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FIELD THEORY OF CHIRAL SYMMETRY

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ABSTRACT

An infinite class of chiral invariant pion-nucleon Lagrange functions is discussed. Each member of this class is shown to be equivalent, under canonical transformation, to the one in which the commutator of the axial current with the meson field is an isotopic scalar. If the chiral symmetry is broken in this special canonical frame in a manner that ensures the partial conservation of the axial current, then the theory is unique.

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The application of chiral $SU(2) \otimes SU(2)$ current algebra techniques to processes involving the emission and absorption of a large number of soft pions can become very cumbersome. Recently, Weinberg[1] has pointed out that this computational complexity can be reduced by employing an effective Lagrangian[2] that is chiral symmetric save for a part that yields a partially conserved axial current. Since such an effective Lagrangian satisfies all the constraints imposed by current algebra, it may be used in lowest order to obtain the kinematical and isotopic spin structure of the current algebra results for the behavior of scattering or decay amplitudes when the four momenta of various emitted (or absorbed) pions vanish. Higher order corrections cannot alter this structure; they can only produce renormalization of coupling constants. The correct values of the coupling constants can be inferred[1] from the general structure of the current algebra method.

Weinberg[1] obtained an appropriate effective Lagrangian by first performing a canonical transformation on the σ-model[3] and then sending the mass of the unphysical σ-particle to infinity so that it is removed from the theory. It is the purpose of this note to investigate an infinite class of chiral invariant pion-nucleon Lagrange functions of the type introduced by Gürsey.[4] This class is restricted only in so far as the canonical pion field momentum is required to involve the first but no higher derivatives of the pion field, with no dependence on the nucleon field. We shall show that in the limit of perfect chiral symmetry[5] every member of this class is equivalent, under canonical transformation,[6] to the one in which the commutator of the axial charge with the pion field is an isotopic spin scalar, a commutation relation characteristic of the σ-model. Thus, in this limit, the physical scattering amplitudes are uniquely defined, although their off-mass-shell values can be altered by canonical transformation. The chiral invariance can be broken in only one way if the divergence of the axial current is to be proportional to the pion field. The theory is therefore ambiguous essentially only to the extent that this strict PCAC is not maintained. For the most part we shall make use of the freedom of canonical transformation and exhibit the theory in its simplest form. Nevertheless, its more general structure is also of interest, particularly with regard to its extension to larger chiral transformation groups such as SU(3) $\otimes$ SU(3). This general structure is outlined in the Appendix.

We begin by considering a free, massless nucleon Lagrange function[7]

$$\mathcal{L}_N^{(0)} = - \bar{\psi} \gamma^\mu \frac{1}{i} \partial_\mu \psi \quad . \tag{1}$$

It is invariant under not only constant isotopic spin rotations,

$$\psi \longrightarrow e^{\frac{i}{2} \underset{\sim}{\tau} \cdot \underset{\sim}{\omega}} \psi , \tag{2}$$

but a constant chiral isotopic spin transformation as well,

$$\psi \longrightarrow e^{-\gamma_5 \frac{1}{2} \underset{\sim}{\tau} \cdot \underset{\sim}{\nu}} \psi , \tag{3}$$

and is thus invariant under a chiral SU(2) $\otimes$ SU(2) group of transformations. The chiral invariance is upset by the addition of a nucleon mass term $m \bar{\psi} \psi$. However, it can be restored if the mass constant m is replaced by a function M of the pion field ϕ_a, if the latter transforms appropriately. We shall require that no derivatives of the pion field occur in M so that the canonical pion field momenta π_a^o is simply related to $\partial_o \phi_a$. Invariance under isotopic spin rotations and parity conservation require that M be a function of $\gamma_5 \underset{\sim}{\tau} \cdot \underset{\sim}{\phi}$, so we write

$$\mathcal{L}_N = - \bar{\psi} \left[\gamma \frac{1}{i} \partial + M(\gamma_5 \underset{\sim}{\tau} \cdot \underset{\sim}{\phi}) \right] \psi \quad . \tag{4}$$

In order that the Lagrange function be Hermitian, $M(z)$ must be a real function,[8]

$$M(z^*)^* = M(z) . \tag{5}$$

Chiral invariance is maintained if the pion field ϕ is transformed into a new field ϕ' which satisfies

$$M(\gamma_5 \underset{\sim}{\tau} \cdot \underset{\sim}{\phi}') = e^{-\frac{1}{2} \gamma_5 \underset{\sim}{\tau} \cdot \underset{\sim}{\nu}} M(\gamma_5 \underset{\sim}{\tau} \cdot \underset{\sim}{\phi}) \, e^{-\frac{1}{2} \gamma_5 \underset{\sim}{\tau} \cdot \underset{\sim}{\nu}} , \tag{6}$$

or, within an eigenstate of γ_5 with eigenvalue $\gamma_5' = i$,

$$M(i \underset{\sim}{\tau} \cdot \underset{\sim}{\phi}') = e^{-\frac{i}{2} \underset{\sim}{\tau} \cdot \underset{\sim}{\nu}} M(i \underset{\sim}{\tau} \cdot \underset{\sim}{\phi}) \, e^{-\frac{i}{2} \underset{\sim}{\tau} \cdot \underset{\sim}{\nu}} \tag{7}$$

[The transformation law Eq. (6) in the other eigenstate $\gamma_5' = -i$ is just the Hermitian adjoint of Eq. (7).]

The implications of the transformation law Eq. (7) are made clear if we define a real, unitary function

$$U(z) = \left[\frac{M(z)}{M(-z)}\right]^{1/2} = U(-z)^{-1} = U(z^*)^* , \tag{8}$$

so that

$$M(z) = U(z) \, H(z^2) \tag{9}$$

with

$$H(z^2) = \left[M(z) \, M(-z)\right]^{1/2} . \tag{10}$$

If this decomposition is inserted into

$$M(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}')\; M(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}')^{\dagger} = e^{-\frac{i}{2}\underset{\sim}{\tau}\cdot\underset{\sim}{\nu}} M(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})\; M(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})^{\dagger}\; e^{\frac{i}{2}\underset{\sim}{\tau}\cdot\underset{\sim}{\nu}} , \tag{11}$$

we find that $H(-\phi^2)$ is invariant under the chiral transformation

$$H(-\phi'^2) = H(-\phi^2)^2 . \tag{12}$$

Since only isotopic spin rotation can preserve the length of the meson field that can take on arbitrary values, this demands that H is a constant, which by a parity convention may be taken to be positive,

$$H(-\phi^2) = m > 0 . \tag{13}$$

Accordingly,

$$\mathcal{L}_N = -\bar{\psi}\left[\gamma\frac{1}{i}\;\partial + m\,U(\gamma_5\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})\right]\psi , \tag{14}$$

with the chiral transformation property of the meson field depending upon the choice of the unitary function $U(z)$ and defined by

$$U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}') = e^{-\frac{i}{2}\underset{\sim}{\tau}\cdot\underset{\sim}{\nu}}\, U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})\; e^{-\frac{i}{2}\underset{\sim}{\tau}\cdot\underset{\sim}{\nu}} . \tag{15}$$

The unitary matrix $U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})$ represents a continuous group of SU(2) transformations. The various different choices of the unitary function $U(z)$ correspond to different parameterizations of this group which are related by canonical transformation of the meson field. We may exhibit this by noting that the unitary matrix may be written as

$$U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\varphi}) = \left[1 - f^2\varphi^2 g(\varphi^2)\right]^{1/2} + i\, f\, \underset{\sim}{\tau}\cdot\underset{\sim}{\varphi}\, g(\varphi^2), \tag{16}$$

where f is a scaling parameter (coupling constant) introduced in order to make the function g dimensionless. This arbitrary function defines a canonical transformation of the meson field from the old field ϕ to a new field Φ by

$$\Phi_a = \phi_a\, g(\phi^2) = V\, \phi_a\, V^{\dagger} , \tag{17}$$

with

$$V = \exp\left\{ - i \int (d\underset{\sim}{r})\, \pi_a^o\ \phi_a\, g(\phi^2)\right\} . \tag{18}$$

Thus the entire class of Lagrange functions which we have considered are equivalent under canonical transformation to one with

$$U(i\tau\cdot\Phi) = \sigma\,(\Phi^2) + i\, f\, \underset{\sim}{\tau}\cdot\underset{\sim}{\Phi} , \tag{19a}$$

where

$$\sigma(\Phi^2) = \left[1 - f^2\Phi^2\right]^{1/2} , \tag{19b}$$

and we may restrict the discussion to this standard form. The simplicity of this standard form is made clear by the structure of the infinitesimal chiral response

$$\Phi \longrightarrow \Phi' = \Phi + \delta^{(C)}\, \Phi \tag{20}$$

defined by the infinitesimal version of Eq. (15)

$$U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\Phi}') = U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\Phi}) - \frac{i}{2}\left\{ \underset{\sim}{\tau}\cdot \delta\underset{\sim}{\nu},\ U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\Phi})\right\} ,$$

namely,

$$\delta^{(C)}\Phi_a = - \frac{1}{f}\,\sigma(\Phi^2)\ \delta\nu_a . \tag{21}$$

We may now easily construct a meson Lagrange function that is invariant under the chiral transformation (21) in addition to the usual isotopic spin invariance

$$\Phi_a \rightarrow \overline{\Phi}_a = \Phi_a + \delta^{(I)} \Phi_a , \tag{22a}$$

$$\delta^{(I)} \Phi_a = - \epsilon_{abc} \delta\omega_b \Phi_c . \tag{22b}$$

We shall write this Lagrange function in first order form so that the correct canonical variables are clearly defined. Since

$$\delta^{(C)} \partial_\mu \Phi_a = \delta\nu_a \sigma(\Phi^2)^{-1} f \Phi_b \partial_\mu \Phi_b , \tag{23}$$

the kinematical term $\Pi_a^\mu \partial_\mu \Phi_a$ that occurs in this function will be invariant if the conjugate field transforms as

$$\delta^{(C)} \Pi_a^\mu = - \Phi_a \sigma(\Phi^2)^{-1} f \delta\nu_b \Pi_b^\mu , \tag{24a}$$

$$\delta^{(I)} \Pi_a^\mu = - \epsilon_{abc} \delta\omega_b \Phi_c . \tag{24b}$$

We shall restrict the meson Lagrange function to depend at most quadratically on the conjugate field Π_a^μ so that this field is linearly dependent on $\partial^\mu \Phi_a$. With this restriction, and a conventional normalization of the field variables, the Lagrange function is uniquely determined by the chiral and isotopic spin invariance:

$$\mathcal{L}_\pi = - \Pi_a^\mu \partial_\mu \Phi_a + \frac{1}{2} \Pi_a^\mu \Pi_{\mu a} - \frac{1}{2} f^2 \Pi_a^\mu \Phi_a \Pi_{\mu b} \Phi_b . \tag{25}$$

The fields Π_{oa}, Φ_a form a canonically conjugate pair

$$\left[\Phi_a(x),\quad \Pi_{ob}(x')\right]_{t=t'} = i\ \delta_{ab}\ \delta(\underset{\sim}{r}-\underset{\sim}{r}') , \tag{26}$$

while the spatial component Π_a^k is a dependent field variable. It is defined by the action principle as

$$-\ \partial^k\Phi_a + \Pi_a^k - f^2\ \Phi_a\ \Pi_b^k\ \Phi_b = 0 ,$$

or

$$\Pi_a^k = \sigma(\Phi^2)\ \ \partial^k\left[\sigma(\Phi^2)^{-1}\ \Phi_a\right] . \tag{27}$$

Vector and axial vector current operators can be defined in the usual fashion by replacing the constant infinitesimal isotopic spin and chiral transformation parameters by space-time functions and computing the response of the total Lagrange function to this extended variation. We compute in this way

$$\delta\ (\mathcal{L}_N + \mathcal{L}_\pi) = -\ V_a^\mu\ \partial_\mu\ \delta\ \omega_a - A_a^\mu\ \partial_\mu \delta\ \nu_a , \tag{28}$$

with

$$V_a^\mu = \bar{\psi}\ \gamma^\mu \tfrac{1}{2}\ \tau_a \psi - \Pi_b^\mu\ \epsilon_{bac}\ \Phi_c , \tag{29}$$

and

$$A_a^\mu = \overline{\psi}\ \ \gamma^\mu\ i\ \gamma_5\ \tfrac{1}{2}\ \tau_a \psi - \tfrac{1}{f}\ \sigma(\Phi^2)\ \ \Pi_a^\mu . \tag{30}$$

The time components of these current operators satisfy local commutation relations appropriate to the chiral $SU(2) \otimes SU(2)$ group. The naive application of the canonical commutation relations gives the more general equal-time commutators

$$\left[V_a^o(x),\ V_b^o(x')\right]_{t=t'} = i\ \epsilon_{abc}\ V_c^o(x)\ \delta(\underset{\sim}{r}-\underset{\sim}{r}') , \tag{31a}$$

$$\left[V_a^o(x),\ V_b^k(x')\right]_{t=t'} = i\ \epsilon_{abc}\, V_c^k(x)\ \delta\,(\underset{\sim}{r}-\underset{\sim}{r}')\ , \tag{31b}$$

$$- i\ \partial^k \left\{ \delta(\underset{\sim}{r}-\underset{\sim}{r}')\left[\delta_{ab}\,\phi^2 - \phi_a\,\phi_b\right]\right\},$$

$$\left[V_a^o(x),\ A_b^o(x')\right]_{t=t'} = i\ \epsilon_{abc}\, A_c^o(x)\ \delta\,(\underset{\sim}{r}-\underset{\sim}{r}')\ , \tag{32a}$$

$$\left[V_a^o(x),\ A_b^k(x')\right]_{t=t'} = i\ \epsilon_{abc}\, A_c^k(x)\ \delta\,(\underset{\sim}{r}-\underset{\sim}{r}') \tag{32b}$$

$$+ i\ \partial_k \left\{ \delta(\underset{\sim}{r}-\underset{\sim}{r}')\ \epsilon_{abc}\ \frac{1}{f}\,\sigma(\phi^2)\ \phi_c \right\},$$

$$\left[A_a^o(x),\ V_b^k(x')\right]_{t=t'} = i\ \epsilon_{abc}\ A_c^k(x)\ \delta\,(\underset{\sim}{r}-\underset{\sim}{r}') \tag{33}$$

$$- i\ \partial_k \left\{ \delta\,(\underset{\sim}{r}-\underset{\sim}{r}')\ \epsilon_{abc}\,\frac{1}{f}\ \sigma(\phi^2)\,\phi_c \right\}\ ,$$

$$\left[A_a^o(x),\ A_b^o(x')\right]_{t=t'} = i\ \epsilon_{abc}\ V_c^o(x)\ \delta\,(\underset{\sim}{r}-\underset{\sim}{r}')\ , \tag{34a}$$

$$\left[A_a^o(x),\ A_b^k(x')\right]_{t=t'} = i\ \epsilon_{abc}\ V_c^k(x)\,\delta\,(\underset{\sim}{r}-\underset{\sim}{r}') \tag{34b}$$

$$- i\ \partial_k \left\{ \delta\,(\underset{\sim}{r}-\underset{\sim}{r}')\left[\frac{1}{f}\ \sigma(\phi^2)^2\ \delta_{ab} + \phi_a\phi_b\right]\right\}.$$

The quantities that involve the spatial gradient are the so-called "Schwinger terms." The structures that we have listed are those that follow from a straightforward application of the canonical commutation relations. However, this method yields no Schwinger terms that are associated with nucleon field, a result in contradiction to general principles, and one that shows that great care must be exercised in handling the bilinear forms that compose the current operator. Therefore, our results must be taken only as a heuristic indication of the nature of the Schwinger terms. We note that the Schwinger term which occurs in the axial-vector commutator $\left[A_a^o, A_b^k\right]$ differs[9] from that which occurs in the vector commutator $\left[V_a^o, V_b^k\right]$.

The commutator of the time component of the axial current with the meson field operator reproduces the structure of the chiral field alteration Eq. (21).

$$\left[A_a^o(x), \Phi_b(x')\right]_{t=t'} = - \delta_{ab} \frac{1}{f} \sigma(\Phi^2) \; i\delta(\underset{\sim}{r}-\underset{\sim}{r}') . \tag{35}$$

The special canonical frame that we have used simplifies this commutator, for in general it also contains terms involving $\Phi_a \Phi_b$ and is not an isotopic spin scalar. This simplicity is reflected in the algebraic closure[10]

$$\left[A_a^o(x), \frac{1}{f}\sigma(\Phi^2)\right]_{t=t'} = i \; \delta(\underset{\sim}{r}-\underset{\sim}{r}') \, \Phi_a(x) . \tag{36}$$

If we require that the chiral symmetry is broken in a manner that makes the divergence of the axial current proportional to the pion field, and if we also require that the commutators of the axial charge with the meson field close in the sense illustrated above, then our model remains unique. We obtain this strict PCAC if we add to our Lagrange function

$$\mathcal{L}_B = \frac{\mu^2}{f^2} \; \sigma(\Phi^2), \tag{37}$$

for then the chiral variation of the complete Lagrange function is

$$\delta^{(C)}(\mathcal{L}_N + \mathcal{L}_\pi + \mathcal{L}_B) = - A_a^\mu \partial_\mu \delta\nu_a + \frac{\mu^2}{f} \; \Phi_a \delta\nu_a ,$$

and the principle of stationary action gives

$$\partial_\mu A^\mu_a = \frac{\mu^2}{f} \Phi_a . \tag{38}$$

The expansion of the symmetry breaking Lagrangian (37) in powers of the meson field gives a meson mass term and multiple meson scattering terms

$$\mathcal{L}_B = \text{const.} - \frac{1}{2}\mu^2 \Phi_a\Phi_a - \frac{1}{8} f^2 (\Phi_a\Phi_a)^2 + \dots \quad . \tag{39}$$

The lowest order meson-meson scattering term which occurs here, together with that contained in the chiral symmetric meson Lagrangian (25), yields immediately the current algebra result of Weinberg[11] for pion scattering lengths. These differ from the scattering lengths computed by Schwinger[2] using an effective Lagrangian method. This discrepancy arises because the axial current-meson field commutator in Schwinger's model is not an isotopic spin scalar, and hence the model does not possess the commutator closure illustrated in Eqs. (35) and (36). A canonical transformation can be performed on this model so that the commutator closure is obtained, but then the PCAC condition is destroyed.

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APPENDIX

We shall outline here a method which does not employ the special canonical frame used in the test. It may prove useful for the extension of the theory to larger chiral symmetry groups. This general treatment requires first a discussion of the properties of the matrix U that occurs in the nucleon Lagrange function (4).

The matrix $U(i\underset{\sim}{\tau} \cdot \underset{\sim}{\phi})$ is not only unitary

$$U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})^{+} = U(-i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) = U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})^{-1} , \tag{A.1}$$

but also unimodular

$$\det U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) = 1 . \tag{A.2}$$

The latter is easily established by noting that the determinant is invariant under isotopic spin rotations, so that ϕ_a may be taken to be non-vanishing only for $a = 2$ where

$$\det U(i\tau_2\phi_2) = \det U(i\tau_2\phi_2)^{T} = \det U(-i\tau_2\phi_2)$$

$$= \left[\det U(i\tau_2\phi_2)\right]^{-1} ,$$

which, with $U(0) = 1$ and continuity in ϕ, implies the unimodularity. If we write, to first order,

$$U\left[i\underset{\sim}{\tau}\cdot(\underset{\sim}{\phi}+\delta\underset{\sim}{\phi})\right] = \left[1 + i\ \delta h(\phi)\right] U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) ,$$

then the unitarity condition (A.1) requires that δh is Hermitian, while the first order formula

$$\det\left[1 + i\ \delta h(\phi)\right] = 1 + i\,\mathrm{tr}\ \delta h(\phi)$$

and the determinantal constraint (A.2) requires that δh has a vanishing trace. Hence[12]

$$\frac{\partial}{\partial \phi_a} \quad U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) = i\ d_{ab}(\phi)\ \tau_b\ U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})\ . \tag{A.3}$$

Since the derivative operation is commutative,

$$\frac{\partial}{\partial \phi_a}\frac{\partial}{\partial \phi_b} \quad U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) = \frac{\partial}{\partial \phi_b}\frac{\partial}{\partial \phi_a}\ U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})\ ,$$

and

$$\left[\tau_a, \tau_b\right] = 2i\,\epsilon_{abc}\ \tau_c\ ,$$

the derivative function $d_{ab}(\phi)$ must satisfy the integrability condition

$$\frac{\partial}{\partial \phi_a}\ d_{bc}(\phi) - \frac{\partial}{\partial \phi_b}\ d_{ac}(\phi) = -\,2\ d_{ad}(\phi)\ d_{be}(\phi)\ \epsilon_{dec}\ . \tag{A.4}$$

Combined infinitesimal isotopic spin rotations of the meson field (22a, b) the nucleon field (2) leave the unitary matrix invariant,

$$U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) = e^{-\frac{i}{2}\underset{\sim}{\tau}\cdot\delta\underset{\sim}{\omega}}\quad U(i\underset{\sim}{\tau}\cdot\overline{\underset{\sim}{\phi}})\ e^{\frac{i}{2}\underset{\sim}{\tau}\cdot\delta\underset{\sim}{\omega}}, \tag{A.5}$$

and imply that

$$\frac{1}{2}\left[U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}),\ \tau_a\right] = \epsilon_{abc}\,\phi_b\ d_{cd}(\phi)\ \tau_d\ \ U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi})\ . \tag{A.6}$$

If we use these results to express

$$\frac{\partial}{\partial \overline{\phi}_a}\ U(i\underset{\sim}{\tau}\cdot\overline{\underset{\sim}{\phi}}) = d_{ab}(\overline{\phi})\ \ i\tau_b\ \ U(i\underset{\sim}{\tau}\cdot\overline{\underset{\sim}{\phi}})$$

in terms of the field ϕ, we learn that the derivative matrix is an isotopic spin invariant in the sense that

$$\epsilon_{cde}\phi_d \frac{\partial}{\partial\phi_e} d_{ab}(\phi) = \epsilon_{acd} d_{db}(\phi) + \epsilon_{bcd} d_{ad}(\phi) . \tag{A.7}$$

We are now in a position to determine the nature of the meson field response

$$\phi_a \longrightarrow \phi'_a = \phi_a + \delta^{(C)} \phi_a \tag{A,8}$$

to the infinitesimal version of the chiral transformation defined by Eq. (15). We have

$$\begin{aligned} \delta^{(C)} \phi_a \frac{\partial}{\partial\phi_a} U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) &= -\frac{i}{2}\left\{U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}),\ \delta\underset{\sim}{\nu}\cdot\underset{\sim}{\tau}\right\} \\ &= -i\ \delta\underset{\sim}{\nu}\cdot\underset{\sim}{\tau}\ U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}) \\ &\quad -\frac{i}{2}\left[U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}),\ \delta\underset{\sim}{\nu}\cdot\underset{\sim}{\tau}\right], \end{aligned} \tag{A.9}$$

and, employing (A.3) and (A.6), secure

$$\delta^{(C)} \phi_a d_{ab}(\phi) = -\delta\nu_b - \delta\nu_a \epsilon_{adc}\phi_d d_{cb}(\phi) ,$$

or

$$\delta^{(C)}\phi_a = -\delta\nu_b\left[d^{-1}_{ba}(\phi) + \epsilon_{bca}\phi_c\right] . \tag{A.10}$$

The structure of this transformation law can be clarified considerably. To this end, we multiply the isotopic spin invariance statement (A.7) and the integrability condition (A.4) by appropriate numbers of the inverse derivative matrix $d^{-1}(\phi)$ to get

$$\epsilon_{cde}\phi_d \frac{\partial}{\partial\phi_e} \, d^{-1}_{ab}(\phi) = \epsilon_{acd}\, d^{-1}_{db}(\phi) + \epsilon_{bcd}\, d^{-1}_{ad}(\phi) \,, \tag{A.11}$$

and

$$d^{-1}_{ad}(\phi) \frac{\partial}{\partial\phi_d} \, d^{-1}_{bc}(\phi) - d^{-1}_{bd}(\phi) \frac{\partial}{\partial\phi_d} \, d^{-1}_{ac}(\phi) = 2 \, \epsilon_{abd}\, d^{-1}_{dc}(\phi) \quad . \tag{A.12}$$

The isotopic spin invariance (A.11) implies that $d^{-1}(\phi)$ may be expanded as

$$d^{-1}_{ab}(\phi) = \delta_{ab}\, A(\phi^2) + \phi_a\phi_b \, B(\phi^2) + \epsilon_{abc}\, \phi_c \, C(\phi^2) \,, \tag{A.13}$$

while the integrability condition (A.12) requires that the scalar coefficients satisfy

$$\left[A + \phi^2 B\right] \left[C - 1\right] = 0 \,, \tag{A.14a}$$

$$B\left[C - 1\right] - \left[A + \phi^2 B\right] \frac{d}{d\varphi^2} \, C = 0 \,, \tag{A.14b}$$

$$AB + C^2 - 2C - 2\left[A + \phi^2 B\right] \frac{d}{d\phi^2} \, A = 0 \quad . \tag{A.14c}$$

These equations possess a solution only if $C = 1$, or $C = 2$ with $A = B = 0$. The latter case must be excluded, for it gives a singular matrix $d^{-1}(\phi)$. Hence $C = 1$ and

$$\delta^{(C)}\phi_a = - \delta\nu_a \, A(\phi^2) - \phi_a \, \delta\nu_b \phi_b \, B(\phi^2) \,, \tag{A.15}$$

with

$$2\left[A(\phi^2) + \phi^2 B(\varphi^2)\right] \frac{d}{d\varphi^2} A(\phi^2) = A(\varphi^2)\, B(\phi^2) - 1 \quad . \tag{A.16}$$

We turn now to the construction of a first order meson Lagrange function. Since

$$\delta^{(C)}\partial_\mu \varphi_a = -\delta \nu_b \left[\frac{\partial}{\partial \varphi_c} d^{-1}_{ba}(\varphi) - \epsilon_{bca}\right] \partial_\mu \phi_c , \tag{A.17}$$

the kinematical term $\pi^\mu_a \partial_\mu \phi_a$ that occurs in this function will be invariant if the conjugate field π^μ_a obeys the chiral transformation law

$$\delta^{(C)} \pi^\mu_a = +\delta \nu_b \left[\frac{\partial}{\partial \phi_a} d^{-1}_{bc}(\phi) - \epsilon_{bca}\right] \pi^\mu_c . \tag{A.18}$$

One can verify, with the aid of the integrability condition (A.12), that the quadratic structure

$$\pi^\mu_a N_{ab}(\phi) \pi_{\mu b} ,$$

with

$$N_{ab}(\phi) = d^{-1}_{ca}(\phi) \; d^{-1}_{cb}(\phi) \tag{A.19}$$

a symmetrical, positive-definite matrix, is a chiral invariant. Accordingly, using a conventional normalization of the field strengths, the chiral invariant meson Lagrangian is given by[13]

$$\mathcal{L}_\pi = - \pi^\mu_a \partial_\mu \phi_a + \frac{1}{2} \pi^\mu_a N_{ab}(\phi) \pi_{\mu b} . \tag{A.20}$$

The vector and axial vector current operators are identified by the response Eq. (28) of the complete Lagrange function to isotopic spin rotations and chiral transformations whose infinitesimal parameters are extended to space-time functions. We find

$$V^\mu_a = \overline{\psi} \gamma^\mu \frac{1}{2} \tau_a \psi - \pi^\mu_b \epsilon_{bac} \phi_c , \tag{A.21}$$

$$A_a^\mu = \overline{\psi}\,\gamma^\mu\, i\,\gamma_5\,\frac{1}{2}\,\tau_a \psi - d_{ab}^{-1}(\phi)\,\pi_b^\mu - \pi_b^\mu\,\epsilon_{bac}\,\phi_c\ . \qquad (A.22)$$

In order to verify that the time components of these operators indeed generate a spatially local group of chiral $SU(2) \otimes SU(2)$ transformations, we couple the currents to external, classical gauge fields B_a^μ and W_a^μ:

$$\mathcal{L} = \mathcal{L}_N + \mathcal{L}_\pi + A_a^\mu\, B_{\mu a} + V_a^\mu W_{\mu a}\ . \qquad (A.23)$$

It is a simple matter to verify that under an isotopic spin rotation

$$\delta^{(I)}\, A_a^\mu = -\,\epsilon_{abc}\,\delta\omega_b\, A_c^\mu\ , \qquad (A.24a)$$

$$\delta^{(I)}\, V_a^\mu = -\,\epsilon_{abc}\,\delta\omega_b\, V_c^\mu\ . \qquad (A.24b)$$

A moderately lengthly calculation that makes use of (A.11) and (A.12) gives the chiral transformation properties

$$\delta^{(C)}\, A_a^\mu = -\,\epsilon_{abc}\,\delta\nu_b\, V_c^\mu, \qquad (A.25a)$$

$$\delta^{(C)}\, V_a^\mu = -\,\epsilon_{abc}\,\delta\nu_b\, A_c^\mu\ . \qquad (A.25b)$$

Hence

$$\begin{aligned}\delta\mathcal{L} = &- V_a^\mu\,\partial_\mu \delta\omega_a - A_a^\mu\,\partial_\mu\,\delta\nu_a \\ &- B_{\mu a}\, A_c^\mu\,\epsilon_{abc}\,\delta\omega_b - B_{\mu a}\, V_c^\mu\,\epsilon_{abc}\,\delta\nu_b \\ &- W_{\mu a}\, V_c^\mu\,\epsilon_{abc}\,\delta\omega_b - W_{\mu a}\, A_c^\mu\,\epsilon_{abc}\,\delta\nu_b\ ,\end{aligned}$$

and the principle of stationary action gives the divergence conditions

$$\partial_\mu V^\mu_a = - \epsilon_{abc} W_{\mu b} V^\mu_c - \epsilon_{abc} B_{\mu b} A^\mu_c , \tag{A.26a}$$

$$\partial_\mu A^\mu_a = - \epsilon_{abc} W_{\mu b} A^\mu_c - \epsilon_{abc} B_{\mu b} V^\mu_c . \tag{A.26b}$$

These divergence conditions imply[14]

$$\left[V^o_a(x),\ V^o_b(x')\right]_{t=t'} = i\ \epsilon_{abc}\ V^o_c(x)\ \delta(\underset{\sim}{r}-\underset{\sim}{r}') , \tag{A.27a}$$

$$\left[V^o_a(x),\ A^o_b(x')\right]_{t=t'} = i\ \epsilon_{abc}\ A^o_c(x)\ \delta(\underset{\sim}{r}-\underset{\sim}{r}') , \tag{A.27b}$$

$$\left[A^o_a(x),\ A^o_b(x')\right]_{t=t'} = i\ \epsilon_{abc}\ V^o_c(x)\ \delta(\underset{\sim}{r}-\underset{\sim}{r}') , \tag{A.27c}$$

the local chiral $SU(2)\otimes SU(2)$ commutation relations.

The commutator of the time component of the axial current with the meson field reproduces the chiral meson field transformation,

$$\begin{aligned}\left[A^o_a(x),\ \phi_b(x')\right]_{t=t'} &= - \left\{ d^{-1}_{ab}(\phi) + \epsilon_{bac}\phi_c \right\} i\,\delta(\underset{\sim}{r}-\underset{\sim}{r}') \\ &= - \left\{ \delta_{ab}\, A(\phi^2) + \phi_a\phi_b\, B(\phi^2) \right\} i\,\delta(\underset{\sim}{r}-\underset{\sim}{r}'). \end{aligned} \tag{A.28}$$

It is not difficult to show that the validity of the Jacobi identity applied to $\left[A^o_a, \left[A^o_b, \phi_c\right]\right]$ is equivalent to the constraint (A.16). This double commutator is generally a complex structure involving various isotopic spin values. However, if we require that $B = 0$, the axial vector-meson field commutator is an isotopic scalar, and the double commutator closes back to the meson field. With $B = 0$ the constraint (A.16) requires that

$$A(\phi^2) = \frac{1}{f} \left[1 - f^2\phi^2\right]^{1/2} = \frac{1}{f}\,\sigma(\phi^2)\,, \tag{A.29}$$

and we recover precisely the special canonical frame used in the text. This is made explicit if we use (A.3) to construct the unitary matrix U. We may "integrate along a straight line" using

$$\begin{aligned} \frac{\partial}{\partial\lambda} U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}\lambda) &= i\ \phi_a\, d_{ab}\,(\phi\lambda)\ \tau_b\ U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}\lambda) \\ &= A(\phi\lambda)^{-1}\ i\,\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}\ U(i\underset{\sim}{\tau}\cdot\underset{\sim}{\phi}\lambda)\,, \end{aligned} \tag{A.30}$$

to obtain immediately

$$U(i\tau\cdot\phi) = \left[1 - f^2\phi^2\right]^{1/2} + i\,\underset{\sim}{\tau}\cdot\underset{\sim}{\phi} f\ . \tag{A.31}$$

REFERENCES

1. S. Weinberg, Phys. Rev. Letters 18, 188 (1967).
2. The utility of an effective Lagrangian method has also been advocated, without regard to current algebra, by J. Schwinger, Phys. Letters 24B, 473 (1967). See also J. A. Cronin, Univ. of Chicago preprint, EFINS 67-19.
3. J. Schwinger, Ann. Phys. (N.Y.) 2, 407 (1957); M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960).
4. F. Gürsey, Nuovo Cimento 16, 230 (1960); Proceedings of the 1960 Rochester Conference, p. 572; Ann. Phys. 12, 91 (1961).
5. It is perhaps worthwhile to observe that if the theory is taken to be of a fundamental kind, not simply an effective Lagrangian that is used in lowest order, then in the perfect chiral symmetry limit the nucleon state occurs as a degenerate mass doublet of opposite parities. The symmetry breaking interaction will remove the mass degeneracy and could possibly extinguish one of the states. There are presently two $(\frac{1}{2}^-)$ candidates for a chiral partner to the nucleon, N(1520) and a less well established N(1700) [Rosenfeld, et al., Rev. Mod. Phys. 39, 1 (1967)].
6. The importance of canonical transformation was emphasized to me in conversation with W. A. Bardeen and B. W. Lee. They have independently obtained results similar to those of this paper.
7. We use a metric tensor $g^{\mu\nu}$ with $g^{oo} = -1$, and γ matrices in a representation such that γ^o is Hermitian, $\gamma^{o\dagger} = \gamma^o$, while $\gamma^{k\dagger} = -\gamma^k$, and $\gamma_5 = \gamma^o\gamma^1\gamma^2\gamma^3 = -\gamma_5^T$.
8. We consider such functions as defined by formal power series.
9. Presumably the corresponding vacuum expectation values also differ, and hence our model does not satisfy a condition employed by S. Weinberg,

Phys. Rev. Letters 18, 507 (1967), in deriving a sum rule on the spectral weights of the vector and axial-vector vacuum correlation functions.

10. It has been observed by Bardeen, et al., Phys. Rev. Letters 18, 1170 (1967), that Eq. (36) follows directly from Eq. (35) if the Jacobi identity is applied to the double commutator $[A^o_a, [A^o_b, \phi_c]]$ and the axial-vector commutator Eq. (34a) together with isotopic spin transformation property

$$\left[V^o_a(x),\ \phi_b(x')\right]_{t=t'} = i\ \epsilon_{abc}\ \phi_c(x)\ \delta(\underset{\sim}{r}-\underset{\sim}{r}')$$

are used.

11. S. Weinberg, Phys. Rev. Letters 17, 616 (1966).

12. We could equally well write

$$\frac{\partial}{\partial\phi_a} U(i\tau\cdot\phi) = U(i\tau\cdot\phi)\ i\ \tilde{d}_{ab}(\phi)\ \tau_b\ ,$$

but the unitary restriction (A.1) shows that $\tilde{d}$ is simply related to d,

$$\tilde{d}_{ab}(\phi) = d_{ab}(-\phi)\ .$$

13. It is not difficult to show that this first order function is equivalent to the manisfestly chiral invariant second order Lagrangian that is proportional to

$$\frac{1}{2}\mathrm{tr}\ \partial^\mu U(i\tau\cdot\varphi)\ \partial_\mu\ U(i\tau\cdot\phi)^{-1} = \partial^\mu\phi_a\ d_{ac}(\phi)\ d_{bc}(\phi)\ \partial_\mu\ \phi_b\ .$$

14. See, for example, D. G. Boulware and L. S. Brown, Phys. Rev. 156, 1724 (1967).