Pub 365

# SOFT PION THEOREMS AND THE $K_{\ell_3}$, $K_{\ell_4}$ FORM FACTORS*

S. M. Berman

and

Probir Roy

Stanford Linear Accelerator Center
Stanford University, Stanford, California

*Work supported by the U. S. Atomic Energy Commission

Using current algebras and soft pion techniques, several attempts have been made during the past few years to relate the $K_{\ell_3}$ decay form factors to the $K_{\ell_2}$ decay[1] as well as the $K_{\ell_4}$ form factors to $K_{\ell_3}$ and $K_{\ell_2}$ decays.[2] It has been customary in these calculations to assume that, except for possible explicit pole terms, both the $K_{\ell_3}$ and $K_{\ell_4}$ form factors have small percentage variations from their values in the physical region to their respective values at the extrapolated points where the soft pion theorems apply (smoothness property). In addition, the very strong assumption is made[2] in $K_{\ell_4}$ decays that the so-called $\sigma$-commutators $\sigma(x)\delta^4(x-y) = [A_0^+(x), \partial_\mu A_\mu^-(y)]\,\delta(x_0 - y_0)$ are either zero or carry no isospin zero component so that they do not affect the $K_{\ell_4}$ decay. Since the vanishing of the $\sigma$-commutator implies the conservation of the pionic type axial currents,[3] which is not assumed, and since models which yield non-zero $\sigma$-terms have them with zero isospin, the conclusions based on the $\sigma$-term assumption must be modified.

In this note we show that to proceed to the soft pion limit simultaneously in both pions for the $K_{\ell_4}$ decay is inconsistent with the nonvanishing of the contribution of the $\sigma$-commutators. Our results do not affect the Weinberg[2] calculation of $F_1$ and $F_2$ which are the pertinent quantities for the $K_{\ell_4}$ decay rate and which is in reasonably good agreement with experiment. However, the equations given by Weinberg[2] from which $F_3$ is deduced are not valid when the contribution from the $\sigma$-commutator is nonvanishing. We give alternative conditions for $F_3$ and show that the parameter $\xi = F_-/F_+$ of $K_{\ell_3}$ decay takes the value $\xi = -1$ and is approximately constant throughout the physical region of $K_{\ell_3}$ decay.

Following the notation of Weinberg[2] we write the vector and axial current matrix elements arising in $K^+ \to \pi^0 e^+ (\mu^+)\nu$ and $K^+ \to \pi^+\pi^- e^+ \nu$ respectively as

$$\frac{1}{\sqrt{2}}\left[F_+(K+p)_\mu + F_-(K-p)_\mu\right] = \langle \pi^0 | V_\mu^{K^-} | K^+ \rangle (2\pi)^3 \sqrt{4p_0K_0} \;,$$

and

$$i\, m_K^{-1}\left[F_1(p+q)_\mu + F_2(p-q)_\mu + F_3(K-p-q)_\mu\right] = \langle \pi^+\pi^- \left| A_\mu^{K^-} \right| K^+ \rangle (2\pi)^{9/2} \sqrt{8p_0 q_0 K_0} \, ,$$

where in $K_{\ell_3}$, K and p are the kaon and pion momentum and in $K_{\ell_4}$, K,p,q are the kaon, positive pion, and negative pion momenta, respectively. For $K_{\ell_3}$ decay the form factors are functions of the three independent invariants $K^2$, $p^2$, $K\cdot p$, $F_\pm(K^2, p^2, p\cdot K)$ and for $K_{\ell_4}$ decay $F_{1,2,3} = F_{1,2,3}(K^2, p^2, q^2, K\cdot q, p\cdot q)$. The smoothness property in pion variables would require these form factors to be essentially constant (unless an explicit pole term is present) in the variables $p^2$, $q^2$, $K\cdot p$, $K\cdot q$, $p\cdot q$ when extrapolating from the physical region to a region where either p or q = 0 or where they are small enough to keep only first order terms in p and q.

Callan and Treiman[1] considered the problem of $K_{\ell_4}$ decay in the limit where first $p \to 0$ and $q^2 = m_\pi^2$ and secondly where $q \to 0$, $p^2 = m_\pi^2$. Applying these two limits and using the usual chiral $SU_3 \otimes SU_3$ algebra of currents they showed that the following two sets of equations must be satisfied:

$$F_1\left(m_K^2, m_\pi^2, 0, K\cdot p, 0, 0\right) + F_2\left(m_K^2, m_\pi^2, 0, K\cdot p, 0, 0\right) = 2C\, F_+\left(m_K^2, m_\pi^2, K\cdot p\right) \quad \text{(1a)}$$

$$F_3\left(m_K^2, m_\pi^2, 0, K\cdot p, 0, 0\right) = C \left\{ F_+\left(m_K^2, m_\pi^2, K\cdot p\right) + F_-\left(m_K^2, m_\pi^2, K\cdot p\right)\right\} \quad \text{(1b)}$$

and

$$F_1\left(m_K^2, 0, m_\pi^2, 0, K\cdot q, 0\right) = F_2\left(m_K^2, 0, m_\pi^2, 0, K\cdot q, 0\right) \quad \text{(2a)}$$

$$F_3\left(m_K^2, 0, m_\pi^2, 0, K\cdot q, 0\right) = 0 \quad \text{(2b)}$$

where $C = m_K/F_\pi$ and $F_\pi$ is the usual coupling constant of $\pi$-decay. Weinberg[2] has shown that if the $\sigma$-commutators vanish or do not contribute to $K_{\ell_4}$ decay then we have in addition to the above equations in the limit of both p and $q \rightarrow 0$ and according to the order of limit

$$\lim_{p \rightarrow 0} \lim_{q \rightarrow 0} F_3\left(m_K^2, p^2, q^2, K\cdot p, K\cdot q, p\cdot q\right) = -m_K F_K / F_\pi^2 \tag{3a}$$

and

$$\lim_{q \rightarrow 0} \lim_{p \rightarrow 0} F_3\left(m_K^2, p^2, q^2, K\cdot p, K\cdot q, p\cdot q\right) = 0 \tag{3b}$$

The different values in the two limits supposedly come about because of the one K-meson intermediate state which contributes to a pole in $F_3$ in the limit when both p and q vanish, thus providing for a rapidly varying function at this double limit. The actual value of $F_3$ in the two limits depends on the residue at the pole which Weinberg takes as the current algebra result neglecting the possible contribution of the $\sigma$-commutator. This residue depends on the pion momenta in such a manner that $F_3$ does not blow up at the double limit and reaches the values given in (3a) and (3b).

However, if the contribution of the $\sigma$ commutator is nonvanishing the limits indicated in (3a) and (3b) are not permitted since the residue at the pole is now nonzero. Thus Eqs. (3a) and (3b) do not follow and in fact $F_3$ will be infinite when q and p both approach zero. Therefore, in the presence of nonvanishing $\sigma$ commutators, Eqs. (1b) and (2b) cannot be continued in their respective arguments to the q and $p \rightarrow 0$ limit.

We now attempt to satisfy Eqs. (1a), (1b), (2a), and (2b) which are independent of any statements about $\sigma$ commutators by simple assumptions, i.e., $F_1$ and $F_2$ are approximately constant and $F_3$ is given by the sum of a pole term plus a constant: $F_3 = T_{K\pi} / \left[m_K^2 - (K-p-q)^2\right] + b$, where $T_{K\pi}$ is proportional to the $K\pi$ elastic scattering amplitude. If $T_{K\pi}$ is evaluated right at the K-meson pole, then it follows from the Adler consistency condition[4] that $T_{K\pi}$ vanishes for either of the limits leading to (1a), (1b), (2a), and (2b), i.e.,

$$\lim_{\substack{p \to 0 \\ q^2 = m_\pi^2, K^2 = m_K^2, (K-q)^2 = m_K^2}} T_{K\pi} = 0 = \lim_{\substack{q \to 0 \\ p^2 = m_\pi^2, K^2 = m_K^2, (K-p)^2 = m_K^2}} T_{K\pi}$$

and thus from (2b) the constant b is also zero. In the actual physical situation in $K_{\ell_3}$ decay $(K-p)^2$ is not $m_K^2$ but varies from $0 \le (K-p)^2 \le (m_K - m_\pi)^2$ and what is required is $T_{K\pi}$ with $q \to 0$ and one of the K-mesons not on its mass shell. We show below that the deviation from the zero in $T_{K\pi}$ required by Adler consistency condition can be estimated on the basis of certain reasonable assumptions and indeed turns out to be quite small. Thus we expect $T_{K\pi}$ to be approximately zero throughout the range of $t = (K-p)^2$ of $K_{\ell_3}$ decay. With the conclusion that $T_{K\pi}$ is approximately zero it follows that the consistent solution of Eqs. (1a) - (1b) which is compatible with $F_1$ and $F_2$ as being approximately constant in its pionic variables is that $F_3 = 0$ at the two limiting points. Moreover, as a consequence of (1b), $\xi$ has the value throughout the physical region of

$$\xi\left(m_K^2, m_\pi^2, K\cdot p\right) = \frac{F_-\left(m_K^2, m_\pi^2, p\cdot K\right)}{F_+\left(m_K^2, m_\pi^2, p\cdot K\right)} = -1 \tag{4}$$

and $F_+, F_-$ are each approximately constant in the physical region.[5]

We note that when both p and $q \to 0$, $T_{K\pi}$ may be non-zero and in fact in that limit with both K-mesons on the mass shell is just the matrix element of the $\sigma$-commutator between K states.[6]

Concerning the deviations of $T_{K\pi}$ from the zero required by the Adler consistency condition we argue as follows: With the assumption as above that $F_3$ is constructed out of a pole term plus a constant we have that as the limit $p \to 0$, $q^2 = m_\pi^2$ ,

$$\lim_{\substack{p \to 0 \\ q^2 = m_\pi^2}} F_3 = b - i m_K F_K \int d^4x \langle \pi^-(q) \left| \left[ A_0^{\pi^-}(x), \phi^{K^-}(0) \right] \delta(x_0) \right| K^+ \rangle \frac{1}{F_\pi}$$

where $\phi^K$ is the interpolating field for the off mass shell K-meson. If we assume that the commutators $\left[ A_0^{\pi^\alpha} , \phi^{K^\beta} \right]$ have no $I = 3/2$ component in an analogous manner to $\sigma$-like commutators having no $I = 2$ components then the requirement that $F_3$ vanish at the limit $p \to 0$, $q^2 = m_\pi^2$ makes the constant in the above expression for $F_3$ vanish. This assumption that the $I = 3/2$ commutator vanish is suggested on physical grounds by the absence experimentally of $I = 3/2$ boson resonances and therefore we take it as a reasonable assumption. In that case we have at the other limit $q \to 0$, $p^2 = m_\pi^2$ only a pole contribution for $F_3$ when given by

$$\lim_{\substack{q \to 0 \\ p^2 = m_\pi^2}} F_3 = - i m_K F_K \int d^4x \langle \pi^+(p) \left| \left[ A_0^{\pi^+}(x), \phi^{K^-}(0) \right] \delta(x_0) \right| K^+ \rangle \frac{1}{F_\pi}$$

We estimate the size of this term by inserting a complete set of states between $A_0^{\pi^+}$ and $\phi^{K^-}$ and considering the contributions from the lowest lying states. Since $q \to 0$ we note that only S-wave intermediate states can contribute to the sum.

The direct term in the equal time commutator has contributions from $\sigma$-like states ($J^P = 0^+$ and $I = 0,2$). Since this kind of term vanishes when both the K-mesons are on the mass shell[7] and since this type of intermediate state carries no spin information so that there is factorization between the couplings of the $\sigma$-like states to the two pions and the two kaons, it should vanish independent of whether one K-meson is either on or off its mass shell. Thus the only nonvanishing contribution comes from the crossed term in the commutator whose major contribution we assume to come from the S-wave $K, \pi$ intermediate state. To compute this contribution we require knowing the S-wave $K, \pi$ elastic amplitude for all energies which is generally unknown. As an estimate we take the threshold

S-wave values for the Feynman amplitude given by current algebra and pionic PCAC as $2m_\pi m_K/F_\pi^2$ pass to the infinite momentum frame and cut off the integral at some mass $\Lambda$. Since this contribution is only logarithmically dependent on the cutoff $\Lambda$, we estimate its contribution by taking the cutoff around the $K^*(888)$ mass. In that case

$$\left| \lim_{\substack{q \to 0 \\ p^2 = m_\pi^2}} F_3 \right| = \frac{m_K^3 F_K m_\pi^2}{2F_\pi^4 \pi^2 (m_K^2 - t)} \int_{1+\alpha}^{\lambda} \frac{dx}{x(x^2-1)} \left(x^4 + 1 + \alpha^4 - 2x^2 - 2\alpha^2 - 2\alpha^2 x^2\right)^{1/2}$$

where $0 \le t = (K-p)^2 \le (m_K - m_\pi)^2$, $\alpha = m_\pi/m_K$ and $\lambda = \Lambda/m_K$

Evaluating the integral we have that

$$\left| \lim_{\substack{q \to 0 \\ p^2 = m_\pi^2}} F_3 \right| \le 0.15$$

where the upper limit is achieved at the largest value of $t$. From Eq. (1b) this last value of $F_3$ leads to a deviation from $\xi = -1$ of amount $\le$ 5%.[8]

Since the theoretical prediction of the $K_{\ell 4}$ rate with $F_1 = F_2 = C\,F_+$ yields a result within 35% or 60%[9] of the measured value, we expect that the accuracy of the statements $\xi = -1$ and $F_+, F_-$ constant should be better than 30%. As for the comparison of $\xi$ with experiment, there are widely scattered values reported varying over both positive and negative values.[10] An experimental clarification of this point would be especially valuable in the light of the results presented here.[11]

## REFERENCES


1. C. G. Callan and S. B. Treiman, Phys. Rev. Letters 16, 193 (1966); V. S. Mathur, S. Okubo and L. K. Pandit, Phys. Rev. Letters 16, 371 (1966); B. d'Espagnat and M. F. Gaillard, Phys. Letters 25B, 346 (1967); J. Iliopoulos and R. Van Royen, CERN Preprint TH800, 1967 (unpublished).

2. S. Weinberg, Phys. Rev. Letters 17, 336 (1966); Phys. Rev. Letters 18, 1178 (1967).

3. W. A. Bardeen, et al., Phys. Rev. Letters 18, 1170 (1967), see footnote 3 of their paper.

4. S. L. Adler, Phys. Rev. 137, B1022 (1965); 139, B1638 (1965). On $K\pi$ scattering this follows immediately from pionic PCAC so long as there is no scalar strange $I = 1/2$ meson degenerate in mass with the kaon.

5. The fact that the simultaneous limit $p, q \longrightarrow 0$ is not permitted here prevents us from deriving the equation relating the $K_{\ell_2}$ decay constant to the non-physical values of the $K_{\ell_3}$ form factors $F_K/F_\pi = 2\left|\left[F_+\left(m_K^2, 0, 0\right) + F_-\left(m_K^2, 0, 0\right)\right]\right|$. This relation nevertheless remains valid and follows directly without consideration of the $K_{\ell_4}$ decay (see Ref. 1). A simple expression for $F_+$ and $F_-$ can be given which satisfies the conditions that they are constant in the physical region, yield the previous equation at the non-physical point and is consistent with Eq. (1a) and with $F_1$ and $F_2$ as constant if we expand to second order in pion variables. In that case $F_+ = a_0$, and $F_- = -a_0 + \left[\left(p^2 - m_\pi^2\right)/m_\pi^2\right] F_K/F_\pi$ where $a_0$ is a constant proportional to $F_1$.

6. S. Weinberg, Phys. Rev. Letters 17, 616 (1966); Y. Tomozawa, Nuovo Cimento 46A, 707 (1966); A. P. Balachandran, G. M. Gundzik and F. Nicodemi, Nuovo Cimento 44a, 1257 (1966).

7. R. W. Brige, et al., Phys. Rev. 139B, 1600 (1965). The theoretical prediction of $F_1$ using the Goldberger-Treiman relation for $F_\pi$ is numerically given by Weinberg at $0.97 \pm 0.06$ while the experimental value is $1.26 \pm 0.26$. The agreement is made slightly poorer if we take $F_\pi$ directly from the pion decay rate rather than using the Goldberger-Treiman relation in which case $F_1 = 0.85 \pm 0.05$.

8. References to the experimental values of $\xi$ are given in the report of N. Cabibbo — see Proceedings of the XIII-th International Conference on High Energy Physics, University of California Press, 1967, p. 44. Values of $\xi \approx -1$ as well as positive $\xi$ values were reported at the recent Princeton Conference on K-Mesons, November 3-4, 1967. Proceedings are unpublished and unavailable.

9. H. D. I. Abarbanal and S. Nussinov (Ann. Phys. 42(3), 467, 1967) show that for elastic scattering of pions on a target, with the incoming and outgoing target particles on the mass-shell, the coefficient of the $\sigma$-term in the S-matrix is proportional to $(p^2 + q^2 - m_\pi^2)$ which vanishes when $q \to 0$ and $p^2 = m_\pi^2$.

10. We note that a general expression for $F_3$ in the physical region of $K_{\ell_4}$ decay which would satisfy the required conditions at the two limits would have to be at least second order in pion momenta p and q.

11. We have been informed of the results of a polarization-measurement of $\xi(t)$ that are consistent with $\xi = -1$ and approximately constant in the physical region of the $K_{e3}$ decay. D. Cutts, R. Stiening, C. Wiegand, M. Deutsch, to be published.