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SEPARABLE OPERATORS IN SCATTERING THEORY*

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ABSTRACT

A general theory of non-local, separable operators in scattering theory is developed on the basis of variational techniques and eigenvalue analysis. The formalism gives as limiting cases the three-particle isobar model as well as the quasiparticle formalism. The extreme sensitivity of off-shell multiparticle theories to calculation approximation schemes is minimized by our methods. The theory provides optimal systematic extension and improvement of several multiparticle models.

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I. INTRODUCTION

The use of separable operators in scattering theory has had many diverse motivations, among them the reduced dimensionality of integral equations (the two-body Lippmann-Schwinger equation reduces to quadrature), simulation of non-local interactions in nuclear shell models, the possibility of convergent perturbation methods for arbitrary coupling strengths, as well as their natural occurrence in isobar models dominated by bound states and resonances. Recent advances in three-particle theories and subsequent efforts at realistic calculations have generated increased interest in separable non-local operators. Although there exists an extensive literature on separable operators most treatments have been essentially phenomenological.

We develop here a rather complete theory of separable operators utilizing several powerful analytic techniques recently developed. Although these techniques per se are of largely academic interest for two-particle scattering, they are of great utility in analyzing multiparticle and multichannel theories where two-particle amplitudes appear as components of the kernels. Three-particle scattering, being a more sensitive probe of pair interactions than on-shell two-particle scattering, requires considerably more sophisticated analysis to avoid severe model dependence from approximation schemes. The occasionally impressive disagreement of some recent three-particle calculations suggest that calculation schemes in such off-shell theories may define rather than approximate the underlying dynamics.

Separable approximations are especially useful in reducing the high dimensionality of three- and multi-particle integral equations. Several recent models have exploited dominant bound state or isobar terms to achieve a simple calculational framework. We show that significantly more information from two particle subsystems may be incorporated while maintaining separability. It is both feasible and necessary to incorporate continuum scattering in such models if in addition to achieving a practical

calculation program we are to respect either particular analytic structure of pair interactions or the known experimental data.

We develop our analysis about three concepts: the Schmidt dissection procedure in Fredholm theory, eigenvalue analysis of the scattering kernel, and variationally optimized non-local approximations to local operators. The resulting formalism gives as limiting cases several models for multiparticle scattering such as Weinberg's quasiparticle formalism and the three-particle isobar model. Our analysis reveals the limitations of such models and provides optimal methods for their systematic improvement and extension.

To motivate our study, Section II presents a brief summary of relevant basic concepts of Fredholm methods and multiparticle scattering theory. Finite rank operators and the Schmidt dissection procedure are introduced in Section III; their relation to the quasiparticle method is derived in Section IV. We treat separable potentials in Section V in a variationally optimized formalism. Finally we discuss the choice of separable operators in Section VI. As a typical application of our formalism the three-particle isobar model is discussed in Section VII.

II. MATHEMATICAL FOUNDATIONS

We begin with a concise review of the relevant aspects of Fredholm theory as applied to the multiparticle Lippman-Schwinger (L-S) equation. Wherever feasible, our notation will parallel that of Smithies,[1] Sugar and Blankenbecler,[2] and of Weinberg.[3] (The reader is referred to these sources for more detailed exposition of the topics to follow.) Obvious or irrelevant variables are suppressed.

The scattering operator T is defined by the L-S equation

$$T = V + K\,T \tag{2.1}$$

for the Hamiltonian $H = H_o + V$; K is the kernel containing the free Green's function G_o for energy s:

$$K \equiv V\left[H_o - s\right]^{-1} = V\,G_o \tag{2.2}$$

The formal solution to Eq. (2.1) is given by

$$T = (1 + R)\,V \tag{2.3}$$

in terms of the resolvent operator R

$$1 + R = (1 - K)^{-1} \tag{2.4}$$

which satisfies the resolvent identity

$$R - K = KR = RK\ . \tag{2.5}$$

Basic to our treatment is the existence of the Schmidt norm

$$\|K\|^2 = \tau(s) \equiv \mathrm{Tr}\left[KK^{\dagger}\right] < \infty\ . \tag{2.6}$$

The existence of this norm guarantees the complete continuity of the operator K, and consequently its uniform approximation by finite rank non-local operators, as is shown in Section III. This norm is a sufficient but not necessary condition; it is however a convenient and general criterion for complete continuity. (See remarks following Eq. (2.18).) In addition, the radius of convergence r(s) of the Born series for the resolvent

$$R = \sum_{n=0}^{\infty} V K^n \tag{2.7}$$

is bounded by

$$r(s) \geq \tau^{-1/2}(s) \tag{2.8}$$

The uniform convergence of the matrix elements of the Born approximation scattering amplitude in energy as well as initial and final momentum is guaranteed by the bound

$$\left| \langle \underline{p}' \right| \left\{ T(s) - V \sum_{n=0}^{N} K(s)^n \right\} \left| \underline{p} \rangle \right| \leq (2\pi)^{-3} (1-\tau^{1/2})^{-1} \tau^{(N+1)/2} \int d^3r \left| V(r) \right| \tag{2.9}$$

The finite norm τ likewise justifies the Fredholm series for the resolvent

$$R(s) = N(s) D^{-1}(s) \tag{2.10}$$

$$N(s) = \sum_{n=0}^{\infty} N_n(s) \; ; \; D(s) = \sum_{n=0}^{\infty} D_n(s) \tag{2.11}$$

with the recursion relations

$$\begin{aligned} D_o &= 1 \quad ; \quad N_o = K \\ D_{n+1} &= -(n+1)^{-1} \mathrm{Tr} \left[N_n - D_n K \right] \\ N_{n+1} &= \left[D_{n+1} + N_n \right] K \end{aligned} \tag{2.12}$$

The series Eq. (2.8) obviously diverges if, for a given energy, one of the eigenvalues

$$K\left|\psi_\nu(s)\right> = \eta_\nu(s)\left|\psi_\nu(s)\right> \tag{2.13}$$

has a norm greater than unity:

$$\left|\eta_\nu(s)\right| \geq 1 \tag{2.14}$$

Eigenvalues satisfying Eq. (2.14) correspond to bound states and resonances of the system and define the minimum reduction of the kernel in the quasiparticle formalism.[3]

In order to guarantee a finite norm in Eq. (2.6) a hierarchy of concepts must be accommodated as the particle number n of the system is increased: (a) for $n = 2$ the finite norm is essentially a constraint on the interaction as we show below, (b) for $n > 2$ the kernel must have a connected structure as we shall discuss, and (c) for $n > 3$ kernels for disjoint subsystems must be coupled by convolutions.[4,5]

Possible infinite norms relevant to our methods may be avoided by appropriate modification of the resolvent, a general representation for which is that of Sugar and Blankenbecler[2]:

$$R = L\left[M(1-K)L\right]^{-1} M \tag{2.15}$$

subject to the conditions that both M^{-1} and $\det\left[M(1-K)L\right]$ exist. Although such modified operators leave the Born and Fredholm series unaltered, they may yield a finite norm or a connected kernel.

As an example of the first situation (a) above, we observe that a local central potential gives for the norm of Eq. (2.6)

$$\tau(s) = \left[8\pi \operatorname{Im} s^{1/2}\right]^{-1} \int d^3r \left|V(r)\right|^2 \tag{2.16}$$

which is seen to be unbounded in the physical scattering region Im $s = 0$, $s > 0$. However appropriate modification of the kernel as in Eq. (2.15) gives the symmetrized kernel

$$K = V^{1/2} G_o V^{1/2} \tag{2.17}$$

with the energy-independent norm

$$\tau(s) = \left[16\pi^2\right]^{-1} \iint d^3 r \, d^3 r' \, \left| V(\underline{r}) \right| \left| V(\underline{r}') \right| \left| \underline{r} - \underline{r}' \right|^{-2} \tag{2.18}$$

This symmetrization[2,3] represents the variationally optimal choice, i.e., it yields the minimum norm in Eq. (2.6) and the largest radius of convergence via Eq. (2.9).

From the energy-independence in Eq. (2.18) we may infer that the Schmidt norm is an excessively stringent condition. This norm and the related radius of convergence in Eq. (2.8) give no indication of the validity of the Born series for sufficiently high energy.

A second source of divergence is seen by naively introducing a sum of pair interactions (we take $n = 3$ as an example, labeling the pair operators by the third or spectator particle, e.g., $V_3 = V_{12}(r_{12})$, etc.)

$$V = \sum_{\alpha=1}^{3} V_\alpha = \sum_{\alpha} V_\alpha(r_\beta - r_\gamma) \quad (\alpha, \beta, \gamma \text{ distinct}) \tag{2.19}$$

This gives infinite contributions to the norm (Eq. 2.6); such terms correspond to disconnect graphs of perturbation theory and represent one (or more) particles propagating freely without interaction.[3]

These infinite terms may be avoided by various choices of the "connecting" operators of Eq. (2.15) a typical example being

$$L = \sum_{\alpha=1}^{3} R_\alpha - 2; \quad M = 1 \tag{2.20}$$

where R_α is the pair resolvent $\left[1-K_\alpha\right]^{-1}$. This modified kernel yields the connected multiparticle kernel of Weinberg[3]

$$(1-K)\,L = 1 - \sum_{\alpha=1}^{3} K_\alpha R_\alpha (K_\beta + K_\gamma) \qquad (2.21)$$

$(\alpha, \beta, \gamma$ distinct)

We obtain another connected kernel by decomposing the scattering operator T in the form

$$\begin{aligned} T^\alpha &\equiv V_\alpha (1 + G_0)\, T \\ T &= \sum_{\alpha=1}^{3} T^\alpha \end{aligned} \qquad (2.22)$$

By using the L-S Eq. (2.1) we see the operators T^α satisfy the coupled integral equations

$$T^\alpha = T_\alpha + T_\alpha G_0 (T^\beta + T^\gamma) \qquad (2.23)$$

These are the Faddeev equations,[6] containing T_α, the two-body scattering operator in the three-particle Hilbert space for total energy s.

$$T_\alpha = \left[1 - K_\alpha\right]^{-1} V_\alpha = T_\alpha \left(s - p_\alpha^2 / 2m_\alpha\right) \qquad (2.24)$$

The existence and domains of analyticity of the amplitudes Eq. (2.23) have been thoroughly investigated by Faddeev.[6] The appearance of two-particle amplitudes is especially advantageous for pairs with bound states since a potential would not exhibit the point spectrum. It should be emphasized that the Faddeev choice for the connected kernel, while advantageous in several respects (e.g., existence analysis, analytic structure) in no sense avoids the underlying potential;

a potential is implicitly contained in any choice for the off-shell continuations in energy and momentum required by the integral equation, Eq. (2.23). We consider this point more fully in Section III. In this context, the choice between non-local approximation to the operator T rather than the potential is a matter of practicality rather than principle. In our treatment the choice of T is especially useful for understanding the formal structure, while separable potentials are the more natural choice for practical utilization of scattering data.

Our development yields both a practical separable formalism and complete utilization of data both for bound states and continuum scattering.

III. FINITE RANK OPERATORS

We derive our methods from the theorem[1,7] that a completely continuous operator K may be uniformly approximated by a finite rank operator K_n; i.e., for a given $\epsilon > 0$ there exists a K_n for which

$$|K(x,y) - K_n(x,y)| < \epsilon \quad . \tag{3.1}$$

The rank is defined as the minimum integer n for which an operator may be expressed in the form

$$K_n(x,y) = \sum_{\nu=1}^{n} |\nu,x><\nu,y| \tag{3.2}$$

where x,y denote vectors of arbitrary dimension.

We now apply the dissection process (Abspaltungsverfahren) of Schmidt[1,8] to obtain a systematic approximation leading to the Fredholm or N/D system of linear equations via a dissection of the kernel into a separable term and a non-separable remainder. A variant of our method, the quasiparticle formalism of Weinberg[3] reduces the remainder term (in the sense of its norm) such that the Born series of Eq. (2.8) is valid. In isobar models the separable terms of the dissection are assumed to be sufficiently dominant that the remainder term may be ignored. These models will be considered in subsequent sections.

A. Dissection Process

Consider the decomposition of the kernel

$$K(x,y) = P(x,y) + Q(x,y) \tag{3.3}$$

where P is a separable operator of rank n:

$$P(x,y) = \sum_{\nu=1}^{n} |\nu,x><\nu,y| \quad . \tag{3.4}$$

The bound of Eq. (3.1) then becomes a bound on the operator Q:

$$\left|K(x,y) - P(x,y)\right| = \left|Q(x,y)\right| < \epsilon \quad . \tag{3.5}$$

In analogy with Eq. (2.4) we define the resolvent of the operator Q

$$\left[1 - Q\right]^{-1} = 1 + R_Q \tag{3.6}$$

and with it express the L-S equation in terms of the dissection of Eq. (3.3):

$$\begin{aligned} T &= V + (P + Q)\,T \\ &= (1 + R_Q)\left[V + PT\right] \\ &\equiv \underset{\sim}{V} + \underset{\sim}{P}\,T \quad . \end{aligned} \tag{3.7}$$

In terms of the eigenvalue analysis of Section II, we reduce the operator Q of Eq. (3.5) such that the norm of its largest eigenvalue is less than unity. The resolvent now is given by

$$R = R_Q + D^{-1} \sum_{\mu,\nu=1}^{n} (1+R_Q)\,|\mu>\, N_{\mu\nu}\, <\nu|\,(1+R_Q)^{\dagger} \tag{3.8}$$

where the inverse of the kernel $1 - \underset{\sim}{P}$ is given in terms of its determinant and adjoint:

$$D = \det\left[1 - \underset{\sim}{P}\right] \quad , \tag{3.9}$$

$$N = \mathrm{adj}\left[1 - \underset{\sim}{P}\right] \quad . \tag{3.10}$$

From Eqs. (3.5, 3.6), we see that in the limit $\epsilon \longrightarrow 0$, the modified resolvent becomes separable. We note that our resolvent differs from the simplest pole approximation in that the separable term of Eq. (3.8) incorporates knowledge of the non-separable portions of the kernel decomposition of Eq. (3.3). As we shall see, this is equivalent to modification of bound state pole terms by the continuum scattering.

B. Spectral Decomposition

We introduce physical content into the preceding formalism by expressing the partial wave scattering operator T

$$T = V + V G V \tag{3.11}$$

in terms of the full Green's function G with the spectral decomposition[10]

$$G_\ell(p,q;s) = \sum_n \frac{|\psi_n(p)><\psi_n(q)|}{s - s_n} + \frac{2}{\pi}\int_0^\infty dk' \frac{|\psi(s',p)><\psi^*(s',q)|}{s - s' + i\epsilon} \tag{3.12}$$

where $s' = k'^2$, ψ_n is the n_{th} bound state wavefunction for a given partial wave, and $\psi(s;p)$ is the p-space Fourier transform of the continuum wavefunction for energy $s > 0$. The scattering operator then becomes:

$$T(p,q;s) = V(p,q) + \sum_n \frac{V|\psi_n(p)><\psi_n(q)|V}{s - s_n} + \frac{2}{\pi}\int_0^\infty dk' \frac{V|\psi(s';p><\psi^*(s',q)|V}{s - s' + i\epsilon}. \tag{3.13}$$

The spectral representation of Eq. (3.13) highlights the advantage of the Faddeev equation Eq. (2.23) in the presence of pair bound states; the meromorphic parts of the two-particle operators are explicitly isolated, facilitating model calculations with bound subsystems.

The spectral representation exhibits the essentially separable nature of the scattering operator for s sufficiently near bound state energies s_n or sufficiently near the energy of a dominant continuum resonance. There is in general no absolute criterion for "sufficiently close"; the concept becomes especially ambiguous in the context of multiparticle formalisms such as Faddeev's for three-particle systems. We see from Eqs. (2.23, 2.24) that as a component of an integral equation kernel, the two-body amplitude T(p,q;s) gives contributions over the domains $p \in [0,\infty]$, $s \in [E, -\infty]$ where E is the maximum energy available to the pair in the three-particle Hilbert space. Thus in this general context the dominant

term or dominant singularity concept has meaning only with knowledge of the behavior of the off-shell pair amplitudes and the domains of the variables in the multiparticle calculation.

C. Fredholm Reduction

As an example of a particular separable decomposition we consider the methods of Noyes[11] and Kowalski[12] whereby the two-particle L-S equation is reduced to a non-singular integral equation with separable and remainder terms analogous to the decomposition of Eq. (3.3).

We define the operator

$$F = W + \Lambda F \tag{3.14}$$

$$R = \Lambda + \Lambda R$$

where

$$W(p, q) = V(p, q)/V(q, q)$$

$$\Lambda(p,q;k) = 2/\pi \frac{q^2}{k^2 - q^2 - i\epsilon} \left[V(p,q) - V(k,q)\, W(p,k)\right] \quad . \tag{3.15}$$

In terms of these operators, the on-shell scattering operator satisfies the non-singular integral equation (in operator notation)

$$T(k) = T(k,k;s=k^2) = -\left[1 + V G_o F\right]^{-1} V \tag{3.16}$$

while the general off-shell operator assumes the form

$$T(p,p';s=k^2) = F(p,k)\, T(s)\, F(k,p') + \frac{\pi}{2} \frac{k^2 - p'^2}{p'^2} R(p,p';k) \quad . \tag{3.17}$$

The amplitude is rigorously separable in the limit $R = \Lambda = 0$; $F = W$, which implies V is a rank one separable operator. In this limit, the on-shell operator Eq. (3.16) yields the Schwinger variational amplitude for plane-wave trial wavefunctions[13]

$$T(s) = V\left[1 - V G_o V\right]^{-1} \quad . \tag{3.18}$$

The remainder term vanishes for the half on-shell operator

$$R(p,k;k) = R(k,p;k) = 0 \tag{3.19}$$

revealing a useful interpretation of the F functions:

$$F(p,k) = \frac{T(p,k;k^2)}{T(k,k;k^2)} \quad . \tag{3.20}$$

From this relation, the analytic structure of F and subsequently the separable term of Eq. (3.17) can be examined in order to avoid possible undesirable singularities in the integration domain of multiparticle kernels. (cf. remarks in Ref. 22.)

The decomposition of Eq. (3.17) has been studied by Guennéguès[14] in a similar formalism for a superposition of Yukawa potentials in order to determine the domains of meromorphy in energy as well as in initial and final momenta. This treatment obtains for T a dissection into a Born term, a meromorphic separable term, and a non-separable homomorphic residue with the advantage of explicitly exhibiting the Jost function in the separable term. The Jost function may likewise be introduced into our spectral representation through the wavefunction representation[10]

$$\psi_\ell(k, r) = \left[k^{\ell+1}/f_\ell(-k)\right]\phi_\ell(k,r) \tag{3.21}$$

where ϕ_ℓ is entire in k. The zeros of $f_\ell(-k)$ give the bound state poles. The bound state wavefunctions of the spectral representation are then given by

$$\psi_n(r) = \phi_\ell(-ik_n, r)/N_n \quad , \tag{3.22}$$

$$N_n^2 = \int_0^\infty dr \left[\phi_\ell(-ik_n, r)\right]^2 \quad .$$

The domain of the variables p, p', and s over which the separable term of

the dissection in Eq. (3.17) will dominate the non-separable residue is potential-dependent and cannot be guaranteed in general; consequently we design our methods to be independent of the choice of dissection. The advantages of the particular representation (3.17) are being investigated by Osborn.[15]

D. Dimension Reduction

We conclude this section with an examination of the feature of separable operators which motivate their use in multiparticle theory. As an obvious example, we note that a rank one potential $V(p,q) = |p><q|$ reduces the L-S equation (2.1) to quadrature:

$$T(p,q;s) = \frac{|p><q|}{1 - <p|G_0(s)|q>} \tag{3.23}$$

with trivial generalization to higher rank. For two-particle amplitudes direct calculation is trivial. However, for three-particle equations the pair amplitude occurs in the typical form

$$<\underline{p}_1,\underline{p}_2,\underline{p}_3|T_{12}(E-p_3^2/2m_3)|\underline{p}'_1,\underline{p}'_2,\underline{p}'_3> = <\underline{p}_1-\underline{p}_2|T_{12}(E-p_3^2/2m_3)|\underline{p}'_1-\underline{p}'_2>\delta^3(\underline{p}_3-\underline{p}'_3) \tag{3.24}$$

where integration is over each of the primed 3-vectors. Energy conservation reduces the integrals to 6 dimensions. Omnès[16] has shown how an optimal choice of energy variables and angular momentum projections reduces the Faddeev equations to three-dimensional integral equations with 3(2J+1) components. Ahmadzadeh and Tjon[17] and Osborn and Noyes[18] have shown that further reductions may be effected in the case of short range pair interactions to give two-dimensional multi-component integral equations. The resulting two-dimensional equations require rather sophisticated analysis to achieve a reliable and practical calculation.[15]

A separable kernel immediately reduces the equations to one-dimension, as has been proposed in several models, such as the dominant pole formalism of Lovelace.[9] In the sections to follow, we develop a separable formalism which may be extended considerably beyond the single pole model.

IV. QUASIPARTICLE THEORY

We may interpret the quasiparticle theory of Weinberg[3] as a variant of our dissection process of Eq. (3.3). This theory emphasizes the formal similarities of the resulting equations to those of quantum field theory and its perturbation expansions in the hope of eventual relativistic generalization so that a convergent perturbation series may be designed for arbitrary coupling strengths. In this context, the terms of the finite rank term P of Eq. (3.4) were shown to correspond formally to fictitious elementary particle introduced into the interaction Lagrangian.

Our goal is an optimal separable system, while Weinberg sought a minimal separable operator to achieve a convergent perturbation series; further reduction improves the convergence. We apply the eigenvalue analysis developed by Weinberg to guide our choice of separable terms.

The quasiparticle transcription of our dissection process follows from our modified resolvent of Eq. (3.8), together with the symmetrization discussed in connection with Eq. (2.16).

$$K(s) = K_Q(s) + \sum_{\alpha} V^{\frac{1}{2}} | \alpha > < \alpha | V^{\frac{1}{2}} \tag{4.1}$$

from which we obtain

$$T(s) = T_Q(s) + \sum_{\alpha,\beta} T_Q(s) | \alpha > \Delta_{\alpha\beta} < \beta | T_Q(s) \tag{4.2}$$

$$\left[\Delta^{-1}(s)\right]_{\alpha\beta} = \delta_{\alpha\beta} - \langle \alpha | T_Q(s) | \beta \rangle$$

where T_Q is determined from the reduced resolvent of Eq. (3.6):

$$T_Q(s) = V^{1/2}\left[1 - K_Q(s)\right]^{-1} V^{1/2} = V^{1/2}\left[1+R_Q\right]V^{1/2} \tag{4.3}$$

subject to the condition that reduction leaves the residual K_Q with sufficiently small norm that the perturbation series

$$T_Q(w) = V^{1/2}\left[\sum_{n=0}^{\infty} K_Q^n(s)\right] V^{1/2} \tag{4.4}$$

converges. As we discussed in Section II, this series diverges only if eigenvalues of the kernel satisfy

$$\begin{aligned} K_Q(s)\left|\psi_\nu\ (s)\right> &= \eta_\nu(s)\left|\psi_\nu(s)\right> \\ |\eta_\nu\ (s)| &\geq 1 \quad . \end{aligned} \tag{4.5}$$

Thus eigenvalues outside the unit circle prevent convergence and define the minimum reduction necessary for the quasiparticle program. On the negative real axis, Ims = 0, $s < 0$ the eigenvalues of Eq. (4.6) correspond to the bound states; elsewhere in the complex s-plane they correspond to resonances. This immediately determines the minimum finite rank term of the dissection: the projection operator which reduces the eigenvalues of Eq. (4.6) to zero (ideally) or at least to a value inside the unit circle.

Thus the eigenvalue analysis determines, in addition to the minimum reduction for the quasiparticle method, a hierarchy of reduction terms to reduce the nonseparable residue K_Q to arbitrarily small norm, which is our goal.

Realistically it is no more difficult to obtain a numerical solution to the two-particle Schrödinger or L-S equation directly than to determine the eigenvalues of Eq. (4.5). However, we see that the formal structure of the eigenvalue analysis

immediately provides a guide to the separable operators.

To illustrate the utility of the eigenvalue analysis, we consider the Hulthén potential

$$V(r) = -\frac{\lambda}{2ma^2}\left[e^{r/a} - 1\right]^{-1} \tag{4.6}$$

which approximates the Coulomb potential for $r/a \ll 1$ and the exponential potential for $r/a \ll 1$. For this potential[18a] we have a closed form for the eigenvalues

$$|\eta_\nu(s)| = \frac{|\lambda|}{\nu\left[\nu^2 + 4a^2 s\right]^{1/2}} \quad . \tag{4.7}$$

From this we see clearly the dominance of the first eigenvalue ($\nu = 1$) at low energies as well as the density of eigenvalues for a typical potential.

In general, we may expect dominance by a small number of eigenvalues for short range potentials as may be inferred by the sum rules[3]

$$\sum_\nu \eta_\nu(s) = -2\,\mathrm{Im}\, s^{1/2} \int r^2\, V(r)\, j_\ell(kr)\, h_\ell^{(1)}(kr)\, dr \tag{4.8}$$

which for s-waves reads

$$\sum_\nu \eta_\nu(s) = -k^{-1} \int_0^\infty V(r) e^{ikr} \sin kr\, dr \quad .$$

We gain further support for the dominance of leading eigenvalues by interpreting the practical quasiparticle calculations of Scadron and Weinberg[18a] in the context of our formalism. For Yukawa, Hulthén, and exponential potentials, the "quasi-Born" approximation (Born amplitude plus a single separable "quasiparticle" term) gives excellent agreement with exact results for low-energy scattering parameters, bound state energies, and coupling constants. The accuracy of such calculations reveals the reason for the phenomenological success

of separable potentials in many nuclear physics applications: interactions (typical of nuclear forces or low-energy limits of field theories) have a single dominant eigenvalue; the remaining eigenvalues are significantly smaller.

A further implication of the success of these calculations is that no single separable term will provide a good approximation for a range of energies; the Born term needed for agreement implies non-negligible contribution from additional eigenvalues. From this we conclude that a formalism which develops successively higher rank operators is more appropriate than one with elaborate rank-one structure. In Section VI we consider practical choices for the separable functions.

V. SEPARABLE POTENTIALS

Having developed a general theory of separable operators, we now concentrate on separable approximations to the interaction operator. Historically, such approximations have been employed for various purposes: to reduce the L-S equation to quadrature, to simulate non-local interactions in the nuclear shell model, and to reduce multiparticle theories to equivalent multichannel two-particle systems.[2,9] Recent developments in determining variationally optimal approximations and the associated error bounds provide techniques for substantially improving and extending the earlier treatment of separable potentials.

Many recent treatments of three-particle systems aspire to relativistic generalization and application to elementary particle systems. Generally, the Faddeev amplitudes of Eq. (2.23) rather than the other possibilities for connected kernels discussed in Section II have been used because they explicitly involve amplitudes rather than potentials, which are considered embarrassing in relativistic contexts. However, most formalisms derive from the original L-S equation and involve the pair amplitudes off-shell; consequently on-shell scattering data expressed through the scattering amplitudes must be supplemented by a prescription for off-shell continuation, i.e., a potential. In this context, choosing the pair amplitude to be separable is in no sense more fundamental (or less non-relativistic) than separating the potential. The powerful techniques for analysis now available suggest that clearer insight and greater formal consistency is achieved by concentrating on the interaction operators in any practical calculation.

The source of the interaction terms may be chosen from a particular model, or to some extent derived from experimental data. Mathematically, the Gel'fand-Levitan theory[10,19] provides a unique potential if we know (a) scattering phase-shifts $\delta(s)$, $s > 0$ for all s, (b) bound state energies $s_n < 0$, and (c) the

normalization of each bound state wavefunction, as determined by Eq. (3.22). Realistically, such extensive knowledge is inaccessible even for the extensively studied three nucleon system,as has been pointed out by Noyes.[20] A choice from among the non-unique interactions (resulting from incomplete data) then defines the calculational model.

Recent three-particle formalisms, if not explicitly involving the potential, implicitly define it through the prescription for off-shell continuation. Failure to appreciate this aspect of the formalism may result in serious inconsistencies with potential theory and the original L-S equation. It may moreover preclude comparison of independent calculations if the off-shell continuation is arbitrarily chosen. The quasiparticle formalism explicitly utilizes the potential; for example the reduction of Section III.C contains the off-shell continuation in the function F(p,q) of Eq. (3.17). The choice for this function is, however, far from arbitrary, since it defines a potential via Eq. (3.14); this in turn is subject to the constraint that the resultant potential in the original L-S equation must reproduce the on-shell data contained in the T(s). The isobar model, as we shall demonstrate in detail in Section VII, actually is a separable potential model; moreover, if taken seriously it requires sufficient knowledge of form factors to determine a local potential.

Clearly the L-S equation determines a separable amplitude from a separable interaction, and vice versa, as seen from the symmetry in Eq. (2.1)

$$V = \sum_{\alpha} |\alpha\rangle\langle\alpha| \tag{5.1}$$

$$T = \sum_{\alpha\beta} |\alpha\rangle\langle\alpha| \, (1-G_o) \, |\beta\rangle^{-1}\langle\beta| \quad .$$

We shall see that for convenient determination of error bounds, systematic improvement of approximations, and incorporation of experimental data, the potential is a superior choice for the separable operator. In particular, we shall see how to extend the isobar model of bound states to accommodate continuum scattering while maintaining separability.

A. Mitra-Mongan Potentials

Earlier use of separable potentials relied upon phenomenological fits to nuclear data. A more theoretical approach was given by Mitra,[21] who suggested that the choice of separable potentials be guided by the analytic structure of the amplitudes as revealed by N/D or Fredholm methods. His potential, and its recent extension by Mongan,[22] are designed to give the exact Born amplitude on the energy shell and thus guarantee the correct analytic structure of the N/D methods.

The simplest such potential is constructed so as to yield the Born approximation for a superposition of Yukawa potentials:

$$V_\ell(p,p') = \lambda\, g_\ell(p) g_\ell(p') \tag{5.2}$$

$$g_\ell(p) = \left[\sum_i \frac{G_i}{4\pi^2} \frac{1}{p^2} Q_\ell\left(1 + \frac{\mu_i^2}{2p^2}\right)\right]^{1/2} .$$

On shell ($p = p'$) this corresponds to the local interaction

$$V(r) = \sum_i G_i \frac{e^{-\mu_i r}}{r} \tag{5.3}$$

subject to the constraint that the summation is of a definite sign for $p > 0$. Other modifications, such as the introduction of energy dependence[22] into the potential have also been proposed to optimize data fitting for this single non-local

interaction. In contrast, our development utilizes operators of arbitrary rank. This enables incorporation of significantly more experimental information.

B. Variational Methods

We now develop variationally a separable approximation to a given potential. Our treatment parallels the methods used by Sugar and Blankenbecler[2] to determine upper and lower bounds on scattering phase shifts and binding energies. For an arbitrary Hilbert space vector $|\psi>$ and an arbitrary subset of vectors $|\alpha>$ we determine the stationary value of the coefficients c_α for the vector

$$|\phi> = |\psi> + \sum_\alpha c_\alpha |\alpha> \tag{5.4}$$

for the expectation value of a positive operator $V > 0$. This yields the operator equation

$$V \geq V|\alpha> A_{\alpha\beta} <\beta|V \equiv V_s \, . \tag{5.5}$$

(Repeated indices imply summation.)

We note that $A_{\alpha\beta}$ is the inverse of the matrix $<\alpha|V|\beta>$ in the subspace of test functions, not the full Hilbert space; thus $A_{\alpha\beta}$ is not the matrix representation for the operator V^{-1}, rather, $\left[A^{-1}\right]_{\alpha\beta} = <\alpha|V|\beta>$. Likewise the operations for the separable operator K_s

$$K_s = -\frac{1}{k} V|\alpha> \left[<\alpha|V - VPG_o V|\beta>\right]^{-1} <\beta|V \tag{5.6}$$

involve determinants taken in the test-function subspace

$$D_s = \det\left[1 - A<\alpha|V\, G_o\, V|\beta>\right] \, . \tag{5.7}$$

The approximate phase shift δ^s is then determined from the plane-wave expectation value

$$\tan \delta^s(p) = < p|K_s|p> \, . \tag{5.8}$$

In the foregoing equations P denotes principal part for the free Green's function $G_0(s)$.

We next prove two valuable features of the operator derived in Eq. (5.5):

(1) If one of the test functions $|\alpha>$ is an exact solution of the L-S equation for a particular energy s, then the phase shift as determined from Eqs. (5.6) and (5.8) is exact for that energy. This follows directly from the L-S equation for the scattering wavefunction $|\psi(p)>$ and plane-wave state $|p>$:

$$K = -\frac{1}{k}<p|V|\psi(p)> \tag{5.9}$$

$$V|p> = (V - V\,P\,G_0\,V)|\psi(p)> \ .$$

Thus for one of the test functions $|\alpha> = |\psi(p)>$, the operator Eq. (5.6) equals that of Eq. (5.9), for the energy $s = p^2$.

(2) Extending the dimensionality of our set $\{|\alpha>\}$ improves the approximation of Eq. (5.5). Consider the space H^N of the N-dimensional set $\{|\alpha>\}$. In this space, the N_{th} rank separable operator defined by Eq. (5.5) is denoted by V^N. For a vector $|\phi> \epsilon H^N$, we see that

$$V^N|\phi> = \sum_{\alpha,\beta=1}^{N} V|\alpha><\alpha|V|\beta>^{-1}<\beta|V|\phi> = V|\phi> \ . \tag{5.10}$$

If we now use Eqs. (5.5) and (5.10) and imbed the operator V^{N+1} in a N-dimensional subspace, we have

$$V > V^{N+1} \geq \sum_{\alpha,\beta=1}^{N} V^{N+1}|\alpha><\alpha|V^{N+1}|\beta>^{-1}<\beta|V^{N+1}$$

$$= \sum_{\alpha,\beta} |\alpha><\alpha|V|\beta>^{-1}<\beta|V = V^N$$

so

$$V \geq V^{N+1} \geq V^{N} \quad . \tag{5.11}$$

The power of our formalism is now apparent. Any knowledge of wavefunctions gives the exact phase shift at the appropriate energies. This considerably extends the simple bound state isobar models which achieved exactly separable terms only at bound state poles. Here continuum wavefunctions (such as those determined by low-energy scattering parameters and prominent resonances) may be appended while maintaining separability. Moreover, the inequality Eq. (5.11) shows that any additional terms improve the approximation for all energies. Of course in the limit as $N \rightarrow \infty$, the functions $|\alpha>$ span the entire Hilbert space, as is evident from our discussion of finite rank operators in Section II.

We remark that our techniques, as did those of Mitra, optimize approximations for on-shell amplitudes; it is evident from the remarks leading to Eq. (5.5) that the crucial element in our derivation is the on-shell expectation value. Off-shell, such proofs and bounds cannot be obtained by these methods; however, our optimization and the improvable approximations will make maximum use of the on-shell scattering information.

We next discuss choices for the subspace of test functions $|\alpha>$; subsequently we consider specific application of our method to the isobar model.

VI. CHOICE OF SEPARABLE FUNCTIONS

Our foregoing analysis provides a guide for choices of optimal test functions for separable approximations. We distinguish between formal and practical choices.

From our spectral representation of Eq. (3.13) we saw that dominant bound states or resonances provide plausible estimates to a separable approximation. The eigenvalue analysis of Section IV formalized such a selection into a hierarchy of eigenvalues and eigenfunctions of the kernel which provides a systematic reduction. In the limit of accommodating all eigenfunctions, the system becomes completely separable. As noted previously, knowledge of the eigenfunction sequence is of no advantage for two-particle calculations, but is useful for approximating the amplitudes in off-shell multiparticle kernels.

The eigenvalue approach also provides a successive approximation scheme for locating dominant eigenfunctions. Consider a trial function

$$|\psi^{(0)}> \; = \sum_{\nu} c_\nu |\psi_\nu > \tag{6.1}$$

which is "close" to a dominant eigenfunction $|\psi_1>$ in the sense that $c_1 \gg c_\nu$, $\nu > 1$. Repeated application of the kernel

$$K^n |\psi^{(0)}> \; = \sum_{\nu} (c_\nu \eta_\nu)^n |\psi_\nu> \equiv |\psi^{(n)}> \tag{6.2}$$

is seen to bring successive $|\psi^{(n)}>$ closer to the leading eigenfunction $|\psi_1>$. A typical example is provided by the s-wave trial function[18a] which incorporates correct threshold and asymptotic behavior:

$$|\psi^{(0)}> = e^{ikr}(1 - e^{-r}) . \tag{6.3}$$

(This wavefunction is exactly the dominant eigenfunction for the Hulthén potential we analyzed in Eq. (4.6).) The iteration procedure of Eq. (6.2) is of more formal then practical value, since direct calculation of the eigenfunctions is more feasible then higher order iteration of the kernel operator.

The most practical choice of separable terms is indicated by the analysis of Section V where we saw that our formalism provides an optimal combination of trial functions which yields exact amplitudes at energies for which one trial function is exact. Typically we know most about bound states wavefunctions, low-energy scattering, and prominent resonance behavior. Each of these elements may be successively incorporated into the separable structure. Numerical verification of the formalism[2] yields impressive agreement with Yukawa scattering lengths using simple polynomial trial functions.

In summary, we develop optimized separable approximations from successive incorporation of known features of the pair scattering data through construction of successively higher rank operators from eigenfunctions of the kernel (formally) or from dominant scattering data (practically).

VII. ISOBAR MODEL

As a particular example to illustrate our methods we conclude with an analysis of the three-particle isobar model of Lovelace.[9] From the standpoint of analyticity we expect any approximation to the amplitudes of the two-particle subsystems to accommodate their meromorphic structure by including the bound state poles. Moreover, such pole terms are, as we have seen, separable in the initial and final momenta. Experience from nucleon-nucleon scattering where the loosely-bound deuteron provides a dominant singularity for low-energy scattering suggests that a model of dominant bound states (or resonances) may be adequate for particular multiparticle kernels. The spectral representation of Eq. (3.13) provides a plausible dominant term near a bound state:

$$T(p,q;s) \approx \frac{V|\psi_n(p)\rangle \langle \psi_n(q)| V}{s - s_n} \qquad (7.1)$$

$$s \approx -|s_n|.$$

Since this expression violates unitarity if simply continued to the continuum domain $s > 0$, Lovelace proposed an amplitude determined for all s from the L-S Eq. (2.1) with a rank one separable potential which gives the correct dominant term of Eq. (7.1) at $s = -|s_n|$:

$$V = \lambda\, g_n(p)\, g_n(q) \qquad (7.2)$$

where λ is the coupling strength parameter and $g_n(p)$ is the form factor from the spectral condition of Eq. (3.13)

$$g_n(p) = V|\psi_n(p)\rangle \qquad (7.3)$$

which obeys the integral equation

$$g_n(p) = 4\pi \int_0^\infty q^2\, dq\, V(p,q)\psi_n(q) = (s_n - p^2)\ \psi_n(p). \qquad (7.4)$$

The L-S scattering amplitude determined from the potential of Eq. (7.2) is then an example of Eq. (3.23):

$$T(p,p';s) = \frac{g(p)\, g(p')}{\lambda^{-1} + 4\pi \int_0^\infty q^2\, dq\, \frac{|g(q)|^2}{q^2 - s - i\epsilon}} . \qquad (7.5)$$

Lovelace[9] has discussed various approximations to the amplitude of Eq. (7.5); numerous calculations have subsequently been undertaken for a wide range of three-particle systems, e.g., 3N, 3π, $N\pi\pi$, etc. We shall not be concerned with particular calculations, but rather consider the present approximation in the context of our general theory.

The obvious limitation of any single pole model has been pointed out in Section III: the pair amplitude contributes to the three-particle kernel over an infinite range of the pair energy so the concept of a dominant term in the spectral representation of Eq. (3.13) is at best vague. These remarks by no means preclude a valid calculation for particular systems of particles and well defined ranges of parameters (e.g., the three-nucleon system), but suggest that considerable attention should be given to the validity of off-shell approximations.

We observe that if the single form factor in Eq. (7.5) is to be valid for all values of the initial or final momentum, as is required in the multiparticle integral equation, we must know $g(p)$ for all p. In this case, its configuration space Fourier transform gives the corresponding wavefunction and finally the potential from the Schrödinger equation $\left(\text{cf. Eq. (7.4)}\right)$:

$$(\nabla^2 + s_n)\ \psi_n = V\psi_n = g_n . \qquad (7.6)$$

The limitations as well as possible improvements and extension of the isobar model are now evident. The single separable potential term will be accurate only near its corresponding singularity in energy; likewise the detailed structure of the form factor will be known for only a limited range. The methods of Section III enable us to supplement the bound state input with detailed knowledge of the continuum scattering; this will clearly be advantageous for calculations with systems for which considerable kinetic energy is available. Thus effective range and scattering length data is easily incorporated in our general framework via the extension of the space of test functions in Eq. (5.5).

The success of typical "quasi-Born" calculations considered in Section IV suggests that even for the simplest two-particle systems a rank-one approximation may be of limited value, even if selected optimally. Our formalism enables convenient "testing" of low-rank isobar approximations by facilitating the incorporation of additional information from the pair subsystem.

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