THE SATURATION OF SUPERCONVERGENCE RBJATIONS AND CURRENT ALGEBRA SUM RUIES FOR FORWARD AMPLITUDES*

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ABSTRACT

We use the algebra of charges and their time derivatives, PCAC, and Regge high energy behavior to derive sum rules for strong interaction forward amplitudes. The saturation of all sum rules by a finite number of states is self-consistert and leads to relations among coupling constants and masses. Saturating all sum rules for $\pi-\rho$ scattering by $\pi, \infty$ and $A_{l}$, we predict: $m_{A_{1}}=1100 \mathrm{MeV}, m_{\omega}=m_{\rho}, \Gamma_{A_{1}}=120 \mathrm{MeV}, g_{\omega \rho \pi}=21 \mathrm{BeV}^{-1}$ in good agreement with experiment.
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A new set of strong interaction sum rules has rocently been proposed by de-Alfaro, Fubini, Furlan and Rossetti ${ }^{(1)}$ who noticed that the high energy behavior of certain amplitudes may lead to superconvergent dispersion relations of the form ${ }^{(2)}$ :

$$
\begin{equation*}
\int \operatorname{Im} A(s, t) d s=0 \tag{1}
\end{equation*}
$$

Other sum rules for strong amplitudes have been previously derived by writing unsubtracted dispersion relations for amplitudes satisfying low energy theorems based on the algebra of currents and PCAC (3). The complete set of all sum rules obtained in this way for a given scattering process represents a significant amount of new dynamical information. In particular, if the sum rules are approximately saturated by the contributions of a small number of s-channel resonances, they lead to sets of equations in the masses and coupling constants of the involved particles (4). In this paper we discuss the possible solutions of such sets of equations and analyse their algebraic properties, restricting ourselves only to forward ( $t=0$ ) amplitudes. We have reached the following conclusions:
(a) If all $t=0$ superconvergence and current algebra sum rules for the scattering of pions on a given hadron $x$ are saturated by states forming an irreducible representation of the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ algebra of charges, the complete set of equations in the masses and coupling constants has a unique non-trivial solution, in the limit of zero pion mass. All intermediate states are then predicted to have a common mass equal to the target mass, while the coupling constants obey the usual results of the chiral algebra. (5)
(b) If we allow additional states to contribute to the sum rules we always find a consistent solution. However, the uniqueness is lost and we can express all masses and coupling constants in terms of a few free parameters. These parameters correspond to the mixing coefficients of the
additional irreducible representations which contribute to the sum rules.
(c) In the particular case of $\pi-p$ scattering the inclusion of the $\pi, \omega$ and $A_{1}$ intermediate states yields a solution of the set of sum rules which agrees very well with experiment.

We use the following set of assumptions:

1. The vector and axial vector charges $Q^{i}$ and $Q_{5}^{i}(i=1,2,3)$ obey the equal time commutation relations of the chiral $\operatorname{SU}(2) \times \operatorname{SU}(2)$ algebra ${ }^{(6)}$.
2. The time derivatives of $Q_{5}^{i}(t)$ satisfy:

$$
\begin{equation*}
\left[D^{i}(t), Q_{5}^{j}(t)\right]=\delta_{i j} S(t) \tag{2}
\end{equation*}
$$

where $D^{i}(t)=\frac{d}{d t} Q_{5}^{i}(t)=-i\left[Q_{5}^{i}, H\right] . \quad S(t)$ does not include an $I=2$ piece and is therefore a pure isoscalar (7).
3. The matrix elements of the divergence of the axial vector current are dominated by the pion pole (PCAC).
4. The high energy behavior for all isospin and helicity amplitudes for $\pi-x$ scattering at $t=0$ is given by the Regge theory expression $s \alpha_{I}(0)-|\Delta h|$ where $\alpha_{I}(0)$ is the $t=0$ intercept of the leading meson trajectory with iso$\operatorname{spin} I$ and $\Delta h$ is the difference between the t-channel helicities of the two $x$-particles ${ }^{(8)}$.
5. All $I=2$ trajectories have $\alpha_{2}(0)<0(1)(9)$.

We discuss here only the example of $\pi-p$ scattering which exhibits most of the interesting features of the complete set of sum rules. Our results can be extended to the full $\mathrm{SU}(3) \times \mathrm{SU}(3)$ case or to other scattering processes such as $\pi-\sum$ or $\pi-N^{*}$ scattering without any difficulty. Details of these calculations will be presented elsewhere.

We write the $\pi-\rho$ scattering amplitude in the form ${ }^{(1)}$ :

$$
\begin{align*}
T(s, t)= & \left(\epsilon_{1} \cdot P\right)\left(\epsilon_{2} \cdot P\right) A(s, t)+\frac{1}{2}\left[\left(\epsilon_{1} \cdot P\right)\left(\epsilon_{2} \cdot Q\right)+\left(\epsilon_{2} \cdot P\right)\left(\epsilon_{1} \cdot Q\right)\right] \cdot B(s, t)+  \tag{3}\\
& +\left(\epsilon_{1} \cdot Q\right)\left(\epsilon_{2} \cdot Q\right) C_{1}(s, t)+\left(\epsilon_{1} \cdot \epsilon_{2}\right) C_{2}(s, t)
\end{align*}
$$

where $P=\frac{1}{2}\left(p_{1}+p_{2}\right), Q=\frac{1}{2}\left(q_{1}+q_{2}\right), p_{1,2}$ are the momenta of the initial and final pion and $q_{1,2}$ and $\epsilon_{1,2}$ are the momenta and polarizations of the $\rho$ mesons. Regge theory predicts that at high energies $A^{(I)}(s, t) \propto s^{\alpha_{I}(t)-2}$; ${ }_{B}{ }^{(I)}(s, t) \propto s^{\alpha_{I}}(t)-1 ; C_{1,2}^{(I)}(s, t) \propto s^{\alpha_{I}(t)}$, where $I$ is the $t$-channel isospin. The only s-channel helicity amplitudes that do not vanish at $t=0$ are $M_{11}$ and $M_{00}$ where the subscripts denote the s-channel helicities of the o's. We find that at $t=0$ :

$$
M_{11}=C_{2} \quad \text { (4) } \quad M_{00}=C_{2}+\left(v^{2}-m_{\pi}^{2}\right) A
$$

where:

$$
\begin{equation*}
v=\frac{P \cdot Q}{m_{\rho}} \tag{6}
\end{equation*}
$$

The complete list of $t=0$ sum rules includes:
(a) Two independent Adler-Weisberger sum rules ${ }^{(10)}$ :

$$
\begin{equation*}
\frac{2}{\pi} \int_{0}^{\infty} \frac{\operatorname{Im} M_{11}^{(I)}(v) d v}{v^{2}-m_{\pi}^{2}}=\frac{8}{f_{\pi}^{2}} \quad \text { (7) } \quad \frac{2}{\pi} \int_{0}^{\infty} \frac{\operatorname{Im} M_{00}^{(1)}(v) d v}{v^{2}-m_{\pi}^{2}}=\frac{8}{f_{\pi}^{2}} \tag{7}
\end{equation*}
$$

where $f_{\pi}=135 \mathrm{MeV}$ is the decay constant of the charged pion, predicted by PCAC to satisfy $f_{\pi}=\sqrt{2} G_{A} m_{N} / g_{\pi N}$. Eqs. (5)-(8) lead to the $I=1 \pi-\rho$ superconvergence relation ${ }^{(1)(11)}$, which is therefore not an independent sum rule:

$$
\begin{equation*}
\int_{0}^{\infty} \operatorname{Im} A^{(1)}(v) d v=0 \tag{9}
\end{equation*}
$$

(b) Two independent sum rules for $M_{11}^{(2)}$ and $M_{00}^{(2)}$ :

$$
\begin{equation*}
\int_{0}^{\infty} \frac{v \operatorname{Im} M_{11}^{(2)}(v) d v}{v^{2}-m_{\pi}^{2}}=0 \quad(10) \quad \int_{0}^{\infty} \frac{v \operatorname{Im} M_{00}^{(2)}(v) d v}{v^{2}-m_{\pi}^{2}}=0 \tag{10}
\end{equation*}
$$

Eqs. (10), (11) are derived by inserting the commutator $\left[D^{+}, Q_{5}^{+}\right]=0$ between pairs of helicity $h=1,0$ o mesons moving with $p_{z}=\infty$. The convergence of these sum rules is guaranteed if $\alpha_{2}(0)<0(9)$ Eqs. (10), (11) together with (5) and (6) lead to an additional superconvergence relation ${ }^{(12)}$ :

$$
\begin{equation*}
\int_{0}^{\infty} v \operatorname{Im} A^{(2)}(v) d v=0 \tag{12}
\end{equation*}
$$

(c) A superconvergence relation of the form ${ }^{(1)}$ :

$$
\begin{equation*}
\int_{0}^{\infty} \operatorname{Im} R^{(2)}(v) d v=0 \tag{13}
\end{equation*}
$$

Strictly speaking, Eq. (13) is not a pure $t=0$ sum rule, since at $t=0$ the $B$ amplitude does not contribute to $M_{11}$ or $M_{00} \quad B(\nu, 0)$ can be experimentally determined only by extrapolating $B(v, t)$ to $t=0$.

We have used PCAC in deriving at least two of the five independent sum rules. We will therefore study the self-consistency of the complete set of equations only in the limit $m_{\pi}=0$. We realize that the superconvergence relations (9), (12) and (13) can be derived without taking this limit. We find, however, that the overall consistency of the saturation assumption requires $m_{\pi}^{e x t}=0$ even if we consider only the superconvergence relations. This may mean that to the extent that these relations give symmetry results, they do so only because of their connection to the algebra of currents. If this is really the case, we clearly have to consider all our sum rules in the limit implied by PCAC or by vector meson dominance which are the crucial links between the algebra of weak and electromagnetic currents and the strong interaction sum rules. Notice, however, that whenever the pion appears as an intermediate state, its mass is not necessarily zero, and we consider it as an additional physical quantity.

We now proceed to discuss the saturation problem. We choose Eqs. (7), (8), (10), (11), and (13) as our five independent sum rules and study three different saturation assumptions. The first two are mostly of theoretical interest while the third yields very good agreement with experiment and is presumably a good approximation of the real world.

1. Assume that the five independent $t=0$ sum rules are saturated by the $\pi$ and $\omega$ intermediate states. The equations are:

$$
\begin{align*}
& g_{\omega \rho \pi}^{2}=\frac{8}{f^{2}} \quad(7 a) \quad \frac{4 g_{\rho \pi \pi}^{2}}{m_{\rho}^{2}}=\frac{8}{f^{2}}  \tag{7a}\\
& \left.-m_{\rho}^{2}\right) g_{\omega \rho \pi}^{2}=0 \quad(10 a) \quad\left(m_{\pi}^{2}-m_{\rho}^{2}\right) g_{\rho \pi \pi}^{2}=0  \tag{8a}\\
&  \tag{10a}\\
& m_{\omega}^{2} g_{\omega \rho \pi}^{2}-4 g_{\rho \pi \pi}^{2}=0
\end{align*}
$$

$g_{\rho \pi \pi}$ and $g_{\omega \rho \pi}$ are defined as in Reference $l$. The unique solution is:

$$
\begin{equation*}
g_{\omega \rho \pi}^{2}=\frac{4 g_{\rho \pi \pi}^{2}}{m_{\rho}^{2}}=\frac{8}{f_{\pi}^{2}} \tag{15}
\end{equation*}
$$

Eq. (15) predicts $\frac{g_{\rho \pi \pi}^{2}}{4 \pi}=5, g_{\omega \rho \pi}=21 \mathrm{BeV}^{-1}$, to be compared with $\frac{g_{\rho \pi \tau \pi}^{2}}{4 \pi}=2.4 \pm 0.2$ as determined from the $\rho$ width and $g_{\omega \rho \pi}=(17 \pm 3) \mathrm{BeV}^{-1}$ as calculated from the Gell-Mann-Sharp-Wagner model ${ }^{(13)}$ for $\omega \rightarrow \pi \gamma$.

While it is clear that Eqs. (14), (15) do not agree very well with experiment, it is still interesting to undexstand algebraically why we have obtained such a solution. In order to do so we notice that our saturation assumption is equivalent to assuming that, at infinite momentum, the $h=1$ components of $\rho$ and $\omega$ are in the ( $\frac{1}{2}, \frac{1}{2}$ ) representation of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ while the $\mathrm{h}=0 \mathrm{o}$ and $\pi$ are in $(1,0) \pm(0,1)$. In this case, the axial charge $Q_{5}$, which is a generator of the algebra, connects $\rho$ only to $\omega$ and $\pi$. The matrix elements of the operator $D^{i}$ between particle states at infinite momentum satisfy ${ }^{(14)}$ :

$$
\begin{equation*}
\lim _{p_{z} \rightarrow \infty} p_{z}(\alpha|\mathrm{D}| \beta)=-\frac{i}{2}\left(m_{\beta}^{2}-m_{\alpha}^{2}\right)\left(\alpha\left|Q_{5}^{i}\right| \beta\right) \tag{16}
\end{equation*}
$$

If $\left(\alpha\left|D^{i}\right| \beta\right)=0$ and $\left(\alpha\left|Q_{5}^{i}\right| \beta\right) \neq 0$, Eq. (16) leads to $m_{\beta}=m_{\alpha}$. The commutation relation (2) implies that the operators $D^{i}$ and $S$ transform according to the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation of $\operatorname{SU}(2) \times \operatorname{SU}(2)$. Consequently, for any irreducible representation ( $k, \ell$ ):

$$
\begin{equation*}
\left((k, l)\left|D^{i}\right|(k, l)\right)=0 \tag{17}
\end{equation*}
$$

We conclude that if $\rho$ and $\omega$ (or $\rho$ and $\pi$, for $h=0$ ) are in the same $\operatorname{SU}(2) \times \operatorname{SU}(2)$ representation, $\left(\rho_{1}|D| \omega_{1}\right)=0$ and $\left(\rho_{0}|D| \pi\right)=0$ where the subscripts denote the helicities. Eq. (16) then leads to the prediction of equal masses for $\rho, \omega$ and $\pi$ (Eq. (14)). This is actually a much more general result: If we saturate all $t=0$ sum rules for $\pi-x$ scattering by states forming an irreducible representation of $\operatorname{SU}(2) \times \operatorname{SU}(2)$, we find that all matrix elements of $D$ vanish. The masses of all intermediate states are then predicted to be the same as the mass of $x$ and the sum rules for $I=2$ t-channel amplitudes become trivial identities (15) while the $I=1$ sum rules lead to the ordinary predictions of the charge algebra.
2. In order to study the case of a reducible, finite, representation we now allow a $\varphi$ meson to contribute (16). The resulting equations can be easily constructed from those of the previous example by adding $\varphi$ terms identical in form to the $\omega$ contributions. The solution depends on two free parameters which can be chosen as the $\varphi$ mass and an arbitrary angle $\theta$ defined as:

$$
\begin{equation*}
\frac{g_{\varphi \rho \pi}}{g_{\omega \rho \pi}}=\tan \theta \tag{18}
\end{equation*}
$$

The general solution is:

$$
\begin{array}{lll}
m_{\pi} & =m_{\rho} & \text { (19) } \\
\frac{4 g_{\rho \pi \pi}^{2}}{m_{\rho}^{2}}=\frac{8}{f_{\pi}^{2}} & m_{\rho}^{2}=m_{\omega}^{2} \cos ^{2} \theta+m_{\varphi}^{2} \sin ^{2} \theta \\
& \text { (21) } \\
& g_{\omega \rho \pi}^{2}=\frac{8}{f_{\pi}^{2}} \cos ^{2} \theta \\
& g_{\varphi \rho \pi}^{2}=\frac{8}{f_{\pi}^{2}} \sin ^{2} \theta \tag{23}
\end{array}
$$

We immediately see that in the limit $g_{\varphi p \rho_{\pi}}=0$, Eqs. (18), (20) give $m_{\rho}=m_{\omega}$. The "bad" predictions (Eqs. (19), (21)) for $m_{\pi}$ and $g_{\rho \pi \pi}$ have not been removed by the $\varphi$ since $\pi$ and $\varphi$ never contribute to the same sum rule (except for Eq. (13) which is not independent, in this approximation).

From the algebraic point of view the solution (19)-(23) can be understood in the following way: The addition of $\varphi$ is equivalent to assigning the $h=\} \omega$ and $\varphi$ to orthogonal mixtures of the $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $(0,0)$ representtions while $\rho_{1}, \rho_{0}$ and $\pi$ are classified as before. We define:
$\left|\varphi_{1}\right\rangle=\cos \theta|(0,0)\rangle+\sin \theta\left|\left(\frac{1}{2}, \frac{1}{2}\right)\right\rangle(24) \quad\left|\omega_{1}\right\rangle=-\sin \theta|(0,0)\rangle+\cos \theta\left|\left(\frac{1}{2}, \frac{1}{2}\right)\right\rangle$
$Q_{5}$ connects $\rho_{1}$ only to states in the $\left(\frac{1}{2}, \frac{1}{2}\right)$ representation while $D$ connects $\rho_{1}$ only to $(0,0)$. We therefore find:

$$
\begin{equation*}
\frac{\left(\rho_{1}^{+}\left|Q_{5}^{+}\right| \varphi_{1}\right)}{\left(\rho_{1}^{+}\left|Q_{5}^{+}\right| \omega_{1}\right)}=\tan \theta \quad \text { (26) } \quad \frac{\left(\rho_{1}^{+}\left|D^{+}\right| \varphi_{1}\right)}{\left(\rho_{1}^{+}\left|D^{+}\right| \omega_{1}\right)}=-\cot \theta \tag{27}
\end{equation*}
$$

Eq. (26) is identical to (18) and leads to (21), (22). Eq. (27) together with (16) leads to Eq. (20). The angle $\theta$ that was arbitrarily introduced in Eq. (18) is now interpreted as the mixing angle between the ( $\frac{1}{2}, \frac{1}{2}$ ) and $(0,0)$ representations. Its experimental value is close to zero, and we will therefore neglect the contribution of the $\varphi$ in the following discussion.
3. We finally consider the contribution of the next $\pi-\rho$ resonance, the $A_{1}\left(J^{P}=I^{+}, I^{C G}=I^{+-}, m=1080 \mathrm{MeV}\right)$. There are two independent $A_{1} p_{\pi}$ couplings and we choose them as the longitudinal coupling
$g_{L}\left(p_{\mu}-\frac{(p \cdot q) q_{\mu}}{q^{2}}\right)\left(q_{\lambda}-\frac{(p \cdot q) p_{\lambda}}{p^{2}}\right) e_{\lambda} e_{\mu}^{\prime}$ and the transverse coupling

,
and polarization of the $A_{1}(\rho)$. The five sum rules give:

$$
\begin{array}{cc}
g_{\omega \rho \pi}^{2}+\frac{v_{A_{1}}^{2}}{m_{A_{1}}^{4}} g_{T}^{2}=\frac{8}{f_{\pi}^{2}} & \text { (Tb) }
\end{array} \frac{4 g_{\rho \pi \pi}^{2}}{m_{\rho}^{2}}+\frac{v_{A_{1}}^{2}}{m_{\rho}^{2} m_{A_{1}}^{2}} g_{L}^{2}=\frac{8}{f_{\pi}^{2}}
$$

$$
\begin{equation*}
m_{\omega}^{2} g_{\omega \rho \pi}^{2}-4 g_{\rho \pi \pi}^{2}-\frac{v_{A_{1}}^{2}}{m_{A_{1}}^{2}} g_{L}^{2}-\frac{v_{A_{1}}^{2}}{m_{A_{1}}^{2}} g_{T}^{2}=0 \tag{13b}
\end{equation*}
$$

where $\nu_{x}=\frac{1}{2}\left(m_{x}^{2}-m_{\rho}^{2}\right)$. The general solution of these equations is $(17)$ :

$$
\begin{array}{rlr}
m_{\omega} & =m_{\rho} & \text { (28) } \\
g_{\omega \rho \pi}^{2} & =\frac{8}{f_{\pi}^{2}} & (30) \\
g_{T}=0 \\
m_{\rho}^{2} & =\frac{4 g_{\rho \pi \pi}^{2}}{f_{\pi}^{2}} \cos ^{2} \psi \tag{31}
\end{array}
$$

Eq. (31) and $\Gamma(\rho \rightarrow \pi \pi)=125 \mathrm{MeV}$ give $\cos ^{2} \psi=0.48$. Inserting this in (33) we predict ${ }^{(18)}: \quad m_{A_{1}}=1100 \mathrm{MeV}$
in remarkable agreement with experiment. Eqs. (29), (32) predict that the $A_{l}$ decays mostly into longitudinal $\rho ' s$ and that:

$$
\begin{equation*}
\Gamma_{A_{l}}=\frac{2 v_{A_{1}}^{3}}{3 \pi f_{\pi^{2}}^{2} A_{1}^{3}} \sin ^{2} \psi=120 \mathrm{MeV} \tag{35}
\end{equation*}
$$

to be compared with $\Gamma=130 \pm 40 \mathrm{MeV}{ }^{(19)}$. The remarkable agreement with experiment of the predictions (28), (30), (34), (35) can be regarded as strong evidence for the validity of our set of assumptions. The inclusion of additional states such as $A_{2}$ or the $J^{P}=I^{+}, I^{C G}=0^{--}$meson predicted for the $q \bar{q}$ I=l system allows us much more freedom in solving the equations and improves the agreement with experiment at the expense of adding many new parameters. The contribution of the $A_{2}$, as calculated from its experimental width, is relatively small in all cases ${ }^{(20)}$.

We have shown that the set of sum rules for forward amplitudes, obtained from a Regge'-type high energy behavior, the algebra of charges, PCAC and specific assumptions on the absence of $I=2$ components, is self-consistert and can be saturated by any finite number of s-channel resonances. The
general solution may depend on some unknown parameters (mixing angles). Additional information is required if we want to determine these parameters and to predict $\frac{111}{}$ the masses and coupling constants of the intermediate states. It is possible that this information can be obtained from additional sum rules a.t $t \neq 0^{(4)}$, or from sum rules similar to Eq. (13), based on extrapolated values at $t=0$. We believe that the complete set of assumptions used in this work is strongly supported by the successful results that we have obtained for the saturation by $\pi, \omega$ and $A_{1}$.

1. V. de-Alfaro, S. Fubini, G. Furlan and C. Rossetti, Phys. Letters 21 , 576 (1966).
2. This will be the case if at high energies $A(s, t) \propto s^{\beta}, \beta<-1$. In some cases we may have stronger relations of the form $\int s^{n} \operatorname{Im} A(s, t) d s=0$.
3. The Adler-Weisberger sum rule is an obvious example. S. L. Adler, Phys. Rev. Letters 14, 1051 (1965), W. I. Weisberger, Phys. Rev. Letters 14, 1047 (1965).
4. The complete, infinite set of all sum rules for all values of $t$ cannot be consistently saturated by a finite number of intermediate states. This does not imply, however, that we should abandon the saturation by a small number of resonances as a valid approximation for a more restricted finite set of sum rules. For a general discussion of the case of infinite sets saturated by an infinite number of states see, e.g. S. Fubini, Proceedings of the $4^{\text {th }}$ Coral Gables Conference, 1967, to be published.
5. It is well known that the saturation of the charge commutator sum rules by the states of an irreducible representation of the chiral algebra leads to the coupling constant relations which are usually referred to as "SU(6) results". We find that these relations, as well as some new relations among the particle masses are predicted by a wider set of sum rules.
6. M. Gell-Mann, Phys. Rev. 125, 1067 (1962).
7. Eq. (2) is valid both in the $\sigma$-model and the free quark model. Notice that the conservation of the vector charges implies that the commutator in (2) does not have an $I=1$ piece.
8. The high energy behavior of helicity amplitudes for a general scattering process is slightly more complicated. See, e.g. T. L. Trueman, Phys. Rev. Letters 17, 2198 (1966).
9. This assumption has already led to a few interesting results and no
contradiction with experiment. The only direct experimental test of this is the energy dependence of $\sigma\left(\pi^{\sim} p \rightarrow \pi^{+} N^{*}{ }^{-}\right)$at small $t$. The poorly known cross-section for this process seems to be negligible and consistent with zero above the resonance region. In fact, the present experimental data is consistent with the absence of the $t$-channel exchange of any system with $I>1$, for all processes.
10. V. S. Mathur and I. K. Pandit, Phys. Letters 19, 523 (1965), S. Gasiorowicz and D. A. Geffen, Phys. Letters 22, 344 (1966), P. H. Frampton, to be published in Nuovo Cimento.
11. S. Fubini and G. Segre, Nuovo Cimento 45, 641 (1966).
12. Eq. (13) was first proposed by F. E. Low, to be published in the proceedings of the $13^{\text {th }}$ conference on high energy physics, Berkeley, 1966.
13. M. Gell-Mann, D. Sharp and W. G. Wagner, Phys. Rev. Letters 8, 261 (1962). We use $\Gamma(\omega \rightarrow \pi \gamma)=1.2 \pm 0.3, f_{\rho}^{2} / 4 \pi=2.5 \pm 0.4$ as determined by J. J. Sakurai, Phys. Rev. Letters 17, 1021 (1966). The quoted errors in our value for $g_{\mu \rho \pi}$ do not include the errors introduced by the model.
14. The operator $D$ was used by $S$. Fubini, G. Furlan and C. Rossetti, Nuovo Cimento 40, 1171 (1965) in deriving $S U(3)$ masis formulae. See also, V. de-Alfaro, S. Fubini, G. Furlan and C. Rossetti, to be published in Nuovo Cimento.
15. Our analysis of the matrix elements of $Q_{5}$ and $D$ applies to the sum rules $(7),(8),(10)$ and (11) which are derived from the commutation relations of the charges or their derivatives. We did not derive Eq. (13) from these considerations. However, in the pure representation case Eq. (13a) is a linear combination of the four other equations and therefore adds no new information.
16. This approximation was first considered by the authors of Ref. l for the
sum rules (9) and (13). They found that $\nu_{\omega}=0$ leads to $g_{\rho \varphi \pi}=0$. Low (Ref. 12) remarked that when we add the sum rule (12), the only consistent solution is $g_{\rho \pi \pi}=g_{\omega \rho \pi}=g_{\varphi \rho \pi}=0$. This would contradict the inhomogenous Eqs. (7), (8). In the limit $m_{\pi}^{e x t}=0$ (and a free intermediate pion mass) Low's objection is not valid, however, and a consistent solution is obtained. R. Oehme and G. Venturi, University of Chicago preprint, have discussed the saturation of Eq. (9). They did not discuss any of the $I=2$ sum rules and consequently could not obtain any relations between masses.
17. The commutator sum rules (7), (8), (10) and (11) give Eqs. (30)-(33) and relate $\nu_{\omega}$ to $g_{T}$ in terms of an additional unknown parameter. The sum rule (13) fixes this parameter and predicts $\nu_{\omega}=g_{T}=0$ (Eqs. (28), (29)).
18. If we use $\cos ^{2} \psi=\frac{1}{2}$ and neglect $\frac{m^{2}}{m_{\rho}^{2}}$ we can rewrite (34) as $m_{A_{1}}=\sqrt{2} m_{\rho}$. This relation was found by $S$. Weinberg, to be published, who derived il from a different set of assumptions, but using the same numerical approximations.
19. A. H. Rosenfeld et al., Rev. Mod. Phys. 39, I (1967).
20. P. H. Frampton and J. C. Taylor, Oxford preprint.
