SLAC-PUB-315
(TH)

Soft Pions and Current Algebras

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1. Comment

The simultaneous application of the hypothesis of partially conserved axial vector current (PCAC) and the $U(3)\times U(3)$ algebra of charges permits a large number of relations between various processes involving the emission and absorption of pions. The relationships provided are usually of the type where the physical amplitude is continued to an unphysical point where one or more pions have zero energy and momentum. In many cases the amplitudes may be assumed to have only a small variation in their value in going from the physical to the non-physical point and the results which are established at the soft pion limit are presumed to apply for the physical process (i.e. the Goldberger-Treiman relation which involves the pion nucleon coupling constant evaluated at the nonphysical point of a zero mass pion and applied to the physical pion decay).

In these lectures an attempt is made to elucidate some of the applications of these soft pion results. The method used here does not involve T-products or retarded commutators but rather makes use of only a single reduction of a pion where accordingly many ambiguities disappear, such as the σ commutators and Schwinger terms. The main tool used is the analogue of the Noether Theorem of statistical mechanics which is carried over here to quantum field theory.

A good set of references is given by N. Cabibbo in his Raporteurs Report of the 1966 Berkeley Conference on High Energy Physics.

2. Introduction

1) Reduction Formula

Basic to the derivation of the soft pion theorems is the reduction formula which allows an S-matrix element which depends only on mass shell quantities to be expressed as the Fourier transform of quantities which depend on space and time. This step permits the introduction of currents about which physically reasonable statements can be made and thus enriches the expression for the S-matrix element enormously.

In its simplest form, which is all that is used in these notes, we

(1967 Tokyo Summer Lectures in Theoretical Physics. Printed in Fundamental Particle Physics, ed. G. Takeda and Y. Hara, 1968, p. 43.)

express the S-matrix element involving a pion of type α in the final state as

$$<b\pi^{\alpha}(\text{out})/a(\text{in})>=i\,[(2\pi)^3(2K)]^{-1/2}\int d^4x e^{iK\cdot x}<b(\text{out})|(\Box+m_\pi^2)\varphi^\alpha(x)|a(\text{in})> \tag{1}$$

where $\varphi(x)$ is the interpolating pion field, K_0 the energy of the pion, and where we have supposed that no pion of type α is contained in the initial state. Eq. (1) can be put in a slightly different form using translational invariance in the form

$$\varphi(x+a)=e^{iP\cdot a}\varphi(x)e^{-iP\cdot a}$$

where P_μ is the momentum operator. Thus

$$<b\pi^{\alpha}(\text{out})/a(\text{in})>=i\,[(2\pi)^3(2K)]^{-1/2}(2\pi)^4\delta^4(p_b+K-p_a)$$
$$\times(K^2-m_\pi^2)<b(\text{out})|\varphi^\alpha(0)|a(\text{in})>\,. \tag{2}$$

The above expression with the δ-function constraint is used in the soft pion theorems to define the S-matrix element even when K is not the physical pion momentum, i.e. when $K\to 0$. (We note that $<b|\varphi(0)|a>$ has a pole at $K^2=m_\pi^2$, otherwise the S-matrix element would be zero for physical pions.)

In the next section we discuss the cornerstone of the soft pion procedure, the Goldberger-Treiman relation.

2) The Goldberger-Treiman Relation

The G-T relation connects the pion decay rate with the axial vector coupling constant of β-decay.

We review here a few kinematic facts about the pion decay and neutron decay.

Leptonic decays of strong particles are supposed to be described by an interaction Lagrangian which is the product of two currents, the lepton current and the hadron current, i.e.,

$$\mathscr{L}_{\text{int}}=\frac{G}{\sqrt{2}}j_\mu(\text{lepton})J_\mu(\text{hadron}),\qquad GM_p^2=(1.023\pm0.002)\times10^{-5}\,.$$

From μ-decay

The lepton current, which only affects leptons, gives rise to the famous matrix element

$$<e(\mu)\nu|\,j_\alpha(\text{lepton})|0>=\bar{u}_{e(\mu)}\gamma_\alpha(1+\gamma_5)u_\nu\equiv L_\alpha$$

where the electron (or muon) spinors and the gamma matrices are as usual,

$$\gamma\cdot P u_s = m_s u_s, \quad \gamma\cdot P u_s = 0$$

$$\gamma P = \gamma_0 E - \boldsymbol{\gamma}\cdot \boldsymbol{P}$$

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \boldsymbol{\gamma} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The pion decay matrix element

$$<e(\mu)\nu|\mathscr{L}_{\rm int}|\pi> = <e\nu|j_\mu(\text{lepton})|0><0|J_\mu(\text{hadron})|\pi>\frac{G}{\sqrt{2}}$$

$$= L_\mu <0|J_\mu(\text{hadron})|\pi> \frac{G}{\sqrt{2}}.$$

Since the pion has no spin

$$<0|J_\mu|\pi> = (\text{const})q_\mu/\sqrt{2(2\pi)^3 E_\pi} = m_\pi f_\pi q_\mu/\sqrt{2(2\pi)^3 E_\pi} \quad (f_\pi \text{ is dimensionless}).$$

The only vector available is q_μ, the momentum of the pion. Further, since the pion is pseudo-scalar, only the axial part of J_μ (hadron) contributes.

The pion decay rate is readily expressed in terms of G and f_π as

$$\Gamma_{\pi\to\mu\nu} = \left(\frac{G^2 m_\pi^4}{8\pi}\right)\left(\frac{m_\mu}{m_\pi}\right)\left[1-\left(\frac{m_\mu}{m_\pi}\right)^2\right]^2 m_\mu f_\pi^2.$$

From the measured rate of pion decay, the constant f_π is found to have the value

$$f_\pi = 0.93.$$

For neutron β-decay, the axial vector current takes the form

$$<P|J_\mu(\text{hadron})|n> = \bar{u}(P)[f_A\gamma_\mu + f_P q_\mu]\gamma_5 u(n)$$

where $q = P - n$ and f_A, f_P are the axial and induced pseudo-scalar coupling constants. (Note that the Fermi constant G is an explicit factor in $\mathscr{L}_{\rm int}$ so that $f_A = G_A/G$.)

(Thus far we have ignored the Cabibbo angle which will be taken into account when it is relevant.)

Consider the process $\nu + n \to \mu(e) + p$, shown in the figure below (Fig. 1).

Consider further the above process for a value of momentum transfer $q = \nu - \mu$,

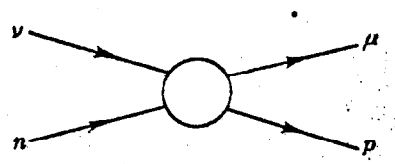

Fig. 1.

which is unphysical and time-like ($q^2>0$) and approximately equal to m_π^2. In this case, we know that the process is dominated by the one-pion pole term shown in the next figure (Fig. 2).

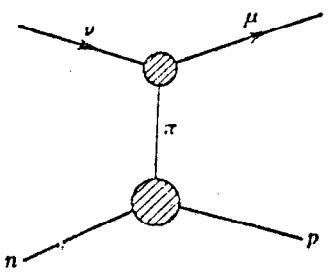

Fig. 2. $q=\nu-\mu=P-n$

This does not mean to imply that such a pole is dominant in the physical region where $q^2<0$ (spacelike). At the unphysical but readily imaginable point $q^2\approx m_\pi^2$, there are two ways of writing the matrix element which are equal.

$$\frac{m_\pi f_\pi q_\mu L_\mu}{q^2-m_\pi^2}\bar{u}(P)\gamma_5 u(n)\sqrt{2}g_{\pi nn}=\bar{u}(P)[f_A\gamma_\mu+f_P q_\mu]\gamma_5 u(n)L_\mu$$
$$+<P|J_\mu^{\text{vec}}|n>L_\mu\,.$$

$\sqrt{2}g_{\pi nn}$ is pion nuclear coupling constant for a charge pion vertex with

$$g_{\pi nn}^2/4\pi=14.6\,,\qquad g_{\pi nn}=13.6\,.$$

Let us further consider the case of forward muons with finite energy loss ($E_\mu\neq E_\nu$); then it is straightforward to show that L_μ is proportional to q_μ, i.e.,

$$L_\mu=Cq_\mu$$

where $C=4\sqrt{\dfrac{E_eE_\nu}{(E_\nu-E_e)^2}}$ and where the muon mass has been neglected. For the forward lepton kinematic situation

$$L_\mu<P|J_\mu^{\text{vec}}|n>=0$$

for a conserved vector current, and we have at the pion pole $q^2\approx m_\pi^2$

$$\frac{m_\pi^3 f_\pi}{q^2-m_\pi^2}\sqrt{2}g_{\pi nn}=[2Mf_A+q^2f_P]\,.$$

The right-hand side of the above equation comes from using the Dirac equation and applying

$$q_\mu \quad\text{to}\quad \bar{u}(P)[\gamma_\mu f_A+q_\mu f_P]\gamma_5 u(n)$$

which is equivalent to considering the matrix element of the divergence

of the axial current matrix element

$$q_\mu \langle P|J_\mu^{axial}|n\rangle = i\partial_\mu \langle P|J_\mu^{axial}|n\rangle .$$

Now the G-T relation is gotten by the following assumption: (The assumption of pole dominance of the divergence of the axial current PDDAC). Let us suppose that the relation valid near the pole $q^2=m_\pi^2$

$$[2Mf_A+q^2f_P]=\frac{\text{residue}}{q^2-m_\pi^2} ;$$

$$\text{residue}=m_\pi^3 f_\pi \sqrt{2}\, g_{\pi nn}$$

remains approximately true all the way to $q^2=0$!

Since there are no particles lighter than the pion, this could be a good approximation. In that case we would have

$$2Mf_A=-m_\pi f_\pi \sqrt{2}\, g_{\pi nn}$$

which is the G-T relation.

Inserting the value of f_π from this expression, one predicts the pion decay rate within 10%, which is a posteriori justification of the PDDAC hypothesis and gives us an idea on what the errors might be in other applications.

Another useful procedure for employing the PDDAC hypothesis due to Feynman is the following: The divergence of the axial current $\partial_\mu A_\mu$ has the same quantum numbers as the pion, i.e., $J^P=0^-$, odd G-parity and isovector. Hence, let us assume that $\partial_\mu A_\mu$ is proportional to the interpolating pion field, i.e.,

$$\partial_\mu A_\mu(x)=a\pi(x)$$

where $\pi(x)$ is the pion field. Then consider $A_\mu{}^+$

$$\begin{aligned}\langle P|\partial_\mu A_\mu{}^+|n\rangle &= [2Mf_A+q^2f_P]\bar{u}(P)\gamma_5 u(n)\\ &= a\langle P|\pi^+|n\rangle = \frac{a}{q^2-m_\pi^2}\langle P|j_\pi{}^+|n\rangle\\ &= \frac{a}{q^2-m_\pi^2}\sqrt{2}\,\bar{u}(P)\gamma_5 u(n) g_{\pi nn}(q^2) .\end{aligned}$$

Again assuming the PDDAC hypothesis

$$a^\pm=-\frac{m_\pi^2 M\sqrt{2}f_A}{g_{\pi nn}(0)} \quad \text{and} \quad a^0=-\frac{m_\pi^2 M f_A}{g_{\pi nn}(0)}$$

the factor $\sqrt{2}$ being absent in the π^0 case because

$$\langle P|A_\mu{}^+|n\rangle = \sqrt{2}\,\langle P|A_\mu{}^0|P\rangle$$

by iso-spin rotation.

The principle difference between the two methods is that in the latter the coupling constant $g_{\pi nn}$ defined as the strength of the matrix

element $<P|J_\pi|n>$ evaluated at the point $g_{\pi nn}(0)$. The constant f_π evaluated by the Goldberger-Treiman relation yields a value 0.83 to be compared with the experimental value of 0.93 which indicates about an 11% error.

3. Charges, Noether Theorem

1) Charges

Associated with every vector current operator $J_\mu(\boldsymbol{x}, t)$ there is a *scalar* quantity known as the charge operator $Q(t)$ which is found by integrating the time component of $J_\mu(J_0)$ over all space, i.e.

$$Q(t)=\int d^3x J_0(\boldsymbol{x}, t) .$$

That Q is a scalar quantity can be made manifest by writing the above equation as a four dimensional scalar product. To this end the notion of a surface element in four space is introduced. Consider any space-like surface rather than the flat space surface defined by fixed t as above. (The reader is reminded that on a space-like surface there are no two points which can be connected by a light signal.) The normal n_μ to a space-like surface is time-like and can be used to define a vector surface element in a similar manner to what one usually does in three dimensions. Thus an element of surface $d\sigma_\mu=n_\mu d\sigma$ defines a four-vector surface element with components in the t, x, y, z directions respectively as

$$\{dx\ dy\ dz,\ dt\ dy\ dz,\ dt\ dz\ dx,\ dt\ dx\ dy\} .$$

In the case where n_μ has no space components and only a time component, $d\sigma_\mu$ has only a time component $dxdydx=d^3x$. If we write for the charge operator

$$Q(\sigma)=\int_\sigma d\sigma_\mu J_\mu(x)$$

then the scalarity of $Q(\sigma)$ is clear. The charge operator $Q(\sigma)$ may have a value which depends on the particular surface σ. It is only when the current J_μ is conserved, i.e., $\partial_\mu J_\mu=0$ that $Q(\sigma)$ does not depend on the choice of surface. This last statement follows from the four-dimensional Gausses theorem which states

$$\int_v d^4\alpha\, \partial_\mu J_\mu(x)=\int_S d\sigma_\mu J_\mu$$

where s is the surface enclosing the volume v. By choosing the currents

to vanish at large spacial distances and choosing the volume to be a four dimensional rectangle whose top and bottom are at t' and t while the sides are at infinite special points yields by Gausses theorem that

$$Q(t')-Q(t)=0 .$$

For non conserved current this is not generally true and $Q(\sigma)$ varies from surface to surface. (The reader may think of $Q(\sigma)$ in the non-conserved case like a scalar temperature which varies from point to point.)

Thus for conserved currents the matrix elements of $Q(t)$ will be a numerical constant independent of the time while for non-conserved current the matrix elements of $Q(t)$ will yield a scalar function varying with time.

2) Noether Theorem

We consider next an extension of Noether's theorem for classical fields (1921) which is applicable for quantum fields and which shall prove extremely useful.

For any operator $0(t, \alpha)$ where t is the time and α any other variable the general quantum (Heisenberg) equation of motion states that

$$\frac{\partial}{\partial t} 0(t, \alpha)=-i[0(t, \alpha),\ H(t)]$$

where $H(t)$ is the total Hamiltonian $H(t)=\int H(x, t)d^3x$. The theorem looks similar to the Heisenberg equation of motion except that it deals with the Hamiltonian density $H(x, t)$ and with $0(t, \alpha)$ replaced the charge operator $Q(t)$. It states that under certain conditions (to be stated below)

$$\partial_\mu J_\mu(x, t)=i[Q(t),\ H(x, t)] \qquad \text{Noether Theorem}$$

where

$$Q(t)=\int d^3x\, J(x, t) .$$

If we integrate the Noether Theorem over all space and use the fact that integration by parts yields

$$\int \nabla \cdot J d^3x=0$$

then we recover the Heisenberg equations of motion for $Q(t)$. Thus the Noether Theorem can be thought of as the unintegrated form of the Heisenberg equation of motion.

The precise conditions under which the Noether Theorem is true are complicated but a sufficient condition which suffices for the purposes of these lectures is that the commutator $[Q, H(x, t)]$ be a scalar which is certainly required if it is to be related to $\partial_\mu J_\mu$ which is generally a scalar operator. Appendix 1 contains a rigorous proof of the Noether Theorem based on Lorentz invariance, causality and the chiral $SU(3)\times SU(3)$ algebra between currents and charges.

A rough proof of the theorem is as follows: Consider the Heisenberg equation of motion expressed as

$$\int d^3x \left\{ [H(\boldsymbol{x}, t),\ Q(t)] - \frac{\partial}{\partial t} J_0(\boldsymbol{x}, t) \right\} = 0 .$$

The quantity under the integral sign is either zero or the three divergence of some current J_μ'.

$$[H(\boldsymbol{x}, t),\ Q(t)] - \frac{\partial}{\partial t} J_0(\boldsymbol{x}, t) = -\nabla \cdot J'(\boldsymbol{x}, t) .$$

But if $[H(\boldsymbol{x}, t), Q(t)]$ is to be a Lorentz scalar then J' must be identified with J in order that

$$\frac{\partial}{\partial t} J_0(\boldsymbol{x}, t) - \nabla \cdot J'(\boldsymbol{x}, t)$$

be a scalar.

Now we have seen that whenever there are pions around the divergence of the axial current $\partial_\mu A_\mu$ appears and through the Noether Theorem this divergence gets related to equal time commutators. This relation is at the heart of current algebra relations and soft pion theorems.

Let us check the Noether Theorem in a simple case where we know the answer. In the absence of electromagnetic interactions the isospin current V_μ is conserved. In particular we have for the charged components of the current

$$\partial_\mu V_\mu^{\pm}(x) = 0 .$$

But we also know that for chaged objects when the electromagnetic interaction is turned on that gauge invariance requires the minimal electromagnetic Hamiltonian be determined by the substitution

$$\partial_\mu \to \partial_\mu \mp e a_\mu$$

where a_μ is the electromagnetic field and where the sign factor depends on sign of the charge of the current. Thus, in the presence of electromagnetism

$$\partial_\mu V_\mu^{\pm}(x) = \pm e a_\mu V_\mu^{\pm} .$$

According to the Noether Theorem the divergence can also be computed by taking the commutator of the associated charge

$$Q^{\pm}(t)=\int d^3x\, V_0^{\pm}(\boldsymbol{x}, t)$$

and commuting it with the Hamiltonian.

With the electromagnetic interaction turned off isospin is a good quantum number and the Hamiltonian of the world is an isoscalar. Hence the charges $Q^{\pm}$ commute with the non electromagnetic part of the Hamiltonian. However, when the electromagnetic interactions are included this is no longer true.

The interaction Lagrangean responsible for electromagnetic interactions has the usual form

$$L_e=ea_\mu J_\mu=-H_e\,.$$

(Since there are no derivatives in L_e it is also-H_e.) J_μ is the electromagnetic current which is neutral, since Hamiltonians are always neutral, and is a sum of an isovector part and an isoscalar part

$$J_\mu=J_\mu^s\neq J_\mu^v=\bar{\psi}_p\gamma_\mu\psi_p+\psi_{\Sigma^+}\gamma_\mu\psi_{\Sigma^+}+\cdots\,.$$

The Noether Theorem then states that

$$\partial_\mu V_\mu^{\pm}(x)=-ea_\mu(x)\,[Q^{\pm}(t),\ J_\mu^v(\boldsymbol{x}, t)]$$

(Note $Q^{\pm}$ and J_μ^v are evaluated at the same time t and that J^s does not contribute to the commutator.)

The electromagnetic field a_μ commutes with $Q^{\pm}$ at equal times but not at different times. This is an assumption which is physically motivated by causality arguments. Since it takes a finite time for any signal to propagate, the electromagnetic field at time t cannot feel the influence of a charge Q at exactly the same time. Therefore the operators must commute.

4. Applications

1) Kroll-Ruderman Theorem

We begin our study of soft pion theorems by considering a relation first derived by Kroll and Ruderman which is purported to relate the threshold photo-pion production to the pion nucleon coupling constant. The derivation given here will use the ideas of PDDAC but in fact the theorem does not really require this assumption and can be shown to follow just from gauge invariance. The gauge invariance argument was used by Kroll and Ruderman in their original derivation but it is

photopion production.

Thus

$$S_{\gamma p\to\pi^+ n}=\frac{1}{a}\int d^4x\, e^{iq\cdot x}(\Box+m_\pi^2)<n|[\partial_\mu A_\mu^+-e\,a_\mu A_\mu^+]|\gamma p>_{\rm in}.$$

The derivative appearing above can be made to act on an exponential factor by using translational invariance and the overall conservation of momentum. This allows the S-matrix element to be written as

$$S_{\gamma p\to\pi^+ n}=\frac{1}{a}\int d^4x\, e^{iq\cdot x}(-q^2+m_\pi^2)<n|[q_\mu A_\mu^+-ea_\mu A_\mu^+]|\gamma p>_{\rm in}.$$

In general this last expression is not any closer to a possible evaluation than the original equation. However, in its present form the above equation can be used to define an analytic expression for $S_{\gamma p\to\pi^+ n}$ for all values of the pion momentum q. Then what can be done is to give value for this S-matrix element evaluated at the unphysical point $q_\mu=0$ for all components. From then on appeal is made to hope and reason that this is a small region of extrapolation and that the resultant S-matrix may still be closely related to the precise physically observable matrix element.

If there are no poles going as $1/q$ in the limit of $q\to 0$ then the first term inside the matrix element will not contribute and

$$\lim_{q\to 0} S_{\gamma p\to\pi^+ n}=-e\frac{m_\pi^2}{a}\int d^4x<n|a_\mu A_\mu^+|\gamma p>_{\rm in}.$$

Furthermore to first order in the fine structure content we can replace the photon field a_μ by $a_\mu^{\rm in}$ since the corrections to $a_\mu^{\rm in}$ come from making pair and higher mass states all of which require one more power of $\alpha\approx 1/137$. Thus

$$\left.\begin{aligned}\lim_{q\to 0} S_{\gamma p\to\pi^+ n}&=\frac{-em_\pi^2}{a}<n|A_\mu^+|P>e_\mu\\&=\frac{-em_\pi^2}{a}f_A\bar{u}(n)\gamma_\mu\gamma_5 u(p)e_\mu\\&=\frac{-eg_{\pi NN}}{2M}\bar{u}(n)\gamma_\mu\gamma_5 u(p)e_\mu\end{aligned}\right\}\times\delta^{(4)}(-p_n+p_p+p_\gamma)$$

where e_μ is the photon linear polarization vector. This is the K-R theorem. The factors involving $f_A=G_A/G$ cancelled out in the last equation and the result is independent of the PDDAC although we used this assumption in the derivation considered here.

Calculating the cross-section from this matrix element gives

much more complicated than the proof using PDDAC. However, in the following we shall see that the K-R theorem does not agree with experiment and that the current algebra approach with PDDAC can be used to mend this disagreement.

The reaction under consideration is

$$\gamma p \to \pi^+ n \ ,$$

$$K p_1 \to q p_2 \ \text{(momenta)} \ .$$

Thus we want the S-matrix element

$$S_{\gamma p \to \pi^+ n} = {}_{\text{out}}\langle \pi^+ n | \gamma p \rangle_{\text{in}} \ .$$

If we use the reduction formula as developed in Section 2 then the pion may be "reduced" out and the above expression becomes

$$S_{\gamma p \to \pi^+ n} = \int d^4x \, e^{iq\cdot x} (\Box + m_\pi^2) \langle n | \pi^+(x) | \gamma p \rangle_{\text{in}} \ .$$

The pion field $\pi^+(x)$ is related to the divergence of the axial current as expressed in Section 2

$$\partial_\mu A_\mu{}^+ = a\pi^+ \ .$$

This is true in the absence of electromagnetic interactions. However, this equation has to be modified in the presence of electromagnetism. According to the Noether Theorem there is the additional term which comes from the commutator

$$[Q_A{}^+, \ H_{\text{elec}}(x)] \ .$$

The electromagnetic Hamiltonian is as before

$$\begin{aligned} H_{\text{elec}}(x) &= e a_\mu(x) [J_\mu{}^s + J_\mu{}^v] \\ &= e a_\mu(x) [J_\mu{}^s + V_\mu{}^3] \ . \end{aligned}$$

Thus in the presence of electromagnetism

$$\begin{aligned} \partial_\mu A_\mu{}^+ &= a\pi^+ + e a_\mu [Q_A{}^+, \ V_\mu{}^3] \\ &= a\pi^+ - e a_\mu A_\mu{}^+ \end{aligned}$$

where we have evaluated the commutator

$$(-)[Q_A{}^+, \ V_\mu{}^3] = A_\mu{}^+ \ .$$

That this is indeed the correct commutator is supported by the fact that the result for $\partial_\mu A_\mu{}^+$ given above is just what one finds by the usual minimal substitution $\partial_\mu \to \partial_\mu - e a_\mu$.

The divergence condition on A_μ then allows us to "solve" for the pion field and insert this expression in the S-matrix element for the

$$\lim_{q,K\to 0}\frac{|K|}{|q|}\frac{d\sigma}{d\Omega\,\mathrm{c.m.}}(\gamma p\to\pi^+ n)=\left(\frac{\alpha}{2}\right)\left(\frac{1}{M_N{}^2}\right)\left(\frac{g_{\pi NN}^2}{4\pi}\right).$$

This is a factor of 2 larger than the experimentally lowest momentum value measured. We consider how to use the current algebra methods to resolve this discrepancy in Appendix 3 and consider now the problem of the term in the photoproduction amplitude which has the form

$$\int e^{iq\cdot x}(-q^2+m_\pi{}^2)<n|q_\mu A_\mu{}^+|\gamma p>_{\mathrm{in}}.$$

What has been stated above was that this term vanished in the limit of $q\to 0$. It is this feature which permits something useful to be calculated from the S-matrix element. In general for finite q the above expression is just unknown as the original photoproduction amplitude but because of the explicit factor of q in the term it could go to zero. (Note that one takes all components of q_μ to zero.) The statement "could go" must be emphasized because it might happen that the matrix element $<n|A_\mu|\gamma p>$ had a pole in q going as q^{-1}. There is a general procedure which may be applied to all cases which can test for the existence of such poles which is the following:

(1) Such poles can only occur when the mass of a possible intermediate state is the same as the initial or final mass. This is analogous to the case of a vanishing energy denominator in non-relativistic perturbation theory and the proof, omitted here, would follow along similar lines.

(2) To check whether there is a pole contribution the special intermediate states has to be put in by hand and the Feynman diagram explicity calculated.

For the photo pion production there is only one such intermediate state which is indicated by the diagram below (Fig.3). The intermediate proton has the same mass as the initial proton (naturally) and this could give a term of order q^{-1}.

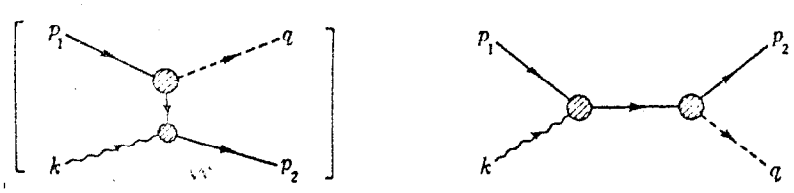

Fig. 3.

If we calculated the proton intermediate state explicitly then this matrix element takes the form

$$<n|A_\mu|\gamma p>\to M_\mu=\bar{u}(n)\frac{\gamma_\mu\gamma_5(\gamma\cdot n+\gamma\cdot q+m)}{n\cdot q}\gamma\cdot eu(p)$$

where coupling constant factors have been dropped. The nucleon-nucleon axial vector current vertex has only the term $\gamma_\mu\gamma_5$ and no term $q_\mu\gamma_5$ since we are going to pass to the limit as $q\to 0$ and this would be higher order. Similarly only the photon proton charge coupling is taken (no magnetic coupling) since the magnetic coupling again brings in momentum factors. The term $(n\cdot q)^{-1}$ looks as if there might be a q^{-1} contribution but will actually be cancelled because the numerator will contain at best one power of q. The precise statement however still requires a little more care. We note that when $q\to 0$ if the overall conservation of momentum $p+k=q+n$ is to remain as true as possible for physical neutron and proton then k must also go to zero. Thus we really want to consider $<n|A_\mu|\gamma p>$ both as q and k go to zero. In that case, using $p+k=n+q$ and the Dirac equation yields

$$M_\mu=\bar{u}(n)\gamma_\mu\gamma_5\frac{(-2p\cdot e)}{n\cdot q}u(p)+0(k)\ .$$

Choosing a gauge where the polarization e has only space components and working in the initial proton rest frame so that $p\cdot e=0$ we get $q_\mu M_\mu=0$ to lowest order in q and k.

There was no special way of knowing that $q_\mu M_\mu\to 0$ other than just working it out, although in all known examples it always vanishes.

2) The Callan-Treiman Relation

Callan and Treiman derived a relation between the decays $K\to\mu\nu$ and $K\to\mu\nu\pi$ or $K\to e\nu\pi$ where the pion is considered soft in a similar manner to the Kroll-Ruderman theorem.

The decay $K\to\mu\nu$ is described by a constant f_K defined by

$$M_{K^+\to\mu^+\nu}=\frac{G}{\sqrt{2}}m_K f_K K_\mu\bar{u}(\mu)\gamma_\mu(1+\gamma_5)u(\nu)$$
$$=\frac{G}{\sqrt{2}}<0|A_\mu{}^-|K^+>L_\mu$$

where K_μ is the K-meson momentum and L_μ is just the lepton spinor factor.

The decay $K\to\mu\nu\pi$ has the matrix element (consider $K^+\to\mu^+\nu\pi^0$ for definitness)

$$M_{K\to\mu\nu\pi}=\frac{G}{\sqrt{2}}<\pi^0|V_\mu{}^-|K^+>L_\mu=<L^+\pi^0|K^+>$$
$$=\frac{G}{\sqrt{2}}[F_+(K+p)_\mu+F_-(K-p)_\mu]\bar{u}(\mu)\gamma_\mu(1+\gamma_5)u(\nu)$$

where K and p are the K-meson and π-meson momenta. The form

factors F_+ and F_- are functions of the invariant momentum transfer $q^2=(K-p)^2$ and the meson masses, i.e.

$$F_\pm=F_\pm(m_K^2,\ m_\pi^2,\ q^2)\ .$$

Following the same procedure as was used in the photoproduction example we reduce the pion in the $K^+\to\mu^+\nu\pi^0$ decay, hence

$$M_{K\to\mu\nu\pi}=\int d^4x e^{ip\cdot x}(\Box+m_\pi^2)<L^+|\pi^0(x)|K^+>\ .$$

The pion field is then related to the divergence of the axial current by

$$\partial_\mu A_\mu{}^0(x)=a_0\pi^0(x)-[Q_5{}^0,\ H_W(x)]$$

or

$$\pi^0(x)=\frac{1}{a_0}\{\partial_\mu A_\mu{}^0(x)+[Q_5{}^0,\ H_W(x)]\}\ ,\qquad a_0=\frac{m_\pi{}^2Mf_A}{g_{\pi NN}}\ .$$

Thus we need the commutator $[Q_5{}^0, H_W(x)]$. Fortunately we know a good phenomenological form of $H_W(x)$ as given by the Cabibbo theory and which appears to be in reasonable agreement with experiment. In this case, the leptonic decay Hamiltonian is given by the sum of two parts; a strangeness conserving part H_1 and a strangeness changing part $H_{1/2}$ where the subscripts refer to the isospin. Thus

$$H_W=H_1+H_{1/2}$$

where

$$H_1=\cos\theta[L_\mu{}^+(V_\mu{}^-+A_\mu{}^-)+L_\mu{}^-(V_\mu{}^++A_\mu{}^+)]G/\sqrt{2}$$

and the currents $V_\mu{}^\pm$ and $A_\mu{}^\pm$ transform as isovectors. Some notation is necessary to clarify this point which we write as $V_\mu{}^\pm(\pi^\pm)$, $A_\mu{}^\pm(\pi^\pm)$, the appearance of $\pi^\pm$ is to emphasize the isospin one strangeness conserving aspect. The strangeness changing part can then be written as

$$H_{1/2}=\sin\theta L_\mu{}^+[V_\mu{}^-(K^-)+A_\mu{}^-(K^-)]+L_\mu[V_\mu{}^+(K^+)+A_\mu{}^+(K^+)]G/\sqrt{2}\ .$$

The $K^\pm$ appearing in parenthesis indicates the isospinor and strangeness changing quality.

For the decays considered here it is $H_{1/2}$ which is responsible and the commutator $[Q_5{}^0, H_{1/2}]$ is required. Just as in the previous example on photoproduction the leptonic part is not involved in the commutator and it may be factored out. However, there is both L^+ and L and since there is positively charged lepton in the first state we want L^+. The convention chosen here shall be that the currents transform like creation operators, i.e.

$$L^+|0>=|L^+>\ .$$

One could choose the opposite convention just as well but the convention chosen here is compatible with the usual choice of signs of the commutation rules.

The commutator that is required is then

$$[Q_5^0,\ (V_\mu^-(K^-)+A_\mu^-(K^-))]\ .$$

Using the fact that the algebra of charges and currents is to be closed the above commutator must then be of the form $[V_\mu^-(K^-)+A_\mu^-(K^-)]$ on grounds of charge and strangeness. To get the correct coefficient we can integrate the commutator

$$[Q_5^0,\ (V_\mu^-(K^-)+A_\mu^-(K^-))]=(\text{coeff})[V_\mu^-(K^-)+A_\mu^-(K^-)]$$

and then sandwich the charge part between a $<K^-|$ state and the vacuum. In which case the coefficient is readily seen to be $-1/2$.

Thus

$$M_{K\to\mu\nu\pi}=\int d^4x(-p^2+m_\pi^2)\frac{e^{ip\cdot x}}{a_0}<L^+\Big|[p_\mu A_\mu^0$$
$$+\frac{G}{\sqrt{2}}L_\mu^+[Q_5^0,\ V_\mu^-(K^-)+A_\mu^-(K^-)]\Big|K^+>$$

to proceed further we pass to the limit of $p\to 0$ and note that there are no pole terms at all so that the $p_\mu A_\mu^0$ term will vanish. In fact there are no one particle states of the type which appeared in the photoproduction problem. If we note further that to lowest order in the weak coupling G the leptons may be treated as free particles and that the matrix element $<0|V_\mu|K>=0$ by parity then inserting the commutator gives

$$\lim_{p\to 0} M_{K\to\nu\mu\pi}=\frac{m_\pi^2}{a_0}\frac{G}{\sqrt{2}}\left(-\frac{1}{2}\right)<0|A_\mu^-|K^+>L_\mu$$
$$=-\frac{m_\pi^2}{2a_0}\frac{G}{\sqrt{2}}[m_K f_K K_\mu L_\mu]\ .$$

But on the other hand from the definition of the matrix element $M_{K\to\mu\nu\pi}$ and the form factors F_+ and F_- we have

$$\lim_{p\to 0}\ [F_+(p+K)_\mu+F_-(K-p)_\mu]L_\mu\frac{G}{\sqrt{2}}=[F_++F_-]\frac{G}{\sqrt{2}}L_\mu K_\mu\ .$$

Combining these equations we have the Callan-Treiman relation

$$\frac{m_K g_{\pi NN}}{2Mf_A}f_K=[F_+(m_K^2,\ 0,\ 0)+F_-(m_K^2,\ 0,\ 0)]\ .$$

The arguments of the form factors are displayed explicitly because the

$K \to \mu\nu\pi$ matrix element has been evaluated at $p \to 0$ hence the pion has zero mass and zero momentum.

To compare with experiment one would like to have the form factors F_+ and F_- at the extrapolated points indicated above. Since this is experimentally not possible one can try to compare with the data on the form factors and hope that the extrapolation will be reasonable. If we compare with the Trilling report of the Argonne Weak Interaction Conference (1965), Callan and Treiman find with $f_K = 0.070 \pm 0.001$ that

$$\frac{2Mf_A}{m_K g_{\pi NN}}[F_+ + F_-] = 0.074 \pm 0.014$$

which indicates rather good agreement.
[The above comparison used the value $\xi = F_-/F_+ = +0.41^{+0.27}_{-0.22}$ as determined by comparing the rates of K_{e_3} and K_{μ_3}. Polarization experiments in K_{μ_3} decay indicate that ξ is more likely in the neighborhood of -1 which would set the Callan-Treiman relation in disagreement with experiment. In that case it is most likely that the extrapolation to the physical region would be incorrect.]

3) The K_{e_4} Decays

A very nice application of soft pion theorems is given by the decay modes

$$K^+ \to \pi^+\pi^- e^+\nu\,, \qquad K^0 \to \pi^0\pi^- e^+\nu\,,$$
$$K^+ \to \pi^0\pi^0 e^+\nu\,, \qquad \bar{K}^0 \to \pi^0\pi^+ e^-\bar{\nu}\,.$$

Since the energy release is not large in these decays, we employ the soft pion theorem on each of the final pions separately. This yields enough information to relate all the decay parameters at the extrapolated point to K_{e_3} decay and K_{μ_2} decay. Assuming that the amplitudes do not vary by very much from the extrapolated points to the physical region, the K_{e_4} decay rate can be calculated and for the $\pi^+\pi^-$ mode is in good agreement with experiment. (The other channels have not yet been measured.)

The original theoretical work on the K_{e_4} decays was done by S. Weinberg.

Consider first the decay

$$K^+(K) \to \pi^+(p)\pi^-(q)\overbrace{e^+\nu}^{L^+}$$

where the momentum labels have been placed in parentheses after the particle. The decay matrix element can be expressed on the form

$$<L^+\pi^+\pi^-|K^+> = \frac{G}{\sqrt{2}} L_\mu[F_1(p+q)_\mu + F_2(p-q)_\mu + F_3(K-p-q)_\mu$$
$$+ F_4\varepsilon_{\mu\nu\sigma\tau}p_\mu q_\nu K_\tau] ,$$
$$L_\mu = \bar{U}(e)\gamma_\mu(1+\gamma_5)U(\nu) .$$

In general, the various form factors F_{1-4} depend on all the possible invariants in the $K\pi\pi$ system, i.e.,

$$F_i = F_i(m_K^2, m_{\pi^+}^2, m_{\pi^-}^2, K\cdot q, K\cdot p, q\cdot p) , \qquad i=1, 2, 3, 4 .$$

The form factor F_4 will be dropped in the considerations here since it multiplies a second order tensor in pion momenta while the other form factors multiply first order tensors. Furthermore, by use of the Dirac equation on the lepton spinor F_3 will come in proportional to the electron mass and will be negligible in its contribution to the rate.

Reducing the π^+ then, as usual, we have

$$<L^+\pi^+\pi^-|K^+> = \int e^{ip\cdot x}(-p^2+m_\pi^2)\frac{1}{a} <L^+\pi^-|\{p_\mu A_\mu^- + [Q_5^-, H_W]\}|K^+> .$$

If there are no poles, a point we return to below, the limit as $p \to 0$ may be taken and the $p_\mu A_\mu$ term will vanish. (Note that since the π^+ was among the *final* particles the operator is Q_5^- rather than Q_5^+.) The weak Hamiltonian

$$H_W = \frac{G}{\sqrt{2}}[L_\mu^+ J_\mu^- + L_\mu^- J_\mu^+]$$

will contribute only a part $G/\sqrt{2}\, L_\mu^+ J_\mu^-$ since the leptons carry a positive charge. However, the commutator

$$[Q_5^-, J_\mu^-] = 0$$

since there are no doubly charged currents and the matrix element must vanish in this limit,

$$\lim_{p\to 0} <\pi^+\pi^- L^+|K^+> = 0$$
$$= \frac{G}{\sqrt{2}} L_\mu[(F_1 - F_2)q_\mu + F_3(K-q)_\mu] .$$

The coefficients of K_μ and q_μ must vanish separately giving

$$F_1(m_K^2, 0, m_\pi^{-2}, K\cdot q, 0, 0) = F_2(m_K^2, 0, m_\pi^{-2}, K\cdot q, 0, 0)$$

and $F_3(m_K^2, 0, m_\pi^2, K\cdot q, 0, 0) = 0$.

Another condition on the form factors may be found by reducing the π^- rather than the π^+. In this case, we have

$$<L^+\pi^+\pi^-|K^+> = \int e^{iq\cdot x}(-q^2+m_\pi^2)\frac{1}{a}<L^+\pi^+|\{q_\mu A_\mu^+ + [Q_5^+, H_W]\}|K^+> .$$

In the limit as $q\to 0$ and if pole terms do not arise (as we show below) the matrix element is proportional to the commutator

$$[Q_5^+, J_\mu^-]=[Q^+, J_\mu]$$

which does not vanish. (Again, we use the fact that $[V+A, V-A]=0$.) The commutator may be evaluated between the π^+ and K^+ states as

$$\begin{aligned}<\pi^+|[Q^+J_\mu^- - J_\mu^- Q^+]|K^+> &= <\pi^+|Q^+J_\mu^-|K^+> \\ &= \sqrt{2}<\pi^0|J_\mu^-|K^+> \\ &= \sqrt{2}[F_+(K+p)_\mu + F_-(K-p)_\mu]\end{aligned}$$

where F_+ and F_- are the two form factors of the K_{e_3} decay.

Thus, we have in the $q\to 0$ limit

$$(F_1+F_2)p_\mu + F_3(K-p)_\mu = \frac{m_\pi^2}{a}\sqrt{2}[F_+(K+p)_\mu + F_-(K-p)_\mu]$$

or

$$F_1(m_K^2, m_\pi^2, 0, 0, K\cdot p, 0) + F_2(m_K^2, m_{\pi^+}^2, 0, 0, K\cdot p, 0) = \frac{2\sqrt{2}\,m_\pi^2}{a}F_+(m_K^2, m_{\pi^+}^2, K\cdot p)$$

and

$$(F_+ + F_-)\sqrt{2}\,m_\pi^2/a = F_3(m_K^2, m_{\pi^+}^2, 0, 0, K\cdot p, 0) .$$

The limit of p and q going to zero are compatible with F_1 and F_2 being smooth functions of their respective variables but F_3 might not be because of the one K-meson intermediate state which gives rise to a pole when both p and q vanish.

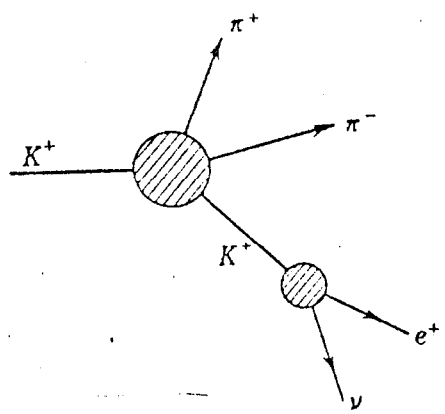

Fig. 4. Pole contribution to F_3 of K_{e_4} decay.

However, for the conditions $p \to 0$ and $q^2 = m_\pi^2$ or $q \to 0$, $p^2 = m_\pi^2$ which are the two limits taken above the $K\pi$ elastic scattering vanishes. This is because of the Adler consistency condition which is merely a statement that $\lim_{p \to 0} p_\mu \langle a|A_\mu|b\rangle = 0$ provided that at least one of the momenta a_μ or b_μ is continued thus allowing the overall δ-function of momentum conservation. Thus the pole part of F_3 does not contribute at the two limits. If F_3 is expanded as a pole term plus a constant this constant must vanish since F_3 is zero at one limit. This means that F_3 is then zero at the $q \to 0$ limit and as a result we have that

$$F_+(m_K^2, m_\pi^2, p\cdot K) = -F_-(m_K^2, m_\pi^2, p\cdot K)$$

which is for their respective physical values. In addition the form factors F_+, F_- are approximately constant in the physical region of K_{l_3} decay if we take F_1 and F_2 as approximately constant in their respective variables in going from the physical region of K_{l_4} decay to the $p \to 0$ and $q \to 0$ limits.

The relation $F_+ = F_-$ in the physical region is not compatible with the Callan-Treiman relation at the point $p \to 0$ if we assume that both F_+ and F_- are smooth functions in all their variables. Since F_+ is related to F_1 and F_2 which are assumed approximately constant the non-smooth behavior can only be in F_-. A simple expression for F_+ and F_- which exhibits all the desired properties is that $F_+(m_K^2, p^2, K\cdot p) = a_0$ and $F_-(m_K^2, p^2, K\cdot p) = [-a_0 + (F_K/F_\pi)(p^2 - m_\pi^2)/m_\pi^2]$ where a_0 is a constant.

To see how good an approximation it is to have $F_1 = F_2 =$ constant we can compute the rate for K_{l_4} decay from the equation $F_1 = F_2 = (\sqrt{2}\, m_\pi^2/a) F_+(m_K^2, m_\pi^2, K\cdot p)$ in which case the experimental value is

$$(F_1/\sqrt{2}) = 1.26 \pm 0.26$$

while the theoretical value from the above equation is

$$F_1/\sqrt{2} = 0.85 \pm 0.05$$

in qualitative agreement.

Appendix 1.

Complete Proof of the Divergence Conditions and the Noether Theorem

(Taken from Physical Review Article of
S. M. BERMAN and Y. FRISHMAN)

In this Appendix, we show that Lorentz invariance, locality and the $SU(3)\times SU(3)$ commutators between charges and charge densities, coupled with PCAC and the usual electromagnetic and weak Hamiltonians, allow divergence equations for vector and axial-vector currents V_μ and A_μ of the form[1]

$$\partial^\mu V_\mu{}^\alpha = -ieb^\alpha a^\mu V_\mu{}^\alpha + GL^{\mu\gamma} C^\alpha_{\beta\gamma}(V_\mu{}^\beta - A_\mu{}^\beta)\ , \tag{1a}$$

$$\partial^\mu A_\mu{}^\alpha = a\pi^\alpha - ieb^\alpha a^\mu A_\mu{}^\alpha - GL^{\mu\gamma} C^\alpha_{\beta\gamma}(V_\mu{}^\beta - A_\mu{}^\beta) \tag{1b}$$

where the indices α, β, γ refer to internal degrees of freedom (γ is a charge index; for α, β, see Ref. 1), a_μ and L_ν are the electromagnetic field and lepton current, respectively, b^α and $C^\alpha_{\beta\gamma}$ are numerical constants, and π^α the pion fields.

The low energy theorems involving soft pions follow simply from Eq. (1b). For example, consider the decay $K\to\pi e\nu$, with the S-matrix element ${}_{\text{out}}\langle\pi e\nu|K\rangle_{\text{in}}$. Reducing the pion and replacing the interpolating pion field by

$$\pi^\alpha = \frac{1}{a}\{\partial^\mu A_\mu{}^\alpha + GL^{\mu\gamma} C^\alpha_{\beta\gamma}(V_\mu{}^\beta - A_\mu{}^\beta)\}$$

leads immediately to the results of Callan and Treiman in the limit where the pion four-momentum vanishes. All low energy theorems follow in a similar manner.

Eqs. (1a)-(1b) were postulated by Veltman[2] as independent of current algebra[3]. We show here that those equations follow essentially from the commutators of charges with charge densities. Thus ETC contains more information than do the "divergence conditions" (1a)-(1b), in case of commutators which yields Schwinger terms. The fact that the low energy theorems may be derived from (1a)-(1b) directly shows that the Schwinger terms in the ETC do not affect the low energy results.[4]

Eqs. (1a)-(1b) are derived in the following manner:
Given a current $j_\mu{}^\alpha(xt)$, we construct its charge $Q^\alpha(t)$ as

$$Q^a(t)=\int d^3x j_0{}^a(xt)\,. \tag{2}$$

Let $H(xt)$ be the energy density. Then

$$\partial^\mu j_\mu{}^a(xt)=i[H(xt),\,Q^a(t)]+\sum_{\substack{n\geq 2\\ m}}\partial^{r_1}\cdots\partial^{r_n}R_{r_1}{}^a{}_{\ldots r_n\underbrace{0\ldots 0}_{m}}(xt) \tag{3}$$

where r_i are 1, 2, 3 (space) indices. This form follows from space-rotation invariance and the requirement that

$$\partial^0 Q^a(t)=i[H,\,Q^a(t)] \tag{4}$$

where H is the total Hamiltonian.[5] The fact that the summation on the right-hand side of Eq. (3) starts from $n=2$ follows from the vector transformation property of $j_\mu{}^a(xt)$ under Lorentz transformations (the generator of a Lorentz transformation in the K direction is $M_{0K}=tP_K-\int d^3x x_K H(xt)$). The summation is usually over a finite number of terms.[6]

Suppose now that

$$H(xt)=H_0(xt)+H_1(xt) \tag{5}$$

such that

$$[H_0(t),\,Q^a(t)]=0 \tag{6}$$

where $H_0(t)=\int d^3x H_0(xt)$. Then, with Eq. (3),

$$\partial^\mu j_\mu{}^a(xt)=i[H_1(xt),\,Q^a(t)]+\sum_{\substack{m\\ n\geq 1}}\partial^{r_1}\cdots\partial^{r_n}\tilde{R}^a_{r_1\ldots r_n\underbrace{0\ldots 0}_{m}}(xt)\,. \tag{7}$$

We now show how to get the "divergence equations" essentially from the charge-charge density commutation relations. Let us start with the case where

$$H_1(xt)=H^{e\cdot m\cdot}(xt)=ej_\mu{}^{e\cdot m\cdot}(xt)a^\mu(xt) \tag{8}$$

and let $Q^a(t)$ be an axial charge. We neglect for the moment other contributions to H_1. We want to calculate the commutator $[H^{e\cdot m\cdot}(xt),\,Q^a_A(t)]$. To this end we note that

$$[Q^a_A(t),\,a_\mu(xt)]=0 \tag{9}$$

and

$$[Q^a_A(t),\,j_\mu{}^{e\cdot m\cdot}(xt)]=b^a A_\mu{}^a(xt)+g_\mu{}^k\sum_{\substack{n\\ m}}\partial^{r_1}\cdots\partial^{r_n}N_{kr_1\ldots r_n\underbrace{0\ldots 0}_{m}}(xt)\,. \tag{10}$$

This form is dictated by space rotation invariance and by

$$[Q_A{}^a(t),\,j_0{}^{e\cdot m\cdot}(xt)]=b^a A_0{}^a(xt)\,. \tag{11}$$

From Eqs. (7)-(10), we get

$$\partial^\mu A_\mu{}^\alpha(xt) = -ieb^\alpha a^\mu A_\mu{}^\alpha(xt) + \{-iea^k(xt) \sum_{\substack{n \\ m}} \partial^{r_1} \cdots \partial^{r_n} N^\alpha_{kr_1\ldots r_n 0\underbrace{\ldots 0}_{m}}(xt) + \sum_{\substack{n\geq 1 \\ m}} \partial^{r_1} \cdots \partial^{r_n} \tilde{R}^\alpha_{r_1\ldots r_n 0\underbrace{\ldots 0}_{m}}(xt)\} . \tag{12}$$

Since the expression in curly brackets cannot be a Lorentz-scalar field, it has to vanish. Thus

$$\partial^\mu A_\mu{}^\alpha(xt) = -ieb^\alpha a^\mu A_\mu{}^\alpha(xt) \tag{13}$$

as the contribution of electromagnetism to the divergence of the $A_\mu{}^\alpha$ current. Similarly, we can calculate the contribution from the weak Hamiltonian, adding to $H_1(xt)$ of Eq. (8) a term $G/\sqrt{2}$ (lepton current) $\times$(hadron current), with the hadron current given as in Ref. 7), and assuming that the hadron charges commute with the lepton current (analogous to Eq. (9) for electromagnetism).[8] The term a π^α in the expression for $\partial^\mu A_\mu{}^\alpha$ is due to the PCAC hypothesis.

Finally, we may further note that each of the two terms in the curly brackets in Eq. (12) must vanish, due to the fact that one is a total divergence, while the other is not. This in turn implies that the Schwinger terms in the commutator $[A_0{}^\alpha(xt), j_K{}^{\alpha\cdot m\cdot}(yt)]$ vanish after the x integration, as follows from Eq. (10), and that $\partial^\mu j_\mu{}^\alpha(xt) = i[H_1(xt), Q^\alpha(t)]$, as follows from Eq. (7).

References for Appendix 1.

1) We consider the divergences of strangeness conserving ($\Delta s=0$) currents only. That means that α in Eqs. (1a)-(1b) corresponds to $\Delta s=0$ only. The index β on the right-hand side, however, includes $\Delta s\neq 0$. If one wants to consider the divergence of a strangeness changing current, one has to include, in the right-hand side, contributions due to the medium-strong Hamiltonian. We also do not consider contributions due to non-leptonic weak decays.

2) M. Veltman, Phys. Rev. Letters **17**, 553 (1966).

3) In Ref. 2), these equations were also written with W-meson fields replacing the lepton currents. It is not clear from that reference what was the formalism employed for the vector fields. This is of importance in applying the reduction technique. See D. G. Boulware and L. S. Brown, Phys. Rev. **156**, 1724 (1967). See also S. G. Brown, Phys. Rev. **158**, 1444 (1967).

4) M. Nauenberg (Phys. Rev. **154**, 1455 (1967)) showed that the electromagnetic contributions to the divergence equations imply commutation relations of the vector charge density with vector and axial currents, with certain Schwinger

terms. D. G. Boulware and L. S. Brown (Phys. Rev. **156**, 1724 (1967)) showed that the weak contributions to the divergence equations, with W-meson fields replacing the lepton currents, lead to commutation relations between vector and axial-vector charge densities with all currents, with certain Schwinger terms.

5) We assume that surface terms at spacial infinity may be neglected, when forming matrix elements.
6) When an infinite number appears, certain relations among the various terms have to hold, in order not to spoil causality.
7) N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963).
8) When W-meson fields are used instead of lepton currents, this no longer holds in general. For example, in a non-abelian gauge formalism (T. D. Lee, S. Weinberg and B. Zumino, Phys. Rev. Letters **18**, 1029 (1967)), we have

$$\left[j_0^{\alpha}(xt),\ W_0^{\beta}(yt) \right] = iC^{\alpha\beta\gamma} \frac{G}{m_0^2} j_0^{\gamma}(xt)\delta^3(x-y) ,$$

where m_0 is the bare mass of the W-meson. However, this introduces G^2 terms in the divergence Eqs. (1a) and (1b), and does not affect lowest order results.

Appendix 2.

Remarks on Isospin Generators and Cabibbo Theory

The Hilbert space of states is defined by the in or out states with definite particle number, momentum, spin, isospin, etc.. In particular, there are the isospin operators I^+, I^-, I^3 which raise, lower and yield the isospin eigenvalue of these states respectively. Invoking a field theoretic idea we make the assumption that these operators can be written as the space integrals of densities. But if these operators are to be only space integrals and to also operate on the asymptotic states then they should be time independent. The charge defined as the integral of the fourth component of a conserved current is just the perfect candidate, i.e. it is a scalar and is time independent. Thus we identify the operators (I^+, I^-, I^3) with the "charges" ($Q_V{}^+$, $Q_V{}^-$, $Q_V{}^3$) formed by integrating the isospin current density

$$I^{\pm}_{3} \equiv Q_V{}^{\pm}_{3} = \int V_0{}^{\pm}_{3}(x, t) d^3x \, .$$

Gell-Mann, Ne'eman Remark on the Cabibbo Theory

The weak current which multiplies the lepton current j_μ to yield the leptonic decay Hamiltonian is, in fact, composed of two separate parts; a part J_μ which transforms as an isovectors ($|J|=1$) and which is responsible for non strangeness changing decays and a part K_μ which transforms as an isospinor ($|J|=1/2$). It is known that these two kinds of decays have different rates and that to emperically incorporate this into the weak Hamiltonian it is written as

$$H_W = \frac{G}{\sqrt{2}} j_\mu{}^+ [a J_\mu{}^- + b K_\mu{}^-] + \text{herm. conj.}$$

$$= \frac{G}{\sqrt{2}} j_\mu{}^+ \mathscr{J}_\mu{}^- + \text{herm. conj.}$$

where

$$\mathscr{J}_\mu{}^- = a J_\mu{}^- + b K_\mu{}^- \, .$$

If $\mathscr{J}_\mu{}^{+,-,3}$ are to form an SU_2 algebra then there results a condition on the constants a and b. The condition is that

$$|a|^2 + |b|^2 = 1 \, .$$

Thus Cabibbo introduced his angle defining .

$$a=\cos\theta, \qquad b=\sin\theta\,.$$

Note that this means there in only one angle and that $\theta_V=\theta_A$ for the vector and axial vector parts. Note further that θ_V and θ_A are phenomenological parameters which are non operator constant quantities which are always the same regardless of any "symmetry breaking effects".

To prove the condition that $|a|^2+|b|^2=1$ some assumption has to be made about the commutator of a K_μ type current with itself. Only one assumption will lead to the above condition which is that the K type charges commute among themselves according to the rules of an SU_3 algebra. We state what the SU_3 result without proof and leave this as an exercise to the reader with a casual familiarity with the subject. We have

$$[Q_{K_\alpha},\ Q_{\bar{K}_\beta}]=[(3/2)\delta_{\alpha\beta}Y+(T_{\beta\alpha}\cdot Q_J)]$$

where Y is the hypercharge operator and where the subscripts α and β refer to the charge state of the K charge, i.e.

$$K_\alpha=\begin{pmatrix} K^+ \\ K^0 \end{pmatrix}, \qquad \bar{K}_\rho=\begin{pmatrix} K^- \\ \bar{K}^0 \end{pmatrix},$$

$$\left.\begin{aligned}
&[Q_{\pi^+},\ Q_{\pi^-}]=2Q_{\pi^0}\,,\\
&[Q_{K^+},\ Q_{K^-}]=Q_{\pi^0}+\tfrac{3}{2}Y\,,\\
&[Q_{\pi^+},\ Q_{K^-}]=Q_{\bar{K}^0}\,,\\
&[Q_{\pi^-},\ Q_{K^+}]=-Q_{K^0}\,,\\
&[Q_{K^+},\ Q_{\bar{K}^0}]=-Q_{\pi^+}\,,\\
&[Q_{K^0},\ Q_{\pi^+}]=Q_{K^+}\,.
\end{aligned}\right\}\quad (T=\sigma)\,.$$

Appendix 3.

Low-Energy Theorem for Pion Photoproduction from the PCAC Hypothesis

by

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1. Introduction

As was first shown by Kroll and Ruderman,[1] the pion photoproduction amplitude at threshold is given, to all orders in the pion-nucleon coupling-constant g, simply by the Born approximation amplitude in the limit as the pion-nucleon mass ratio m_π/M_N approaches zero. For positive pion production from protons, $\gamma+p\to n+\pi^+$, the calculated cross section in the c.m. system gives at threshold,

$$\frac{|k|}{|q|}\frac{d\sigma}{d\Omega_{\text{c.m.}}}(\gamma+p\to n+\pi^+)=\frac{e^2}{4\pi}\frac{g^2}{4\pi}\frac{1}{2M_N{}^2}=23.1\ \mu\text{barns/ster.}$$

where $|k|$ and $|q|$ are the photon and pion c.m. momenta, respectively, $\frac{e^2}{4\pi}\approx\frac{1}{137}$ is the fine structure constant, and $\frac{g^2}{4\pi}\approx 14.4$.

However, the experimental result[2] is

$$\frac{|k|}{|q|}\frac{d\sigma}{d\Omega_{\text{c.m.}}}(\gamma+p\to n+\pi^+)=(15.6\pm 0.5)\ \mu\text{barns/ster.},$$

at threshold, which suggests that corrections to the Kroll-Ruderman theorem of order m_π/M_N may not be neglected.

The proof of Kroll and Ruderman is based essentially on the gauge invariance of the photoproduction amplitude. Their result can also be obtained by relating the pion field to the divergence of the weak axial current through the partially conserved axial current hypothesis[3] (PCAC). We wish to point out here that by using both gauge invariance and PCAC, the first order terms in an expansion of the threshold amplitude in powers of m_π/M_N may also be calculated. The agreement with the experimantal results is then considerably improved.

2. The Low-Energy Theorem

Consider the process $\gamma+p\to n+\pi^+$. The S-matrix amplitude has the form

$$<\pi^+ n_{p'}\,\mathrm{out}|\gamma_k P_p\,\mathrm{in}> = i(2\pi)^4\delta^{(4)}(p+k-p'-q)\mathscr{M}_+$$

where $\mathscr{M}_+ \equiv <n_{p'}|j_{\pi^+}(0)|\gamma_k\ P_p\,\mathrm{in}>$ and $j_{\pi^+}(x)$ is the source of the pion field $\phi_{\pi^+}(x)$, i.e., $(\Box+m_\pi^2)\ \phi_{\pi^+}(x)=j_{\pi^+}(x)$. According to the PCAC hypothesis, including electromagnetic interactions to first order in e,

$$\partial_\mu A_+^\mu(x)+ie\,\mathscr{A}_\mu(x)A_+^\mu(x)=ia\,\phi_{\pi^+}(x) \tag{1}$$

where $A_+^\mu(x)$ is the positive-charged component of the weak axial current, $\mathscr{A}_\mu(x)$ is the electromagnetic potential, and

$$a=\sqrt{2}\,M_N m_\pi^2 F_A(0)/g(0)\ .$$

$F_A(0)\approx 1.18$ is the weak axial coupling constant and $g(0)$ is the off-mass-shell pion-nucleon coupling constant $[g^2(m_\pi^2)/4\pi\approx 14.4]$.

Taking matrix elements of this expression between states $<n_{p'}|$ and $|\gamma_k P_p\,\mathrm{in}>$, we have

$$a\frac{1}{m_\pi^2-q^2}<n_{p'}|j_{\pi^+}(0)|\gamma_k P_p\,\mathrm{in}>=-q^\mu<n_{p'}|A_\mu^+(0)|\gamma_k P_p\,\mathrm{in}>$$
$$+e<n_{p'}|\mathscr{A}_\mu(0)A_+^\mu(0)|\gamma_k P_p\,\mathrm{in}>\ . \tag{2}$$

In the first term on the right side of Eq. (2) we separate out the pion pole contribution to the axial current matrix element,

$$q^\mu<n_{p'}|A_\mu^+(0)|\gamma_k P_p\,\mathrm{in}>=\frac{q^2}{m_\pi^2-q^2}\,\frac{\sqrt{2}\,MF_A(0)}{g(0)}<n_{p'}|j_{\pi^+}(0)|\gamma_k P_p\,\mathrm{in}>$$
$$-q^\mu<n_{p'}|A_\mu^+(0)|\gamma_k P_p\,\mathrm{in}>'$$

where the prime on the second term indicates that the pion pole term has been subtracted. Inserting this in Eq. (2), we then obtain

$$<n_{p'}|j_{\pi^+}(0)|\gamma_k P_p\,\mathrm{in}>\frac{a}{m_\pi^2}=-q^\mu<n_{p'}|A_\mu^+(0)|\gamma_k P_p\,\mathrm{in}>'$$
$$+e<n_{p'}|\mathscr{A}_\mu(0)A_+^\mu(0)|\gamma_k P_p\,\mathrm{in}>\ . \tag{3}$$

To lowest order in e, $\mathscr{A}_\mu(x)=\mathscr{A}_\mu^{\mathrm{in}}(x)$, so that

$$e<n_{p'}|\mathscr{A}_\mu(0)A_+^\mu(0)|\gamma_k P_p\,\mathrm{in}>=e\epsilon_\mu(k)<n_{p'}|A_+^\mu(0)|P_p>+O(e^2)$$
$$=e\epsilon_\mu(k)\,\bar{U}(p')\Big[\gamma^\mu\gamma_5 F_A\Big((q-k)^2\Big)+(q-k)^\mu\frac{2M_N F_A(0)}{(q-k)^2-m_\pi^2}\gamma_5$$
$$+O\Big(\frac{(q-k)^2}{M_N^2}\Big)\Big]U(p)+O(e)^2$$
$$=eF_A(0)\,\bar{U}(p')\Big[\gamma\cdot\epsilon\gamma_5+\frac{2M_N q\cdot\epsilon}{-2q\cdot k}\gamma_5\Big]U(p)+O\Big(\frac{(q-k)^2}{M_N^2}\Big)+O(e^2) \tag{4}$$

assuming $F_A((q-k)^2)=F_A(0)+O\left(\frac{(q-k)^2}{M_N^2}\right)$. Here $\epsilon_\mu(k)$ is the polarization vector of the photon ($k\cdot\epsilon=0$).

Also, by isolating the Born contribution to the first term on the right in Eq. (3), we may write*

$$q^\mu<n_{p'}|A^+(0)|\gamma_k P_p \text{ in}>'=e\bar{U}(p')\Big[\gamma\cdot q\gamma_5\frac{\gamma\cdot p+\gamma\cdot k+M_N}{(p+k)^2-M_N^2}\Big(\gamma\cdot\epsilon-\frac{\kappa_p}{2M_N}\gamma\cdot\epsilon\gamma\cdot k\Big)$$
$$-\frac{\kappa_n}{2M_N}\gamma\cdot\epsilon\gamma\cdot k\frac{\gamma\cdot p'-\gamma\cdot k+M_N}{(p'-k)^2-M_N^2}\gamma\cdot q\gamma_5\Big]U(p)F_A(q^2)+q^\mu\epsilon^\nu N_{\mu\nu}$$
$$+e(q\cdot\epsilon)(a/m_\pi^2)\sqrt{2}g\,\bar{U}(p')\gamma_5U(p)[m_\pi^2/(k-q)^2-1]^{-1} \quad (5)$$

where κ_p and κ_n are the proton and neutron anomalous moments. The non-Born amplitude $N_{\mu\nu}$ is finite as q, $k\to 0$ (with $m_\pi^2=q^2\to 0$) so we have

$$q^\mu N_{\mu\nu}=q^\mu N_{\mu\nu}(q^2=m_\pi^2=0,\quad q=0,\quad k=0)+O\left(\frac{q^2}{M_N^2},\quad \frac{q\cdot k}{M_N^2}\right).$$

Combining Eqs. (3), (4) and (5), we obtain

$$\mathscr{M}_+\equiv<n_{p'}|j_\kappa^+(0)|\gamma_k P_p \text{ in}>=e\frac{g(0)}{\sqrt{2}M_N}\bar{U}(p')\Big[\gamma\cdot\epsilon\gamma_5-\frac{2M_Nq\cdot\epsilon}{q\cdot k}\gamma_5$$
$$-\gamma\cdot q\gamma_5\frac{\gamma\cdot p+\gamma\cdot k+M_N}{2p\cdot k}\Big(\gamma\cdot\epsilon-\frac{\kappa_p}{2M_N}\gamma\cdot\epsilon\gamma\cdot k\Big)-\frac{\kappa_n}{2M_N}\gamma\cdot\epsilon\gamma\cdot k\frac{\gamma\cdot p-\gamma\cdot k+M_N}{-2p'\cdot k}\gamma\cdot q\gamma_5\Big]U(p)$$
$$+q^\mu\epsilon^\nu N_{\mu\nu}(q^2=0,\ q^\mu=k^\mu=0)+O\left(\frac{q^2}{M_N^2},\quad\frac{q\cdot k}{M_N^2}\right)+O(e^2). \quad (6)$$

Now, writing $\mathscr{M}_+=\epsilon_\mu M_+^\mu$, gauge invariance of the S-matrix amplitude requires that $k^\mu M_\mu^+=0$. Since the first term in Eq. (6) is separately gauge invariant, we must have

$$k^\nu N_{\mu\nu}(q^2=0,\ q^\mu=k^\mu=0)=0,\ \text{which implies}\ N_{\mu\nu}(q^2=0.\ q^\mu=k^\mu=0)=0\,.$$

Thus, we have shown that, neglecting terms of order $\frac{q^2}{M_N^2}$ and $\frac{q\cdot k}{M_N^2}$, the S-matrix amplitude $\mathscr{M}_+$, for $\gamma+p\to n+\pi^+$, is given by the first term in Eq. (6), which can be re-written in the form

$$\mathscr{M}_+=-\sqrt{2}\,g\bar{e}\bar{U}(p')\Big\{\Big[\frac{\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}-\Big(\frac{p\cdot\epsilon}{p\cdot k}-\frac{q\cdot\epsilon}{q\cdot k}\Big)-\frac{\kappa_p}{2M_N}\frac{\gamma\cdot q\gamma\cdot\epsilon\gamma\cdot k}{2p\cdot k}$$
$$+\frac{\kappa_n}{2M_N}\frac{\gamma\cdot\epsilon\gamma\cdot k\gamma\cdot q}{2p'\cdot k}\Big]+\frac{\kappa_p+\kappa_n}{4M_N^2}\gamma\cdot\epsilon\gamma\cdot k\Big\}\gamma_5U(p) \quad (7)$$

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* The last term in (5) comes from the Born term involving the π-γ-axial vector vertex which is evaluated using PCAC for the axial vector current.

where we have assumed that $g(0)=g(m_\pi^2)+O\left(\frac{m_\pi^2}{M_N^2}\right)$. The factor in square brackets is just the usual Born amplitude. The additional term is of order $(\kappa_p+\kappa_n)\frac{m_\pi}{M_N}$, which may be neglected since $(\kappa_p+\kappa_n)\frac{m_\pi}{M_N}<\left(\frac{m_\pi}{M_N}\right)^2$. The anomalous moment terms in the Born amplitude also contribute to the cross section a term of order $(\kappa_p+\kappa_n)\frac{m_\pi}{M_N}$, and hence they may be ignored.

3. Comparison with Experimental Results

A. π^+ Production

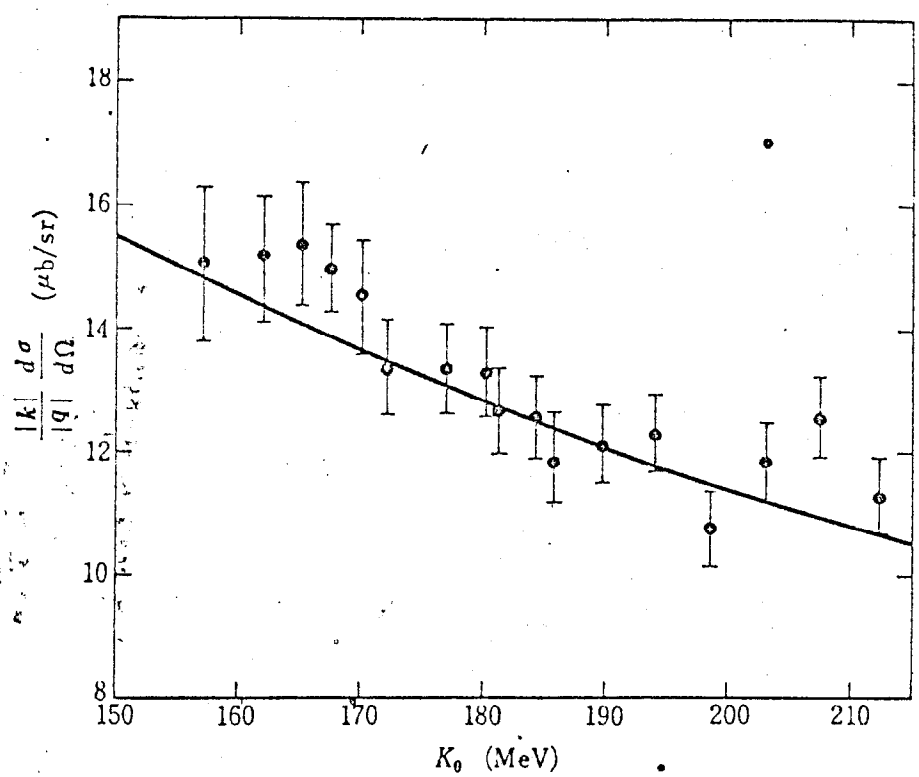

Fig. 5. The differential cross section in the c.m. system (times a kinematical factor $|k|/|q|$) for photoproduction of π^+ mesons from protons near threshold. The momentum transfer is held fixed at its value at threshold as the photon energy is varied. The experimental points are taken from Ref. 2).

The differential cross section in the c.m. system obtained from Eq. (7) gives

$$\frac{|k|}{|q|}\frac{d\sigma}{d\Omega_{\text{c.m.}}}(\gamma+p\to n+\pi^+)=\frac{e^2}{4\pi}\frac{g^2}{4\pi}\frac{1}{2(p_0+k_0)^2}\left[\frac{p'\cdot k}{p\cdot k}+\frac{1}{2}q^2\left(\frac{p^\mu}{p\cdot k}-\frac{q^\mu}{q\cdot k}\right)^2\right]. \tag{8}$$

Fig. 5 shows the experimantal data[2] for $\frac{|k|}{|q|}\frac{d\sigma}{d\Omega_{\text{c.m.}}}$ near threshold for the momentum transfer fixed at its value at the threshold, together with the theoretical curve predicted by Eq. (8). At threshold ($|q|=0$), we find from Eq. (8),

$$\frac{|k|}{|q|}\frac{d\sigma}{d\Omega_{\text{c.m.}}}(\gamma+p\to n+\pi^+)=15.5\ \mu\text{b/ster..}$$

This value is consistent with the experimental result[2] of $(15.6\pm0.5)\mu$b/ster..

We see from Fig. 5 that Eq. (8) correctly predicts the slope of the cross section near threshold. The angular distribution has been observed experimentally[4] in the region just above threshold and it does not agree with Eq. (8). However, since the angular variations are small, this discrepancy is not surprising, due to the approximate nature of the PCAC relation. The observed distribution is presumably due to the tail of the N^* (1236) resonance.

B. π^- Production

A calculation for π^- photoproduction from neutrons similar to the one in Section 2 gives the result

$$R\frac{\frac{d\sigma}{d\Omega_{\text{c.m.}}}(\gamma+n\to p+\pi^-)}{\frac{d\sigma}{d\Omega_{\text{c.m.}}}(\gamma+p\to n+\pi^+)}=\left(\frac{p\cdot k}{p'\cdot k}\right)^2\approx1.3 \text{ at threshold.}$$

A recent experimental value[2] is

$$R=1.265\pm0.075$$

which agrees with our result, whereas the Kroll-Ruderman limit gives $R=1$.

C. π^0 Production

For π^0 photoproduction the amplitude vanishes in zeroth order (the Kroll-Ruderman limit). Calculation of the first order terms gives

$$\frac{|k|}{|q|}\frac{d\sigma}{d\Omega_{\text{c.m.}}}(\gamma+p\to\pi^0+p)=\frac{e^2}{4\pi}\frac{g^2}{4\pi}\frac{1}{4M_N^2}\left(\frac{m_\pi}{M_N}\right)^2=0.24\ \mu\text{b/ster.} \tag{9}$$

at threshold. Furthermore, $\frac{|k|}{|q|}\frac{d\sigma}{d\Omega_{\text{c.m.}}}$ should be approximately constant as a function of photon energy just above threshold. Experimentally[5] this is not the case. $\frac{|k|}{|q|}\frac{d\sigma}{d\Omega_{\text{c.m.}}}$ increases quadratically with $|q|$, and at 160 MeV is still over twice as large as Eq. (9). Also the angular distribution disagrees with the calculated result. Clearly, then, for π^0 production near threshold, the N^* resonance may not be ignored because of the vanishing of the Born amplitude in the limit as $m_\pi/M_N \to 0$.

4. Conclusion

We have shown[6] that, by using gauge invariance and the PCAC hypothesis, one is justified using the Born approximation for pion photoproduction near threshold if we neglect terms of order $\frac{m_\pi^2}{M_N^2}$ and $(\kappa_p+\kappa_n)\times\frac{m_\pi}{M_N}$ and if the $N^*(1236)$ resonance can be ignored. For charged pion production the agreement with experiment is good, showing our assumptions are justified. For neutral pions, due to the smallness of the Born amplitude, the N^* resonance apparently dominates near threshold.

References for Appendix 3.

1) N. M. Kroll and M. A. Ruderman, Phys. Rev. **93**, 233 (1954).
2) J. P. Burq, Am. Phys. (Paris) **10**, 363 (1965).
3) M. Gell-Mann and M. Levy, Nuovo Cimento **26**, 705 (1960); S. L. Adler, Phys. Rev. **139**, B1638 (1965).
4) M. J. Bazin and J. Pine, Phys. Rev. **132**, 830 (1963).
5) W. Hitzeroth, Proc. Int. Symp. on Electron and Photon Interactions at High Energies (Hamburg, 1965) Vol. 2, 207.
6) After this work had been completed, our attention was called to a paper by S. Ragusa (EFINS preprint 67-9, January 1967) in which essentially the same results were obtained. Our derivation of the low energy theorem is some what simpler and avoids taking the pion off the mass shell.