THREE POLE FORMULA FOR THE PROTON ELECTROMA GNETIC FORM FACTORS*
by

Vladimir Wataghin ${ }^{\dagger}$<br>Stanford Linear Accelerator Center Stanford University, Stanford, California

[^0]Experiments on electron-proton elastic scattering as a function of the square of the four-momentum transfer $t$ have shown that the empirical formula:

$$
\begin{equation*}
G_{E_{p}}(t)=\frac{1}{\mu_{p}} \quad G_{M_{p}}(t)=\frac{a^{2}}{(t-a)^{2}} \tag{1}
\end{equation*}
$$

is in reasonable agreement with the experimental data up to very large momentum transfer. ${ }^{1}$
$\mathrm{G}_{\mathrm{E}_{\mathrm{p}}}(\mathrm{t})$ and $\mathrm{G}_{\mathrm{M}_{\mathrm{p}}}(\mathrm{t})$ are the electric and magnetic form factors, respectively; $a=0.71 \mathrm{GeV}^{2}$ and in our metric $t$ is negative for elastic scattering. We use units e for $G_{E_{p}}(t), e / 2 M$ for the proton magnetic moment $\mu_{p}$ and $G e V^{2}$ for $t$.

It would be interesting to establish a relation between (1) and a physical model of the dispersion type with a linear combination of vector boson resonance terms. Among these we shall consider at first only the $\rho^{0}, \omega$ and $\phi$ since they are the least massive ones.

It is the main purpose of this letter to give a short account of the results obtained trying to fit the experimental data up to $-t=25$ for $G_{E_{p}}(t)$ or $G_{M_{p}}(t) / \mu_{p}$ with a three-pole formula of the type:

$$
\begin{equation*}
G_{E_{p}}(t)=\sum_{k=1}^{3} \beta_{k} \frac{1}{t-a_{k}+\gamma_{k} \sqrt{t_{k}-t}} \tag{2}
\end{equation*}
$$

where $\mathrm{a}_{1}, \mathrm{a}_{2}, \mathrm{a}_{3}$ are equal to $\mathrm{m}_{\rho}^{2} \mathrm{o}, \mathrm{m}_{\omega}^{2}, \mathrm{~m}_{\phi}^{2}$, respectively; $\gamma_{1}, \gamma_{2}, \gamma_{3}$ are the corresponding resonance widths and $t_{1}, t_{2}, t_{3}$ are equal to $\left(2 \dot{m}_{\pi}\right)^{2},\left(3 m_{\pi}\right)^{2}$ and $\left(2 \mathrm{~m}_{\mathrm{K}}\right)^{2}$, respectively.

A more detailed discussion of this expression together with a preliminary fit for $-10<t<0$ will appear elsewhere ${ }^{2}$ and for greater details we refer the reader to it.

The presence of the widths of the resonances in expression (2) is extremely important both for theoretical reasons and in order to obtain a good fit. The square
root dependence and the presence of the thresholds make the single pole terms real for $t<t_{k}$ and complex for $t>t_{k}$ allowing an analytical continuation into the $t>0$ region. In Ref. 2 it is proved that expression (2) satisfies dispersion relations.

For the $\rho$ pole term one would expect theoretically a behaviour of the width near threshold as $\left(t_{k}-t\right)^{3 / 2}$. However, we note that far from resonance the width dependence on $t$ would then dominate and the single $\rho$ pole term would decrease as $t^{-3 / 2}$, rather than as $t^{-1}$ which is what dispersion theory would predict.* We remark also that $G_{E_{p}}(t)$ is in practice insensitive to the behaviour near threshold. Therefore, we assume that the width behaviour given in (2) is correct. The values of the unknown parameters $\beta_{\mathrm{k}}$ have been determined by a least squares method using the available experimental data in the range $-25<t<0$.

In so doing we have weighted the experimental data with their error and we have imposed the two conditions:
(a) $\quad \sum_{k} \beta_{k}=0$
(b) $\quad \mathrm{G}_{\mathrm{F}_{\mathrm{p}}}(0)=1$

Condition (a) is necessary in order that $\mathrm{G}_{\mathrm{E}_{\mathrm{p}}}(\mathrm{t})$ given by (2) should decrease faster than $t^{-1}$ for $t \longrightarrow-\infty$. (b) is a normalization condition expressing the experimental fact that the charge of the proton is $e$. The values of the constants used in the numerical calculation are:

$$
\begin{aligned}
& \mathrm{a}_{1}=(0.760)^{2} ; \mathrm{a}_{2}=(0.780)^{2} ; \mathrm{a}_{3}=(1.020)^{2} \\
& \gamma_{1}=0.12 ; \gamma_{2}=0.012 ; \gamma_{3}=0.003 \\
& \mathrm{t}_{1}=(0.28)^{2} ; \mathrm{t}_{2}=(0.42)^{2} ; \mathrm{t}_{3}=(0.99)^{2}
\end{aligned}
$$

[^1]We have determined different sets of values of the $\beta_{k}$ by either (1) requiring the two conditions (a) and (b) to hold exactly (one free parameter fit); or (2) requiring that only (a) holds exactly (two free parameters fit). The values of $\beta_{\mathrm{k}}$ obtained were:

$$
\begin{array}{ll}
\beta_{1}=0.626 \pm 0.005 ; \beta_{2}=-2.231 ; \beta_{3}=1.605 & \text { in case (1) } \\
\beta_{1}=0.531 \pm 0.013 ; \beta_{2}=-2.08 \pm 0.02 ; \beta_{3}=1.553 & \text { in case (2). }
\end{array}
$$

In Ref. 2, we have taken only approximately into account conditions (a) and (b). In Figs. 1a, 1b, 1c we show the fit to the experimental data by plotting the ratio of the magnetic form factor to the empirical form factor (1). If the validity of the relation $G_{M_{p}} / \mu_{p}=G_{E_{p}}(t)$ is assumed to hold even at large values of $t$ then one can consider the data to represent also the values of $\mathrm{G}_{\mathrm{E}_{\mathrm{p}}}(\mathrm{t})$ up to $-\mathrm{t}=25$. The reason for plotting $G_{M_{p}} / \mu_{p}$ is that (a) at medium $t$ values the experimental errors may be ten times larger in $G_{E_{p}}(t)$ than in $G_{M}(t)$, and that (b) at very large t's the contribution of $\mathrm{G}_{\mathrm{E}_{\mathrm{p}}}(\mathrm{t})$ to the cross section is very small.

It is interesting to note a difference in the asymptotic behaviour of the empirical formula (1) and of our three-pole formula (2). As shown in Ref. 2, formula (2) can be written in the following form:

$$
G_{E_{p}}(t)=-\frac{A^{2}}{(t-a)^{2}}
$$

with

$$
\begin{aligned}
& A^{2}=\sum_{k} \beta_{k} \delta_{k} \frac{1}{1-\frac{\delta_{k}}{a-t}} ; \\
& \delta_{k}=a-a_{k}+\gamma_{k} \sqrt{t_{k}-t}
\end{aligned}
$$

One can see from this expression that asymptotically for $t \rightarrow-\infty, \mathrm{G}_{\mathrm{E}_{\mathrm{p}}}{ }^{(\mathrm{t})}$ decreases as $\mathrm{t}^{-3 / 2}$ while the empirical formula (1) in this case goes to zero as $\mathrm{t}^{-2}$ : This means that the curves representing the fit with expression (2) in Figs. 1a, 1b, ic if continued to higher values of ( -t ) would finally bend and start to increase. At present, however, it seems unlikely that experimentally one could extend the measurements to higher momentum transfer.

It can be seen that the fit is better than the so-called "dipole" fit given by expression (1) and which is represented by the horizontal straight line. In Fig. 1a, one sees that at very small values of $t$ the fit is not so satisfactory as for higher values of $t$. It is worth pointing out that the deviations between experiment and the empirical "dipole" formula (1), especially at large values of $t$, were predicted in Ref. 2 on the basis of expression (2).

As final comments of our results we would like to make two remarks. The first is to emphasize the practical importance of taking into account the width of the resonances in order to obtain a satisfactory fit with experiment. ${ }^{4}$ The second comment is that if one tries to apply a formula of type (2) to the neutron electric (or magnetic) form factor one gets into difficulty at $t=0$. Namely, the isovector part should change $\operatorname{sign}\left(\beta_{1} \rightarrow-\beta_{1}\right)$ with respect to the proton case while the isoscalar part should not. But then the normalization condition

$$
\mathrm{G}_{\mathrm{E}_{\mathrm{n}}}(0)=0 \quad\left(\text { or } \quad \mathrm{G}_{\mathrm{M}_{\mathrm{n}}} / \mu_{\mathrm{n}}=1\right)
$$

is not satisfied. Until this point is clarified one should therefore consider (2) as a phenomenological formula, i.e., we cannot give a physical meaning to the coefficients $\beta_{k}$ and therefore test the vector dominance hypothesis. ${ }^{5}$ However, it may be considered a three parameters phenomenological formula with the correct analyticity properties and which: (a) fits well the data over an extremely wide region of values of momentum transfers, (b) gives correctly the experimentally well-established slope shown by the data for $-t>5$.

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## FIGURE CAPTION

The ratio of the experimental magnetic moment form factor of the proton to the so-called "dipole" formula (1) is compared to the ratio of the theoretical expression (2) to the same formula (1). The dashed line represents the one free parameter fit to the experimental data; the solid line shows the two free parameter fit to the experiment. The region $-25<\mathrm{t}<0$ has been for convenience divided into the three regions: $-0.5<\mathrm{t}<0$ (Fig. 1a); $-2.5<\mathrm{t}<-0.5$ (Fig. 1b); $-25<\mathrm{t}<-2.5$ (Fig. 1c).


FIG. I(a)


FIG. 1 (b)


FIG. 1 (c)


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    $\dagger$ Visiting NATO Fellow, on leave of absence from Istituto di Fisica Teorica-Universita' di Torino.

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