# COMPIEX PROPAGATOR INSERTIONS IN THE SQUARE DIAGRAM* 

Edward S. Sarachik<br>Stanford Linear Accelerator Center, Stanford University, Stanford, California

## ABSTRACT

The problem of inserting a complex propagator in the crossed channel of the square diagram is considered. 'lhe propagator is spectrally represented and the square diagram with full propagator insertion is expressed as a spectral integral of' square diagrams over an internal mass variable. The analytic properties of such diagrams in two complex variables, one internal mass and one external energy, are investigated in detail. These properties are then used to find the analytic properties of the full diagram with propagator insertions.
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## I. Introduction

We wish to consider the problem of the insertion of a composite particle into the diagrams of perturbation theory and the singularities so generated. The case in which insertions are made into the selfenergy and triangle single loop graphs have been treated in great detail by Aitchison and Kacser ${ }^{(1)}$. The extension to insertions into the single loop square diagram is distinctly non-trivial and will be the specific problem treated here. The work of Aitchison and Kacser then enters as the lower order contraction singularities of the square diagram.

The insertion of a full propagator in place of a simple line in a graph leads to an integral of the diagram over an internal mass, weighted by the spectral function of the inserted propagator. An investigation of the singularities of the integral requires an analysis of the analytic properties of the graph in the usual external energy variable and in the internal mass, both treated as complex variables. Such treatment of the square diagram has appeared only once before in the literature, in the work of Barton and Kacser ${ }^{(2)}$. They looked at the square diagram with two internal masses, in the decay region of one of the external particles. Thus our work will not overlap with theirs.

We consider the diagram of Fig. la, to be called $F\left(s, t ; M^{2}\right)$, where the heavy line, the propagator $P\left(\mathrm{q}_{2}{ }^{2}\right)$, will represent an infinite sum of graphs. The propagator will be assumed to have a bound state mass at $\mu<2 \mathrm{~m}$. Note that all the particles are taken as neutral and spinless.

The usual Feynman rules give, up to a factor,

$$
\begin{equation*}
F\left(s, t ; M^{2}\right)=\int a^{4} q_{1} \frac{1}{q_{1}^{2}-m^{2}+i \epsilon} P\left(q_{2}^{2}\right) \frac{1}{q_{3}^{2}-m^{2}+i \epsilon} \frac{1}{q_{4}^{2}-m^{2}+i \epsilon} \tag{I}
\end{equation*}
$$

A specific model will be used for the propagator, viz. a sum of bubble diagrams strung in linear chains (see the Appendix). A typical diagram contributing to the sum of insertions is shown in Fig. Ic.

The propagator is explicitly written

$$
\begin{equation*}
P\left(q_{2}^{2}\right)=\lambda_{0}^{2} \pi\left(q_{2}^{2}\right)=-\lambda_{0}-\int_{0}^{\infty} \frac{\sigma\left(\lambda^{2}\right) d \lambda^{2}}{\lambda^{2}-q_{2}^{2}-i \epsilon} \tag{2}
\end{equation*}
$$

where $\lambda_{0}$ is the direct coupling and

$$
\begin{align*}
\sigma\left(\lambda^{2}\right) & =g^{2} \delta\left(\lambda^{2}-\mu^{2}\right)+\sigma^{2}\left(\lambda^{2}\right) \sigma\left(\lambda^{2}-4 m^{2}\right) \\
\sigma^{2}\left(\lambda^{2}\right) & =\sqrt{\frac{\lambda^{2}-4 m^{2}}{\lambda^{2}}} \frac{1}{\left(\lambda^{2}-\mu^{2}\right)^{2}} \frac{1}{\left|\Delta_{f}\left(\lambda^{2}\right)\right|^{2}} \tag{3}
\end{align*}
$$

$\mu$ here is the mass of the bound state and $g$ is its coupling to the original particles. More realistic propagators will retain this general structure.

The expression for (1) becomes

$$
F\left(s, t ; M^{2}\right)=-\lambda_{0} I(s)+\int_{0}^{\infty} \sigma\left(\lambda^{2}\right) f\left(s, \lambda^{2} ; t, M^{2}\right) d \lambda^{2}
$$

where

$$
I(s)=\int d^{4} q_{1} \frac{1}{q_{1}^{2}-m^{2}+i \epsilon} \frac{1}{q_{3}^{2}-m^{2}+i \epsilon} \frac{1}{q_{4}^{2}-m^{2}+i \epsilon}
$$

is simply the triangle graph of Fi . $2 b$ and
$f\left(s, \lambda^{2} ; t, M^{2}\right)=\int d^{4} q_{1} \frac{1}{q_{1}^{2}-m^{2}+i \epsilon} \frac{1}{q_{2}{ }^{2}-\lambda^{2}+i \epsilon} \frac{1}{q_{3}^{2}-m^{2}+i \epsilon} \frac{1}{q_{4}{ }^{2}-m^{2}+i \epsilon}$
is the square diagram with one line of mass $\lambda$, Fig. lb .

Note that the contribution $I(s)$ is in fact a triangle contraction of the square diagram so that its singularities will already be included in the analysis of the square diagram and need not be considered separately. We have then, excluding the $I(s)$ term,

$$
\begin{equation*}
F\left(s, t ; M^{2}\right)=g^{2} f\left(s, \mu^{2} ; t, M^{2}\right)+\int_{4 m}^{\infty} \sigma^{\prime}\left(\lambda^{2}\right) f\left(s, \lambda^{2} ; t, M^{2}\right) d \lambda^{2}, \tag{5}
\end{equation*}
$$

so that the singularities of the graph with full propagator insertion are generated partly by the bound state approximation, $f\left(s, \mu^{2} ; t, M^{2}\right)$, and partly by the continuum parts of the propagator. The usual procedure in treating a graph with bound state (or resonance) insertion is replacing the composite particle by a simple line of real (or complex) mass, which is equivalent to retaining only the first tern in (5). We will be interested in analyzing the integral term in (5) and thereby seeing which singularities are neglected by the bound state (or resonance) approximation.

More specifically, we will be interested in those singularities of $F\left(s, t ; M^{2}\right)$ in the complex variable $s$, for $t . f i x e d$ in the physical region of $s$ channel scattering, which survive the $\lambda^{2}$ integration in (5). The conditions for the singularities of such an integral are by now very well known (3). The method requires knowing the analytic properties of $f\left(s, \lambda^{2} ; t, M^{2}\right)$ in the complex variables $s$ and $\lambda^{2}$ and then analyzing the integral for end-point and pinch singularities. We will, in fact, specialize to the case of equal mass scattering, $M^{2}=m^{2}$. Since this is a fairly degenerate case, the limit $M^{2} \rightarrow m^{2}$ will not be taken until near the end of the analysis.

The sections following will begin with the kinematics of the process, continue on to a study of the singularity structure of $f\left(s, \lambda^{2} ; t, m^{2}\right)$ as a
function of the complex variables $s$ and $\lambda^{2}$ for $t$ and $M^{2}$ fixed and real, and conclude with the properties of the full graph $F\left(s, t ; M^{2}\right)$ as $M^{2} \rightarrow \mathbb{I n}^{2}$.

## II. Kinematics

For a gencral scattcring process, say that of Fig. la, the physical region is given by det $p_{i} \cdot p_{j}>0$ where the $p_{i}$ are the external momenta, all taken incoming (4). For the process $m+m \rightarrow m+M$, this condition becomes

$$
\text { st } u>m^{2}\left(M^{2}-m^{2}\right)^{2}
$$

where

$$
u=m^{2}+3 m^{2}-s-t
$$

The boundary of the physical region is given by the vanishing of the Gram determinant det $p_{i} \cdot p_{j}$. This vanishing is also the condition for the non-Landau singularities ${ }^{5}$ ). Thus the physical region is bounded by the real section of the leading non-Landau curve

$$
\text { st } u=m^{2}\left(M^{2}-m^{2}\right)^{2}
$$

or

$$
\begin{equation*}
\sigma\left(s, t ; M^{2}\right)=s^{2} t+s t\left(t-3 m^{2}-M^{2}\right)+m^{2}\left(M^{2}-m^{2}\right)^{2}=0 \tag{6}
\end{equation*}
$$

This curve is shown in Fig. 4.
We may note that $\sigma\left(s, \lambda^{2} ; M^{2}\right)=0$ is in turn the leading Landau curve for the triangle diagram with an internal mass $\lambda$, Fig. 2d.

In the eventual limit $M^{2}=m^{2}$, the physical region degenerates into the regions bounded by the lines $s=0, t=0$ and $s+t=4 m^{2}$.

We will be looking as s channel processes whose physical region is labeled by I in Fig. 4. t will be kept fixed, real, and negative.
III. The function $f\left(s, \lambda^{2} ; t, M^{2}\right)$

In this section we will study the singularity structure of the square diagram of $\operatorname{Fig}$. lb , in terms of the two complex variables $s$ and $\lambda^{2}$. The techniques used in the analysis of one internal and one external invariant is a straightforward generalization of those used in more usual analyses ${ }^{(6)}$.

In particular, after feynman parametrizing and loop integrating, we have, up to inessential factors,

$$
\begin{equation*}
f\left(s, \lambda^{2} ; t, M^{2}\right)=\int_{0}^{1} \frac{d \alpha_{1} d \alpha_{2} d \alpha_{3} d \alpha_{4}}{D^{2}} \tag{7}
\end{equation*}
$$

where $D$ is the discriminant, with respect to $q_{1}$, of

$$
\begin{aligned}
\psi=\sum_{i=1}^{4} \alpha_{i}\left(q_{i}{ }^{2}-m_{i}{ }^{2}+i \epsilon\right) \quad m_{i}^{2} & =m^{2} \quad i=1,3,4 \\
m_{2} & =\lambda^{2}
\end{aligned}
$$

Introducing the usual variables

$$
\begin{gathered}
y_{11}=y_{22}=y_{33}=y_{44}=1 \\
y_{14}=y_{34}=\frac{1}{2} \quad y_{13}=\frac{2 m^{2}-s}{2 m^{2}} \quad y_{12}=\frac{\lambda}{2 m} \\
y_{23}=\frac{m^{2}+\lambda^{2}-m^{2}}{2 m \lambda} \quad y_{24}=\frac{m^{2}+\lambda^{2}-t}{2 m \lambda}
\end{gathered}
$$

gives

$$
\begin{equation*}
D=-\sum_{i, j=1}^{4} \alpha_{i} m_{i} \alpha_{j} m_{j} y_{i j}-\sum_{i=1}^{4} \alpha_{i}^{2} m_{i}{ }^{2} \tag{8}
\end{equation*}
$$

In terms of these variables, the stability conditions at the $\ell_{k}{ }^{\prime}$ th vertex are

$$
\begin{array}{ll}
y_{\ell_{k}}>-1 & \text { External stability } \\
y_{\ell_{k}}<1 & \text { Internal stability }
\end{array}
$$

The conditions for a singularity of the integral (7) are

$$
\begin{equation*}
\alpha_{k} \frac{\partial D}{\partial \alpha_{k}}=0=\alpha_{k}\left[-2 \alpha_{k k}^{m_{k}^{2}}-2 m_{k} \sum_{j \neq k} \alpha_{j} m_{j} y_{k j}\right] \quad k=1,2,3,4 \tag{9}
\end{equation*}
$$

These four conditions then guarantee that $D$ vanish at a singularity since $D$ is second order homogeneous in the $\alpha^{\prime}$ s.

Define the matrix $Y$ whose elements are

$$
\begin{equation*}
Y_{i j}=m_{i} m_{j} y_{i j} \tag{10}
\end{equation*}
$$

The conditions (9) are equivalent to the determinant of v and the principle minors of $Y$ vanishing in turn. Each of these vanishings give the singularity surface of a contraction graph everything on its mass shell. The leading Landau surface is generated by the contraction graph of Fig. 1b, and the various lower order contractions graphs are shown in Figs. 2 and 3.

The definition of the physical sheet of the function $f\left(s, \lambda^{2} ; t, M^{2}\right)$ in fact depends on the singularities of the lower order contractions. Thus the physical sheet of the box diagram is defined with respect to the cuts provided by the branch points due to the triangle and self-energy contractions. The singularities due to the leading contraction are then located in the topological product of cut $s$ and $\lambda^{2}$ complex planes.

## IV. The Contraction Singularities

As preliminary remarks, let us recall some standard terminology and known theorems that will be used in the following.

The physical limit onto real values of an invariant is taken as $s+i \epsilon$ for all external invariants and $\lambda^{2}$ - ie for all internal invariants. This is the standard Feynman prescription. It may be seen explicitly from (8) that $D \neq 0$ when real values are approached in this limit.

The surfaces attached to the real sections of Landau curves will be analyzed by the method of search lines ${ }^{(6)}$. Thus if a search line of positive (negative) slope touches a real section of a Landau curve, the complex surface attached to that section is in corresponding (opposite) half'-planes. If a real section whose attached surface lies in corresponding (opposite) half-planes is reached from corresponding (opposite) halfplanes, the limit onto the section is said to be the curve limit, otherwise, the non-curve limit.

For single loop graphs, surfaces due to contractions of successive order can only intersect by touching and the touches are necessarily effentive (the $\alpha$ 's at the touch are the same for both surfaces). (3,7)

The fundamental property to be used in the following concerns the way singularities pass from one part of the surface to another. In particular, in a given limit, (corresponding or opposite half-planes) the singularity character of a surface changes at an effective touch with a contraction curve of one lower order if the lower order curve is singular in that limit. It does not change if the lower order curve is non-singular in that limit.

We will consider, in turn, the lower order contraction singularities, and the definition of the various sheets of the function $f\left(s, \lambda^{2} ; t, M^{2}\right)$.
a) The total contraction, $\lambda^{2}=0$, due to Fig. 3d, is a logarithmic singularity on all sheets of $f\left(s, \lambda^{2} ; t, M^{2}\right)$. Its cut runs along the negative real $\lambda^{2}$ axis from - $\infty$ to 0 .
b) The self-energy contraction, $s\left(s-4 m^{2}\right)=0$, due to Fig. $3 a$, has $s=4 m^{2}$ as a singular square root branch point on the physical sheet. $s=0$ is a square root branch point not singular on the physical sheet.
c) The self-energy contraction $\left(\lambda^{2}-(M-m)^{2}\right)\left(\lambda^{2}-(M+m)^{2}\right)=0$, due to Fig. 3b, has the real section of Fig. 5. In this figure, $\lambda^{2}=0$ is the singular one lower order curve for this graph. The parabola lies outside the region of the cuts so that the curve and non-curve limits have the same singularity nature.

Arc $B C$ is not singular in the curve limit $\lambda^{2}-i \epsilon, M^{2}+i \in$ since this is also the physical limit. Nothing happens at $B$ so that $A B$ is nonsingular in both limits. The intersection at $C$, however, is an effective intersection with a singular curve of one lower order so that the singularity nature of the arc changes and $C D$ must be singular in both limits. In particular, the complex surfaces attached to $C D$, which lie in corresponding half planes, are singular on the physical sheet. Since we are restricting ourselves to $M^{2}$ real and not less than $\mathrm{m}^{2}$, we have a branch cut along the negative $\lambda^{2}$ real axis from $-\infty$ to $(M-m)^{2}$. The cut is of square root type.
d) The self-energy contraction $\left[t-(\lambda+m)^{2}\right]\left[t-(\lambda-m)^{2}\right]=0$, Fig. $3 c$, leads to a complex value of $\lambda^{2}$ for fixed real negative $t$ and would, therefore, lead to a complex branch cut it if were singular.

The real section is given by Fig. $>$ but with $M^{2}$ replaced by $t$. We see immediately that there are no real singularities in $\lambda^{2}$ for real negative $t$. The limit onto real $t$ is taken as $t+i \in$ in accordance with the Feynman rule for external variables. To reach a point on the surface with real $t+i \epsilon<0$, we can move off arc $A B$ or arc $C B$, neither of which is
singular in the curve limit. The former will reach points on the surface with $\operatorname{Im} \lambda^{2}>0$ and the latter with $\operatorname{Im} \lambda^{2}<0$. The important point to notice is that the singular surface attached to the arc $C D$ in fact does not connect to the surface with Re $t<0$. We conclude, therefore, that the complex points $\lambda^{2}=(\sqrt{t} \pm m)^{2}$ can only be reached over non-singular surfaces and are thus not singular on the physical sheet.
e) The triangle contraction of Fig. 2a also leads to complex values of $\lambda^{2}$ for real negative $t$. The real section of its Landau surface is an ellipse and is show in Fig. 6. Its lower order singular contractions are the lines $\lambda^{2}=0$ and $\lambda^{2}=(M-m)^{2}$ and the arc $C D$. Note that the ellipse lies above all the singular $\lambda^{2}$ cuts (only the arc $C D$ of the parabola is singular) so that the curve and non-curve limits onto the ellipse must be of the same singularity character.

The axis $c d$ and $a b$ are not singular in the physical limit + - , i.e. $t+i \epsilon, \lambda^{2}-i \in$, which is also the curve limit onto these arcs. At point $c$ we have an effective intersection with a non-singular one lower ordor contraction so that ce is non-singular in both the +- and ++ limits, the latter being the curve limit. We can continue along the ellipse all the way to point $f$ without becoming singular. But as we continue onto the arc dfeither at $d$ or $f$, we have an effective intersection with a singular one lower order contraction so that the singularity character changes at these points. Thus the arc df is singular in both limits and there is a complex surface of singularities, lying in corresponding half planes, attached to df.

We require a point with real negative $t$ approached from $t+i \in$. Such a point can only be reached from the ellipse over the surfaces attached to
arcs ce or cd which are not singular.
We conclude that the complex value of $\lambda^{2}$ lying on the Landau surface having real negative $t$ is in fact not singular on the physical sheet. This result also holds for the contraction graph of Fig. lc which is a special case with $M^{2}=m^{2}$.
f) The triangle contraction, $s\left(s-3 m^{2}\right)=0$, Fig. 2 b , has neither $s=0$ nor $s=3 m^{2}$ singular on the physical sheet. Both, however, are singular branch points in the first unphysical sheet reached by continuing downward through the normal threshold cut.
g) The real section of the Landau surface due to the triangle contraction of Fig. 2d is given by Fig. 4 with $t$ replaced by $\lambda^{2}$. The analysis for this contraction for $M^{2}>9 m^{2}$ was given in Ref. (1). For $m^{2}<M^{2}<9 m^{2}$ the analysis is very similar.

In particular, the ellipse is connected to the three disconnected ares by complex surfaces. Thus surface $A B$ ab connects the arc ab to the arc $A B$, etc. ${ }^{(8)}$ The singular one lower order contractions are the branch points $s=4 m^{2}$ and $\lambda^{2}=(M-m)^{2}$, the points $s=0$ and $\lambda^{2}=(M+m)^{2}$ being nonsingular on the physical sheet. The central ellipse is thus outside the region of the $-\infty<\lambda^{2}<(M-m)^{2}$ cut so that both curve and non-curve limits onto the ellipse must be the same.

The surfaces $A B a b, C D$ and $D E$ ed are not singular because the limit onto the ellipse along these surfaces is $+-\left(s+i s, \lambda^{2}-i \eta\right)$ which is also the physical limit. By examination, we can tell that the discriminant for this graph cannot vanish on the undistorted $\alpha$ contour for $s \leq 2 m^{2}$, $\lambda^{2} \geq M^{2}+m^{2}$ so that $A E$ is not singular in either limit and we can continue along the surface $A E$ ae onto the are ae without becoming singular. Thus
none of the arcs ba, ae, ed, or dc are singular nor are the complex surfaces attached to them.

The points $b$ and $c$ are effective intersections with singular normal thresholds so that the singularity nature changes at these points. Thus the complex surface $B C$ bc is in fact singular: this would lead to complex singularities. In the limit we eventually take, $M^{2}=m^{2}$, this surface will in fact disappear and cause no trouble.

We can summarize the results of this section in the limit $M^{2}=m^{2}$. The physical sheet is the topological product of two cut planes, the $s$ plane cut only from $4 m^{2}$ to $\infty$ and the $\lambda^{2}$ plane cut from - $\infty$ to 0 . This topological product will be denoted by pp. As we continue through the $s=4 m^{2}$ cut, we reach the sheet $q p$, keeping $\lambda^{2}$ in its $p$ sheet. This $q$ sheet in $s$ has superimposed left hand cuts from $-\infty$ to 0 , a right hand cut from $3 m^{2}$ to $\infty$, and the normal threshold cut from $4 m^{2}$ to $\infty$. The singularities due to the leading Landau surface will thus be defined with respect to this sheet structure provided by the lower order contractions.

## V. The Leading Landau Curve

The leading curve, due to the full contraction of Fig. lb, can be represented as

$$
\begin{align*}
\lambda_{ \pm}^{2}=\frac{-B \pm \sqrt{\Sigma}}{2 A} \text { with } & =s\left(s-4 m^{2}\right) \\
B & =2 s\left[2 m^{2} t+3 m^{4}-s t+m^{2} M^{2}-m^{2} s\right] \tag{11}
\end{align*}
$$

and

$$
\Sigma=B^{2}-4 A C=16 m^{2} s\left(s-3 m^{2}\right) v\left(s, t ; M^{2}\right)
$$

where $\sigma\left(s, t ; M^{2}\right)$ is the non-Landau curve already given in Eq. (6).

The real section in the $s \lambda^{2}$ plane is therefore limited by the condition $\Sigma>0$ which obtains for $s_{-}<s<0$ and $3 m^{2}<s<s_{+}$where $s_{ \pm}$are the roots of

$$
\begin{equation*}
\sigma\left(s_{ \pm}, t ; M^{2}\right)=0 \tag{12}
\end{equation*}
$$

We can determine the location of $s_{ \pm}$by inspection of Fig. 4: we see that for $t<0$, the points always arrange themselves in the order

$$
s_{-}<0<3 m^{2}<s_{+}
$$

By differentiating (11) we see that the vertical tangents appear at the zeros of $A$ and the zeros of $\Sigma$. The zeros of $A$ are at $s=0$ and $s=4 m^{2}$ and these vertical tangents are in fact asymptotes. This is expected since the tangents at the normal thresholds, at least for single loop graphs, are at infinity (9). The vertical tangents at finite points, $s=3 \mathrm{~m}^{2}$ and $s=s_{ \pm}$ are the zeros of $\Sigma$. The meaning of the location of the tangents can be seen by the following arguments based directly on the Landau equations.

The explicit form of the discrimanant D, given by Eq. (8), can be rewritten

$$
D=s f(\alpha)+\lambda^{2} g(\alpha)-K\left(\alpha, M^{2}, t\right)
$$

where

$$
\begin{aligned}
& f=\alpha_{1} \alpha_{3} \\
& g=-\alpha_{2}\left(\alpha_{1}+\alpha_{2}+\alpha_{3}+\alpha_{4}\right)
\end{aligned}
$$

and $K$ is everything else.
The tangents to the Landau curve are (9)

$$
\frac{d \lambda^{2}}{d s}=-\frac{f}{y}=\frac{\alpha_{1} \alpha_{3}}{\alpha_{2}\left(\alpha_{1}+\alpha_{2}+\alpha_{3}+\alpha_{4}\right)}
$$

where the $\alpha$ 's are considered to be those functions of s obtained by solving the Landau equations. The vertical tangents occur at $\mathrm{y}=0$; i.e. the triangle contraction curve $\alpha_{2}=0$ and the self-energy contractions $\alpha_{2}, \alpha_{i}=0, i=1,3,4$. These are the curves given by Fig. $2 b$ and Fig. $3 a$ giving vertical tangents at $s=0, s=3 \mathrm{~m}^{2}$ and $\mathrm{s}=4 \mathrm{~m}^{2}$. But not at $\mathrm{s}=\mathrm{s}_{ \pm}$.

The points $s_{ \pm}$arise not in the usual way, $g=0$, but by having $f=\infty$. This corresponds to the hyper-contour of the $\alpha$ integration in (7) being dragged to infinity thereby producing a singularity. This is the usual mechanism for the generation of non-Landau singularities. The points $s_{ \pm}$are in fact, for $t<0$, the non-Landau singularities for the fourth order curve. In a similar way, the third order curve, Fig. 4, has vertical tangents at the third order non-Iandau curves $s=(M \pm m)^{2}$.

The horizontal tangents are given by $\mathrm{f}=0$ and are thus due to the single contractions $\alpha_{1}=0$ and $\alpha_{3}=0$ and the double contraction $\alpha_{1}, \alpha_{3}=0$, given by Figs. 2a, $2 c$ and $3 c$ respectively. For $t<0$, these curves are complex in the $s \lambda^{2}$ space so that the leading curve can have complex horizontal tangents. These complex horizontal tangents are, as we have seen in the previous sections, not singular for $t<0$ and will cause no trouble in what follows.

The intersections of the leading surface with the third order surface are touches on the real section and are necessarily effective ${ }^{(7)}$. There are two asymptotic touches ( $s=0, \lambda^{2}= \pm \infty$ ) and three in the finjte plane. The real sections, for $M^{2}<9 m^{2}$, are shown in Figs. 7 and 8. The real. sections for $M^{2}>9 m^{2}$ are similar except that the touch is on the underside of the ellipse.

In the limit $M^{2}=m^{2}$, the third order curve reduces to

$$
\begin{equation*}
s \lambda^{2}\left(4 m^{2}-\lambda^{2}-s\right)=0 \tag{13}
\end{equation*}
$$

and the leading curve to

$$
\begin{equation*}
\lambda_{ \pm}^{2}=\frac{\left(s-4 m^{2}\right) m^{2}+\left(s-2 m^{2}\right) t \pm 2 \sqrt{\Sigma}}{s-4 m^{2}} \tag{14}
\end{equation*}
$$

where

$$
\Sigma=m^{2} t\left(s-3 m^{2}\right)\left(s+t-4 m^{2}\right) .
$$

The cuts of $\sqrt{\Sigma}$ are chosen to make $\lambda_{ \pm}{ }^{2}$ real analytic and run from $-\infty<s<3 m^{2}$ and $4 m^{2}-t<s<\infty$. The real section of this limiting situation is shown in Fig. 9. We see that at point L, the leading curve in fact intersects the third order curve. This occurs because at $M^{2}=m^{2}$, the third order curve develops a cusp at $s=4 \mathrm{~m}^{2}$ so that the theorem about effective intersections being touches no longer holds. The intersection, while no longer a touch, is nevertheless still effective. The limit $M^{2}=m^{2}$ is clearly the limit of a touching situation, Fig. 10.

The third order curve (13) is in fact not singular for s on sheet p but is for $s$ on sheet $q$ as can be seen from Fig 10 - the $\operatorname{arcs} D E$, adc, and $B D$ become the straight line $\lambda^{2}+s=4 m^{2}$. All are non-singular on $p$ but become singular on $q$.

For $s$ on sheet $p$ and $\lambda^{2}$ on its sheet $p$, the function $f p p\left(s, \lambda^{2} ; t, m^{2}\right)$ will have additional singularities due to the leading surface. There can be no singularity on a surface in the physical limit $s=i \in, \lambda^{2}$ - ie for $t<0$ taken onto real values in the limit $t+i \in$. Thus on sheet $p p$, the surface attached to KLPN in Fig. 9 a is not singular. At $K$ nothing happens so that KI is not singular in the + - limit and also not in the curve limit since $K J$ if outside the region of the cuts. We can continue from $K J$ to $N Q$ by means of the complex surface connecting them so that the curve
limit onto $N Q$ is not singular. There are therefore no complex singularities of $f_{p p}\left(s, \lambda^{2} ; t, m^{2}\right)$ and it must satisfy a Mandelstam type representation in the variables $s$ and $\lambda^{2}$.

In order to analyze the function $f_{q p}$, we begin at the point $T, F i g .9$, which is that point in the leading curve with $\lambda^{2}=m^{2}$. The properties of point $T$ are well known from the properties of the box diagram with all masses, both internal and external, equal. In particular, $T$ is not singular on $p$ but is singular in both limits on the $s$ sheet $q$. Thus the arc KTL is singular in the curve limit on $q$. The intersections at points $L$ and $P$ are with triangle contractions singular on $q$. Therefore, the surfaces attached to $L P$ are not singular and those attached to $P N$ are. $K$ is also an intersection with a singular triangle contraction curve on $q$ so that $K J$ is non-singular in both limits and continuing onto the curve $N Q$ over the complex surface indicates $\mathbb{N Q}$ is non-singular in the curve limit.

To summarize, $f_{p p}$ has no complex singularities. $f_{q p}$ has complex singularities on the surfaces attached to $K I$ and $P N$, all other surfaces being non-singular.
VI. The Function $F\left(s, t ; m^{2}\right)$

We have already seen, in Eq. 5, that the square diagram with full propagator insertion, Fig. la, is given by
$F_{p}\left(s, t ; m^{2}\right)=g^{2} f_{p p}\left(s, \mu^{2} ; t, m^{2}\right)+\int_{4 m}^{\infty} \sigma^{\prime}\left(\lambda^{2}\right) f_{p p}\left(s, \lambda^{2} ; t, m^{2}\right) d \lambda^{2}$.

The subscript $P$ denotes the physical sheet of the function $F$. $f\left(s, \mu^{2} ; t, m^{2}\right)$ is the bound state approximation, i.e. the square diagram with a simple line of mass $\mu$, the bound state mass, replacing the infinite sum of diagrams comprising the complex propagator. The integral term is
the correction due to keeping the continuum contributions to the complex propagator and will contain singularities not present in the bound state approximation alone.
$F_{P}$ certainly has the normal threshold cut from $s=4 m^{2}$ to $\infty$ since it is a cut of $f_{p p}$ independent of $\lambda^{2}$. The physical sheet $p$ is then well defined just above this real $s$ axis where the contour of the integral in (15) runs just below the real $\lambda^{2}$ axis. The continuation of $F_{p}$ into the complex plane is performed by moving counterclockwise off the $s$ axis. Since, as we have seen in the previous section, $f_{p p}$ has no complex singularities, the $\lambda^{2}$ contour is never distorted or pinched and we conclude that the physical sheet function, $F_{P}$, is defined everywhere in the complex plane by (15) with the contour undistorted.

We move onto sheet $Q$ by crossing the normal threshold cut in a clockwise direction. The definition is now

$$
\begin{equation*}
F_{Q}\left(s, t ; m^{2}\right)=E^{2} f_{q p}\left(s, \mu^{2} ; t, m^{2}\right)+\int_{C} \sigma^{1}\left(\lambda^{2}\right) f_{q p}\left(s, \lambda^{2} ; t, m^{2}\right) d \lambda^{2} \tag{16}
\end{equation*}
$$

where $C$ is a contour running from $4 m^{2}$ to $\infty$ but which may have been distorted by an advancing singularity $\lambda^{2}(s)$ of $f_{q p}$ as s moves throughout the Q sheet. A singularity of the function occurs when the distortion of the contour is no longer possible, either because the singularity hits the endpoint or because it is pinched. The pinch itself can occur two ways either the contour can be pinched between coincident singularities of $f_{q p}$ or a singularity of $f_{q p}$ can pinch with the douple pole of $\sigma^{\prime}\left(\lambda^{2}\right)$.

This latter distinction can be sharpened by a consideration of the diagram of Fig . Id, the square diagram with a single bubble insertion.

This diagram is given by

$$
\begin{equation*}
F_{0}\left(s, t ; m^{2}\right)=\int_{4 m^{2}}^{\infty}\left(\frac{\lambda^{2}-4 m^{2}}{\lambda^{2}}\right)^{1 / 2} f\left(s, \lambda^{2} ; t, m^{2}\right) d \lambda^{2} \tag{17}
\end{equation*}
$$

It is known from the "tautening" conditions for the dual diagrams of such graphs (10), that the singularities of $F_{0}$ are in fact also the singularities of any graph of the type in Fig. lc, where the number of bubbles in the chain is finite. We see that the full diagram $F$ will certainly contain the singularities of $F_{0}$ and in addition will contain singularities due to the binding of the particles into a particle of mass H. The singularities due to the bound state, of course, are not present in any finite order of perturbation theory but arise only through the summation of an infinity of graphs.

The physical sheet function $F_{O P}\left(s, t ; m^{2}\right)$ has the normal threshold cut and no other singularities. If we continue downward through this cut onto sheet $Q$, we see that $F_{O Q}$ has in addition to the normal threshold cut, a left hand cut with $-\infty<s<0$ due to the various contractions of $f_{q p}$, and a right hand cut $3 m^{2}<s<\infty$ due to the triangle contraction already mentioned. The other triangle contraction, $\lambda^{2}=4 m^{2}-s$, does give an endpoint singularity at $s=0$ which simply superimposes another left hand cut on those already there. $F_{O Q}$ also has possible singularities due to the leading Landau curve of $f_{q p}$.

In order to study these singularities due to the leading Iandau surface, we have to know the location of the $\lambda^{2}$ singularities on the surface as s ranges over the complex plane, i.e. we need the mapping from the $s$ to the $\lambda^{2}$ plane provided by $E q$. (13). The essentials of the mapping
are relatively simple: $\lambda_{+}{ }^{2}$ maps the entire $s$ plane into the boundary and interior of a circle of radius $m^{2}$ - $t$ centered at the origin of the $\lambda^{2}$ plane, and $\lambda_{-}{ }^{2}$ maps the entire $s$ plane into the boundary and exterior of the circle.

Referring to Fig. Il, the specifics of the mapping are:

$$
\lambda_{+}^{2}
$$

Real maps:

$$
\begin{array}{cc}
\left(-\infty<s<3 m^{2}\right)_{+} \rightarrow c b & \left(-\infty<s<3 m^{2}\right)_{+} \rightarrow d b \\
\left(-\infty<s<3 m^{2}\right)_{-} \rightarrow d b & \left(-\infty<s<3 m^{2}\right)_{-} \rightarrow c b \\
\left(3 m^{2}<s<4 m^{2}\right)_{ \pm} \rightarrow(b 0)_{\mp} & \left(3 m^{2}<s<4 m^{2}\right)_{ \pm} \rightarrow(b \infty)_{ \pm} \\
\left(4 m^{2}<s<4 m^{2}-t\right)_{ \pm} \rightarrow(0 a)_{\mp} & \left(4 m^{2}<s<4 m^{2}-t\right)_{ \pm} \rightarrow(-\infty a)_{ \pm} \\
\left(4 m^{2}-t<s<\infty\right)_{+} \rightarrow a c & \left(4 m^{2}-t<s<\infty\right)_{+} \rightarrow a d \\
\left(4 m^{2}-t<s<\infty\right)_{-} \rightarrow a d & \left(4 m^{2}-t<s<\infty\right)_{-} \rightarrow a c
\end{array}
$$

## Complex maps:

$$
\begin{array}{ll}
\text { l.h.p. } \rightarrow C_{+} & \text {l.h.p. } \rightarrow\left(\text { l.h.p. }-C_{-}\right) \\
\text {u.h.p. } \rightarrow c_{-} & \text {u.h.p. } \rightarrow\left(\text { u.h.p. }-C_{+}\right)
\end{array}
$$

The notation $\left(-\infty<s<3 m^{2}\right)_{ \pm}$means the upper lower edge of the real axis below $3 \mathrm{~m}^{2}$.

Note that $\lambda_{+}{ }^{2}$ belongs to the surface attached to arc KIN and $\lambda_{-}{ }^{2}$ to the surface attached to KJ and $N Q$ of Fig. 9 a . Note also that the inverse map $s=s\left(\lambda^{2}\right)$ given by Eq. (13) is rational so that $s$ is real whenever $\lambda^{2}$ is. Thus we can already be assured that in the bound state case, pinches or endpoint singularities will lead to s singularities on the real axis.

We have already indicated that $\mathrm{F}_{\mathrm{Op}}$ has only the normal threshold cut and is defined throughout the complex physical s sheet by (17) with an undistorted contour. Fig. 12a shows the search of the $Q$ sheet reached by continuing clockwise down through the normal threshold cut. Fig. 12b shows the corresponding motion of the $\lambda^{2}$ singularities - corresponding points are labelled by the same letter. The notation $a(+)$ means that this point corresponds to the map provided by $\lambda_{+}{ }^{2}$. Note that along a $\lambda^{2}$ path, a point will change names, say from $(+)$ to ( - ). This indicates that a smooth motion in the $s$ plane in fact is quite complicated in terms of motion on the Landau surfaces, the path changing surfaces as certain regions are entered. By continuity, however, the singularity character cannot change as we go from surface to surface unless we cross a cut. Then the singularity can move into a sheet other then the one under consideration.

It is clear from Fig. l2b that an endpoint singularity arises when point $m(+)$ hits the endpoint $\lambda^{2}=4 m^{2}$. This occurs for

$$
\begin{equation*}
S_{E}=4 m^{2} \frac{8 m^{4}-4 t m^{2}+\left(t-m^{2}\right)^{2}}{8 m^{4}-8 t m^{2}+\left(t-m^{2}\right)^{2}} \tag{18}
\end{equation*}
$$

with $3 \mathrm{~m}^{2}<\mathrm{S}_{\mathrm{E}}<4 \mathrm{~m}^{2}$ on the upper edge of the $3 \mathrm{~m}^{2}$ triangle cut. This endpoint singularity can only occur (on sheet $Q$ ) by $m(+)$ hitting the endpoint, not $m(-)$, since only $m(+)$ is singular in sheet $Q(m(+)$ belongs to the singular, on $Q$, surface $K L$ in Fig. $9 a) . m(+)$ can only hit the endpoint of $\lambda^{2}=4 m^{2}$ lies inside the circle, i.e. $|t| \leq 3 m^{2}$. In terms of Fig. 9a, we see that for $|t| \leq 3 m^{2}, \lambda^{2}=4 m^{2}$ intersects the singular surface $K L$
while for $|t|>3 m^{2}$, it intersects the non-singular (on sheet $Q$ ) surface $K J$.

It is also clear from Fig. $12 b$ that nowhere, on sheet $Q$, does a moving $\lambda^{2}$ singularity cause the contour to be distorted. Thus (17) has no pinch singularities on sheet $Q$. We see then that the only additional singularity of (17) due to the leading surface is $S_{E}$ of $E q$. (18) and this occurs on sheet $Q$.

If now we turn to the diagram with the full spectral function inserted, Eqs. (15) and (16), and look only at the integral term, we can see the extra effects due the double pole at $\lambda^{2}=\mu^{2}$ in the spectral function.

Again on the sheet $P$, there is no effect at all. If we search to the point

$$
\begin{equation*}
S_{p}=4 m^{2} \frac{\mu^{4}-2 m^{2} \mu^{2}-\mu^{2} t+\left(t-m^{2}\right)^{2}}{\mu^{4}-2 m^{2} \mu^{2}-2 \mu^{2} t+\left(t-m^{2}\right)^{2}} \tag{19}
\end{equation*}
$$

on sheet $Q$ in Fig. 12a, we have a $\lambda^{2}$ singularity moving to $\lambda^{2}=\mu^{2}$ but without distorting the contour. Thus there are no extra pinch singularities on sheet $Q$. We can in fact only get a pinch of the contour at the double pole by continuing through the $3 \mathrm{~m}^{2}$ cut for $3 \mathrm{~m}^{2}<\mathrm{s}<4 \mathrm{~m}^{2}$ from above. On the sheet so reached, call it $Q^{\prime}$, there will be a singularity of the integral at $S=S_{p}$. This will occur only when $t>m^{2}-\mu^{2}$ so that point $K$ in Fig. 9a appears below $\lambda^{2}=\mu^{2}$. The result is that the first singularity of the integral term in (5), due to the double pole, appears in a sheet very far from the physical region.

## VII. Conclusions

The function we have been considering is given by Eq. (5) and represents the equal mass square diagram with a full propagator inserted in the cross channel. The propagator is assumed to have a bound state pole at a mass $\mu$.

The approximation usually made, the bound state approximation, is obtaincd by setting $F\left(s, t ; m^{2}\right)=g^{2} f\left(s, \mu^{2} ; t, m^{2}\right)$. This corresponds to the replacement of the full graph of Fig. la by a simple square graph containing a simple line of mass $\mu$ replacing the propagator.

On the physical sheet $P$ (in the $s$ variable) the approximation becomes $F_{p}\left(s, t ; m^{2}\right)=g^{2} f_{p p}\left(s, \mu ; t, m^{2}\right)$ which, as we have already seen, contains no singularities other than the normal threshold cut. On the first unphysical sheet, $F_{Q}\left(s, t ; m^{2}\right)=g^{2} f_{q p}\left(s, \mu^{2} ; t, m^{2}\right)$, and there is a branch point at $S=S_{p}$ on the upper edge of the real axis when $t>m^{2}-\mu^{2}$.

The corrections to the bound state approximation are contained in the integral term in Eq. (5). On sheet $P$, the integral in fact has no additional singularities and can be calculated as a weight over a simple elastic unitarity dispersion relation. On sheet $Q$, the intcgral term has a single additional branch point on the upper edge of the trianglc cut at position $3 m^{2}<S_{E}<4 m^{2}$ which occurs only for $t<-3 m^{2}$. This branch point occurs whether or not the propagator has a bound state pole, but is fairly far from the physical region. Therc is, in addition, a singularity of the integral on a still further shcet $Q^{\prime}$ as described in the last section.

## Appendix

The model for our bound state will be an infinite sum of chains of bubble diagrams.

The single bubble, composed of two neutral scalar particles of mass m , will be denoted by $\triangle(\mathrm{s})$ where

$$
\begin{equation*}
\Delta(s)=-\frac{1}{16 \pi} \int_{4 m}^{\infty} 2 \frac{p\left(s^{i}\right) d s^{1}}{s^{i}-s} \tag{20}
\end{equation*}
$$

and

$$
p\left(s^{\prime}\right)=\left(\frac{s^{2}-4 m^{2}}{s^{2}}\right)^{1 / 2}
$$

The quantity that acts like a propagator will be called $\pi(s)$ and is given by the sum of bubbles in Fig. 13.

$$
\pi(s)=\frac{\Delta(s)}{1-\lambda_{0} \Delta(s)}
$$

where $\lambda_{0}$ is a direct bare four particle coupling.
If we are to insert tho propagator into a diagram, it will have to be attached at both ends so that $P(s)=\lambda_{0}^{2} \pi(s)$ is more properly the propagator. Wc know that for a critical range of the renormalized coupling constant (1l), $\lambda_{0}^{2} \pi(s)$ will have a bound state pole at $s=\mu^{2}<4 m^{2}$.

The integral for the bubble, Eq. (20), in fact diverges and we can extract the infinite constant by subtracting at the bound state pole:

$$
\Delta(s)=-B-\left(s-\mu^{2}\right) \Delta f(s)
$$

where $B=-\Delta\left(\mu^{2}\right)$ is a positive logarithmically infinite constant and

$$
\Delta_{f}(s)=\int_{4 m^{2}}^{\infty} d s^{\prime} \frac{p\left(s^{i}\right) d s^{i}}{\left(s^{3}-s\right)\left(s^{i}-\mu^{2}\right)}
$$

is a well defined function.

The condition that $\lambda_{0}{ }^{2} \pi(s)$ has a pole at $s=\mu^{2}$ is $I+\lambda_{0} B=0$ which reduces the propagator to

$$
P(s)=\lambda_{0}^{2} \pi(s)=\frac{1}{s-\mu^{2}} \frac{1}{\Delta_{f}(s)}-\lambda_{0}
$$

The residue at the bound state pole is $g^{2}=\frac{1}{\Delta_{f}\left(\mu^{2}\right)}$, where $g$ is the coupling of the bound state to the two external ${ }^{1}$ particles.

We can represent

$$
\frac{1}{\Delta_{f}(s)}=g^{2}-\left(s-\mu^{2}\right) \int_{4 m^{2}}^{\infty} \frac{p\left(s^{i}\right)}{\left(s^{i}-\mu^{2}\right)^{2}} \frac{1}{\Delta_{f}\left(s^{i}\right)^{2}} \frac{d s^{1}}{s^{1}-s}
$$

so that

$$
P(s)=-\lambda_{0}-\int_{0}^{\infty} \frac{\sigma\left(\lambda^{2}\right) d \lambda^{2}}{\lambda^{2}-s-i \epsilon}
$$

where

$$
\sigma\left(\lambda^{2}\right)=g^{2} \delta\left(\lambda^{2}-\mu^{2}\right)+\sigma^{\prime}\left(\lambda^{2}\right) \theta\left(\lambda^{2}-4 m^{2}\right)
$$

and

$$
\sigma^{\prime}\left(\lambda^{2}\right)=\frac{p\left(\lambda^{2}\right)}{\left(\lambda^{2}-\mu^{2}\right)^{2}} \frac{1}{\left|\Delta_{f}\left(\lambda^{2}\right)\right|^{2}}
$$

In the case that the sum of chains forms a resonance rather than a bound state, only the spectral weight changes:

$$
\sigma_{r e s}\left(\lambda^{2}\right)=\frac{p\left(\lambda^{2}\right)}{\left(\lambda^{2}-I\right)\left(\lambda^{2}-I^{*}\right)} \frac{1}{\left|\Delta_{f}\left(\lambda^{2}\right)\right|^{2}}
$$

where I is the position of the resonance in the complex plane.

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## Figure Captions

Fig. 1 (a) The function $F\left(s, t ; M^{2}\right)$, the square diagram with full propagator insertion. $p_{12}^{2}=p_{14}^{2}=p_{13}^{2}=M^{2}, P_{23}^{2}=M^{2}$, $s \equiv p_{13}^{2}=\left(p_{14}+p_{34}\right)^{2}, t \equiv p_{24}^{2}=\left(p_{23}+p_{34}\right)^{2}$ (b) The function $f\left(s, \lambda ; t, M^{2}\right)$ - the square diagram with one internal mass $\lambda$ (c) A typical diagram contributing to (a) (d) Square diagram with single bubble insertion.

Fig. 2 The triangle contractions of Fig. lb.
Fig. 3 The lowest order contractions of Fig. Ib.
Fig. 4 The non-Landau curve for $m^{2}<M^{2}<9 m^{2}$.
Fig. 5 The real section of the Landau curve due to the contraction of Fig. lb.

Fig. 6 The real section of the Landau curve due to the contraction of Fig. 2a.

Fig. 7 The leading Landau curve for $M^{2}<9 m^{2}$, $t>m(-M+m)$. The intersections of the third and fourth order curves occur at points G, L and P. The third order curve labelled as Fig. 4.
Fig. 8 The leading Landau curve for $M^{2}<9 m^{2}, t<m(-M+m)$.
Fig. 9 (a) The leading Landau curve for $M^{2}=m^{2}$, $t<-m^{2}$
(b) The leading curve for $M^{2}=m^{2},-m^{2}<t<0$.

Fig. 10 The leading Landau curve in the limiting case $M^{2}=m^{2}+\epsilon$.
Fig. 11 The regions of the complex $\lambda^{2}$ plane.
Fig. 12 (a) Search of the sheet $Q$ of the $S$ plane.

Fig. 12 (b) Corresponding motion of the $\lambda_{ \pm}{ }^{2}$ given by the mapping of $s$ to $\lambda^{2}$ provided by the leading surface. $n(+)$ means the point corresponding to $n$ in the $s$ search having the name $\lambda_{+}{ }^{2}$.
Fig. 13 Sum of graphs comprising the model propagator.


Fig. 1


FIG. 2


b.



C.
d.

Fig. 3


FIG. 4


Fig. 5


Fig. 6


FIG. 7


FIG. 8


Fig. 9a


Fig. 9b


Fig. 10


FIG. 11
$S$ PLANE

(a)

(b)

FIG. 12


FIG. 13


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