\partial A_6}{\partial\nu} \mp \frac{\partial A_7}{\partial\nu}\right) \longrightarrow 0$

(Δ_0Q_i) $\quad Q_0\left(A_6 \mp A_7 + Q^2\,\frac{\partial A_8}{\partial\delta}\right) \longrightarrow 0$

$(\Delta_0^2Q_i)$ $\quad Q^2\left(\frac{\partial A_6}{\partial\delta} \mp \frac{\partial A_7}{\partial\delta}\right) \longrightarrow 0$

$(P_0^2\Delta_0Q_i)$ $\quad Q_0^3\,\frac{\partial^2 A_6}{\partial\nu^2} \longrightarrow 0$

$(\Delta_0^3Q_i)$ $\quad Q_0^3\,\frac{\partial^2 A_6}{\partial\delta^2} \longrightarrow 0$

$(P_0\Delta_0^2Q_i)$ $\quad Q_0^3\,\frac{\partial^2 A_6}{\partial\nu\partial\delta} \longrightarrow 0$

(Δ_i) $\quad Q^2(A_6 \pm A_7) \longrightarrow 2if^{\alpha\beta\gamma}\,F_2^{\gamma}(t) \pm s(t)$

$$(P_0\Delta_i) \qquad Q_0\left(A_4 \pm A_5 + Q^2 \frac{\partial A_6}{\partial \nu} \pm Q^2 \frac{\partial A_7}{\partial \nu}\right) \longrightarrow 0$$

$$(P_0^2\Delta_i) \qquad Q^2\left(\frac{\partial A_4}{\partial \nu} \pm \frac{\partial A_5}{\partial \nu} + \frac{Q^2}{2} \frac{\partial^2 A_6}{\partial \nu^2}\right) \longrightarrow 0$$

$$(P_0\Delta_0\Delta_i) \quad Q^2\left(\frac{\partial A_4}{\partial \delta} \pm \frac{\partial A_5}{\partial \delta} + Q^2 \frac{\partial^2 A_6}{\partial \nu \partial \delta} + \frac{\partial A_9}{\partial \nu}\right) \longrightarrow 0$$

$$(\Delta_0\Delta_i) \qquad Q_0\left(A_9 + Q^2 \frac{\partial A_6}{\partial \delta} \pm Q^2 \frac{\partial A_7}{\partial \delta}\right) \longrightarrow 0$$

$$(\Delta_0^2\Delta_i) \qquad Q^2\left(\frac{Q^2}{2} \frac{\partial^2 A_6}{\partial \delta^2} + \frac{\partial A_9}{\partial \delta}\right) \longrightarrow 0 \tag{A.2}$$

The $\pm$ refer to $M_{0i}^{\alpha\beta}$ and $M_{i0}^{\alpha\beta}$ respectively.

Given these equations, they can be disentangled into the following relations.

$$Q_0 A_1 \longrightarrow 0 \qquad Q^2 \frac{\partial A_1}{\partial \nu} \longrightarrow 0 \qquad Q^2 \frac{\partial A_1}{\partial \delta} \longrightarrow 0$$

$$Q^2 A_2 \longrightarrow 2if^{\alpha\beta\gamma} F_1^{\gamma}(t) \qquad Q_0^3 \frac{\partial A_2}{\partial \nu} \longrightarrow 0 \qquad Q_0^3 \frac{\partial A_2}{\partial \delta} \longrightarrow 0$$

$$Q^2 A_3 \longrightarrow 0 \qquad Q_0^3 \frac{\partial A_3}{\partial \nu} \longrightarrow 0$$

$$Q_0 A_4 \longrightarrow 0 \qquad Q^2 \frac{\partial A_4}{\partial \nu} \longrightarrow 0 \qquad Q^2 \frac{\partial A_4}{\partial \delta} \longrightarrow 0$$

$$Q_0 A_5 + Q_0^3 \frac{\partial A_3}{\partial \delta} \longrightarrow 0 \qquad Q^2 \frac{\partial A_5}{\partial \nu} \longrightarrow 0 \qquad Q^2 \frac{\partial A_5}{\partial \delta} \longrightarrow 0$$

$$Q^2 A_6 \longrightarrow 2if^{\alpha\beta\gamma} F_2(t) \qquad Q_0^3 \frac{\partial A_6}{\partial \nu} \longrightarrow 0 \qquad Q_0^3 \frac{\partial A_6}{\partial \delta} \longrightarrow 0$$

$$Q^4 \frac{\partial^2 A_6}{\partial \nu^2} \longrightarrow 0 \qquad Q^4 \frac{\partial^2 A_6}{\partial \nu \partial \delta} \longrightarrow 0 \qquad Q^4 \frac{\partial^2 A_6}{\partial \delta^2} \longrightarrow 0$$

$$Q^2 A_7 \longrightarrow s(t) \qquad Q_0^3 \frac{\partial A_7}{\partial \nu} \longrightarrow Q_0^3 \frac{\partial A_3}{\partial \delta} \qquad Q_0^3 \frac{\partial A_7}{\partial \delta} \longrightarrow 0$$

$$Q^2 A_8 \longrightarrow s(t) \qquad Q_0^3 \frac{\partial A_8}{\partial \nu} \longrightarrow 0 \qquad Q_0^3 \frac{\partial A_8}{\partial \delta} \longrightarrow 0$$

$$Q_0 A_9 \longrightarrow 0 \qquad Q^2 \frac{\partial A_9}{\partial \nu} \longrightarrow 0 \qquad Q^2 \frac{\partial A_9}{\partial \delta} \longrightarrow 0$$

$$A_{10} \longrightarrow - s(t) \qquad Q^2 \frac{\partial A_{10}}{\partial \nu} + Q^4 \frac{\partial A_8}{\partial \nu} \longrightarrow - 2if^{\alpha\beta\gamma} F_1^{\gamma}(t)$$

$$Q^2 \frac{\partial A_{10}}{\partial \delta} + Q^4 \frac{\partial A_8}{\partial \delta} \longrightarrow - 2if^{\alpha\beta\gamma} F_2^{\gamma}(t) \qquad (A.3)$$

Finally, calculation of the limit for M_{ij} gives a few additional results:

$$(\Delta_0 P_i Q_j) \qquad Q^2 \frac{\partial A_3}{\partial \delta} \longrightarrow 0$$

$$(P_i \Delta_j) \qquad Q_0 A_5(0) \longrightarrow 0$$

$$(Q_i \Delta_j P_0) \qquad Q^2 \frac{\partial A_7}{\partial \nu} \longrightarrow 0$$

$$(\delta_{ij}) \qquad Q_0 \left[A_{10}(0) + s(t)\right] \longrightarrow 0$$

$$(\delta_{ij} P_0) \qquad Q^2 \frac{\partial A_{10}}{\partial \nu} \longrightarrow - 2cif^{\alpha\beta\gamma} F_1^{\gamma}(t)$$

$$(\delta_{ij} \Delta_0) \qquad Q^2 \frac{\partial A_{10}}{\partial \delta} \longrightarrow - 2cif^{\alpha\beta\gamma} F_2^{\gamma}(t) \qquad (A.4)$$

These, combined with (A.3), give equations (3.10a) and (3.10b).

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