## ERRATUM

## QUARK MODEL AND LOCAL CURRENT ALGEBRA

## J. Weyers

The statement (p. 4) that all the requirements of relativity and the algebra can be satisfied with Eqs. (7), (8) and (9) is incorrect. In fact

$$
\left[U_{2}^{-1} \times U_{1}, U_{2}^{-1}(-x) U_{2}\right] \neq 0
$$

for any non-constant potential, which means that the algebra cannot be satisfied except for free quarks.

Discussions with Professor M. Gell-Mann and Dr. D. Horn are gratefully acknowledged.

## CORRECTION

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Contrary to what is stated at the bottom of page 4, the formulas (7), (8) and (9) satisfy only the angular condition. If, in addition one wants the algebra to be satisfied, or

$$
\left[e^{i S_{1}} \times e^{-i S_{1}}, e^{i S_{2}} \times e^{-i S_{2}}\right]=0
$$

one finds that this condition can only be fulfilled for a constant potential $\left(\frac{1}{r} \frac{d V}{d r}=0\right)$. This means that a mass spectrum given by $M^{2}=4\left(m^{2}+\vec{p}^{2}+v\right)$ is nol allowed for any non-trivial potential.

# QUARK MODEL AND LOCAL CURRENT ALGEBRA* 

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$$
\begin{aligned}
& \text { to goince. Pugorits due 3/29. }
\end{aligned}
$$

Recently Dashen and Gell-Mann ${ }^{1}$ have suggested that the local $\mathrm{SU}_{3} \times \mathrm{SU}_{3}$ algebra be used to classify hadronic states and make predictions for the weak and electromagnetic form factors.

More specifically, their program has been formulated as follows:
A. Under equal time commutation, the octets of vector and axial-vector charges generate the $\mathrm{SU}_{3} \times \mathrm{SU}_{3}$ algebra. ${ }^{2}$ If one assumes that the equal time commutators of the charge densities contain only $\delta$ functions (no Schwinger terms), ${ }^{3}$ then the Fourier transforms $F_{i}(\overrightarrow{\mathrm{k}})$ and $\mathrm{F}_{\mathrm{i}}^{5}(\overrightarrow{\mathrm{k}})$ of, respectively, the vector and axial-vector densities obey the following commutation relations ("local" $\mathrm{SU}_{3} \times \mathrm{SU}_{3}$ algebra):

$$
\begin{align*}
& {\left[F_{i}(\vec{k}), F_{j}\left(\overrightarrow{k^{\prime}}\right)\right]=i f_{i j k} F_{k}\left(\vec{k}+\vec{k}^{\prime}\right)} \\
& {\left[F_{i}(\vec{k}), F_{j}^{5}\left(\overrightarrow{k^{\prime}}\right)\right]=i f_{i j k} F_{k}^{5}\left(\vec{k}+\vec{k}^{\prime}\right)}  \tag{1}\\
& {\left[F_{i}^{5}(\vec{k}), F_{j}^{5}\left(\overrightarrow{k^{\prime}}\right)\right]=i f_{i j k} F_{k}\left(\vec{k}+\vec{k}^{\prime}\right)}
\end{align*}
$$

B. These commutation relations are then sandwiched between infinite momentum states of the form ${ }^{4}$ :

$$
\begin{equation*}
\left|\mathrm{N}, \mathrm{~h} ; \overrightarrow{\mathrm{p}}_{\perp}, \mathrm{p}_{\mathrm{z}}=\infty\right\rangle \tag{2}
\end{equation*}
$$

$\vec{p}_{\perp}$ is the transverse momentum, $h$ the helicity and $N$ is a label referring to all other quantum numbers necessary to define the state (mass $M$, angular momentum $\vec{J}$, baryonic number, etc....).

In the infinite momentum frame, the matrix elements of the charge densities between states with transverse momentum $\overrightarrow{\mathrm{p}}_{\perp}$ and $\overrightarrow{\mathrm{p}}_{\perp}^{\prime}$ do not depend on the sum $\overrightarrow{\mathrm{p}}_{1}+\overrightarrow{\mathrm{p}}_{1}^{\prime}$ but only ${ }^{1}$ on the difference $\overrightarrow{\mathrm{k}}=\overrightarrow{\mathrm{p}}_{1}-\overrightarrow{\mathrm{p}}_{1}^{\prime}$. (This would not be true for any finite value of $\mathrm{p}_{\mathrm{z}}$.)
C. The next step is to find out what the angular momentum properties of $\mathrm{F}_{\mathrm{i}}(\overrightarrow{\mathrm{k}})$ and $\mathrm{F}_{\mathrm{i}}{ }^{5}(\overrightarrow{\mathrm{k}})$ are. First, one defines ${ }^{1}$ the angular momentum operator in the $p_{z}=\infty$ frame: $J_{z}=h$. The matrix elements of $J_{x}$ and $J_{y}$ between helicity states are given by the usual Clebsch-Gordan coefficients.

Starting from

$$
\begin{aligned}
\left\langle N^{\prime}, h^{\prime}\right| F_{i}(\vec{k}) \mid N h>\equiv<N^{\prime}, h^{\prime}, p_{x}^{\prime} & =\frac{k}{2}, p_{y}^{\prime}=0, p_{z}^{\prime}=\infty\left|F_{i}(\vec{k})\right| N, h, p_{x} \\
& =-\frac{k}{2}, p_{y}=0, p_{z}=\infty>
\end{aligned}
$$

$\vec{k}$ has been taken in the x direction) and transforming to a Breit frame where the angular propertics of the matrix elements are well known, ${ }^{5}$ it is possible to show ${ }^{1,6}$ that ${ }^{7}$ with

$$
\begin{aligned}
\phi & =\arctan \frac{M^{\prime}-M}{k}-\arctan \frac{k}{M^{\prime}+M} \\
\phi^{\prime} & =-\arctan \frac{M^{\prime}-M}{k}-\arctan \frac{k}{M^{\prime}+M}
\end{aligned}
$$

the matrix element

$$
\begin{array}{r}
<N^{\prime} h^{\prime} \mid \exp \left\{i J_{y} \phi\right\} F_{i}^{\prime}(\vec{k}) \cdot  \tag{3}\\
\quad \exp \left\{+i J_{y} \phi^{\prime}\right\} \mid N h>
\end{array}
$$

has $\Delta J_{x}=0, \pm 1 .{ }^{8}$ The same condition holds for the axial current density $F_{i}^{5} \overrightarrow{(k)}$.
D. This angular condition can be rewritten in the following way:

$$
\begin{equation*}
\left[J_{x},\left[J_{x},\left[J_{x}, \exp \left(\mathrm{iJ}_{y} \phi\right) F_{i}(\vec{k}) \operatorname{cxp}\left(\mathrm{iJ}_{y} \phi^{\prime}\right)\right]\right]\right]=\left[J_{x}, \exp \left(i J_{y} \phi\right) F_{i}(\vec{k}) \exp \left(\mathrm{iJ} \mathrm{y}_{\mathrm{y}} \phi^{\prime}\right)\right] \tag{4}
\end{equation*}
$$

Defining

$$
\begin{gathered}
K=e^{-\mathrm{iJ} J_{y} \phi} J_{x} e^{i J_{y} \phi} \\
K^{\prime}=e^{-\mathrm{iJ} y^{\phi^{\prime}}} J_{x} e^{i J_{y} \phi^{\prime}}
\end{gathered}
$$

Equation (4) leads to

$$
\begin{equation*}
K^{3} F_{i}-3 K^{2} F_{i} K^{\prime}+3 K F_{i} K^{\prime 2}-F_{i} K^{\prime}=K F_{i}-F_{i} K^{\prime} \tag{5}
\end{equation*}
$$

This equation is supposed to hold only when sandwiched between states of masses M' and M. However, it is easy to rationalize Eq. (5) and write it as an operator equation between $F_{i}, \vec{J}$ and $M$ (the mass operator) which has to be satisfied for any representation of the algebra.

So, one is led to the problem of representing simultaneously $F_{i}(\vec{k}), F_{i}^{5}(\vec{k})$, $\vec{J}$ and $M$ in such a way that Eqs. (1) and (5) are verified. An important question is then to find what restriction, if any, is induced on the mass spectrum by Eq. (5). The purpose of this letter is to show that the quark model for mesons suggests that it is always possible to satisfy the angular condition and the algebra for any mass spectrum.

A representation of the algebra corresponding to a quark-antiquark structure for mesons is given by ${ }^{6}$

$$
\begin{gather*}
\left.F_{i} \overrightarrow{\mathrm{k}}\right)=U_{1}^{-1}\left(\frac{\lambda_{\mathrm{i}}^{(1)}}{2} e^{\mathrm{i} \overrightarrow{\mathrm{k}} \cdot \overrightarrow{\mathrm{x}} / 2}\right) \mathrm{U}_{1}=U_{2}^{-1}\left(\frac{\lambda_{\mathrm{i}}^{(2)}}{2} e^{-\mathrm{i} \overrightarrow{\mathrm{k}} \cdot \overrightarrow{\mathrm{x}} / 2}\right) \mathrm{U}_{2}  \tag{6a}\\
\mathrm{~F}_{\mathrm{i}}^{5}\left(\overrightarrow{\mathrm{k})}=\mathrm{U}_{1}^{-1}\left(\frac{\lambda_{\mathrm{i}}^{(1)}}{2} \sigma_{\mathrm{z}}^{(1)} \mathrm{e}^{\mathrm{i} \overrightarrow{\mathrm{k}} \cdot \overrightarrow{\mathrm{x}} / 2}\right) \mathrm{U}_{1}-\mathrm{U}_{2}^{-1}\left(\frac{\lambda_{\mathrm{i}}^{(2)}}{2} \sigma_{\mathrm{Z}}^{(2)} \mathrm{e}^{-\mathrm{i} \overrightarrow{\mathrm{k}} \cdot \overrightarrow{\mathrm{x}} / 2}\right) \mathrm{U}_{2}\right. \tag{6b}
\end{gather*}
$$

$\vec{x}(-\vec{x})$ is the position operator for the quark (antiquark); $\vec{\sigma}^{(1)}$ and $\vec{\sigma}^{(2)}$ their spin operators and $\lambda_{i}^{(1)}$ and $\lambda_{i}^{(2)}$ the matrices of the 3 and $\overline{3}$ representations of $\mathrm{SU}_{3}, \mathrm{U}_{1}$ and $\mathrm{U}_{2}$ are unitary operators chosen in such a way that the angular condition is satisfied together with the algebra. For the algebra it is sufficient that the first terms of Eqs. (6) commute with the second ones (as they do for $U_{1}=U_{2}=1$ ).

In this quark-antiquark picture (exact $\mathrm{SU}_{3}$ limit) it is easy to represent ${ }^{6}$ $M$ and $\vec{J}$ :

$$
\begin{aligned}
M^{2} & =4\left(\mathrm{~m}^{2}+\mathrm{p}^{2}+\mathrm{V}\right) \\
\vec{J} & =\frac{\vec{\sigma}^{(1)}}{2}+\frac{\vec{\sigma}^{(2)}}{2}+\overrightarrow{\mathrm{x}} \wedge \overrightarrow{\mathrm{p}}
\end{aligned}
$$

where $\vec{p}$ is the momentum conjugate to $\vec{x}, m$ the quark mass and $V$ the potential between the quarks.

Finding what restrictions on the mass spectrum are induced by the angular condition is equivalent to finding the potentials V for which the unitary transformations $U_{1}$ and $U_{2}$, with the required properties, exist. Gell-Mann has shown ${ }^{6}$ that the case $\mathrm{V}=0$ can be solved exactly. What potentials $\mathrm{V} \neq 0$ are allowed? To answer that question we expand ${ }^{6} \phi$ and $\phi^{\prime}$ in power series in $1 / \mathrm{m}$ and put

$$
\begin{gathered}
U_{1}=e^{i S}, \quad U_{2}=e^{i T} \\
S_{m}=\frac{s^{(1)}}{m}+\frac{s^{(2)}}{m^{2}}+\frac{s^{(3)}}{m^{3}}+\ldots \quad T=\frac{t^{(1)}}{m}+\frac{t^{(2)}}{m^{2}}+\frac{t^{(3)}}{m^{3}}+\ldots
\end{gathered}
$$

We find then that for a central potential $V \equiv V(r)(r=|\vec{x}|)$ we can satisfy all the requirements of relativity and the algebra with the following potential dependent terms ${ }^{9}$ :

$$
\begin{gather*}
s^{(1)}=0  \tag{7}\\
s^{(2)}=\frac{\left(\sigma_{x}^{(1)}-\sigma_{x}^{(2)}\right) y-\left(\sigma_{y}^{(1)}-\sigma_{y}^{(2)}\right) x}{8} \quad z \frac{1}{r} \frac{d V}{d r}-\frac{1}{8} \vec{p} \cdot \vec{x} z^{2} \frac{1}{r} \cdot \frac{d V}{d r}+\text { h.c. } \tag{8}
\end{gather*}
$$

$$
\begin{align*}
& s^{(3)}=\frac{1}{16} z^{2} p_{z} \vec{p} \cdot \vec{x} \frac{1}{r} \frac{d V}{d r}+\frac{1}{12} z^{3} p^{2} \frac{1}{r} \frac{d V}{d r}+\frac{1}{48} z^{3}(\vec{p} \cdot \vec{x})(\vec{p} \cdot \vec{x}) \frac{1}{r} \frac{d}{d r}\left(\frac{1}{r} \frac{d V}{d r}\right) \\
& -\frac{1}{32} \mathrm{z}^{2} \mathrm{p}_{\mathrm{z}} \overrightarrow{\mathrm{p}} \cdot \overrightarrow{\mathrm{x}} \mathrm{r} \frac{\mathrm{~d}}{\mathrm{dr}}\left(\frac{1}{\mathrm{r}} \frac{\mathrm{dV}}{\mathrm{dr}}\right)-\frac{1}{32} \overrightarrow{\mathrm{p}} \cdot \overrightarrow{\mathrm{x}}\left\{\left(\sigma_{\mathrm{x}}^{(1)}-\sigma_{\mathrm{x}}^{(2)}\right) \mathrm{y}-\left(\sigma_{\mathrm{y}}^{(1)}-\sigma_{\mathrm{y}}^{(2)}\right) \mathrm{x}\right\} \mathrm{z}^{2} \frac{1}{\mathrm{r}} \frac{\mathrm{~d}}{\mathrm{dr}}\left(\frac{1}{\mathrm{r}} \frac{\mathrm{dV}}{\mathrm{dr}}\right) \\
& -\frac{1}{32}\left\{\left(5 \sigma_{\mathrm{x}}^{(1)}-\sigma_{\mathrm{x}}^{(2)}\right) \mathrm{p}_{\mathrm{y}}-\left(5 \sigma_{\mathrm{y}}^{(1)}-\sigma_{\mathrm{y}}^{(2)}\right) \mathrm{p}_{\mathrm{x}}\right\} \mathrm{z}^{2} \frac{1}{\mathrm{r}} \frac{\mathrm{dV}}{\mathrm{dr}}+\frac{1}{16}\left(\sigma_{\mathrm{x}}^{(1)} \mathrm{y}-\sigma_{\mathrm{y}}^{(1)} \mathrm{x}\right) \mathrm{z} \mathrm{p}_{\mathrm{z}} \mathrm{r} \frac{\mathrm{~d}}{\mathrm{dr}}\left(\frac{1}{\mathrm{r}} \frac{\mathrm{dV}}{\mathrm{dr}}\right) \\
& +\frac{1}{16}\left(\vec{\sigma}^{(1)} \cdot \overrightarrow{\mathrm{x}} \sigma_{\mathrm{z}}^{(2)}-\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} \mathrm{z}\right) \frac{1}{\mathrm{r}} \frac{\mathrm{dV}}{\mathrm{dr}}-\frac{1}{16}\left(\sigma_{\mathrm{x}}^{(1)} \mathrm{y}-\sigma_{\mathrm{y}}^{(1)} \mathrm{x}\right) \mathrm{z} \mathrm{p}_{\mathrm{z}} \frac{1}{\mathrm{r}} \frac{\mathrm{dV}}{\mathrm{dr}} \\
& -\frac{1}{8}\left(\sigma_{\mathrm{x}}^{(1)} \mathrm{p}_{\mathrm{y}}-\sigma_{\mathrm{y}}^{(1)} \mathrm{p}_{\mathrm{x}}\right) \cdot \mathrm{V}+\frac{1}{16} \overrightarrow{\mathrm{p}} \cdot \overrightarrow{\mathrm{x}}\left(\sigma_{\mathrm{x}}^{(1)} \mathrm{y}-\sigma_{\mathrm{y}}^{(1)} \mathrm{x}\right) \frac{1}{\mathrm{r}} \frac{\mathrm{dV}}{\mathrm{dr}}+\text { h.c. } \tag{9}
\end{align*}
$$

The $t^{(i)}$ are obtained from the $s^{(i)}$ by the parity operation and the exchange $\vec{\sigma}^{(1)} \rightarrow \vec{\sigma}^{(2)}$.

Expanding each $s^{(i)}$ in a power series in $k$ (for $s^{(n)}$, the $k$ series will start at $\left.\left(\frac{1}{\mathrm{k}}\right)^{\mathrm{n}}\right)$ it is easy, if somewhat lengthy, to show with Eqs. (7-9) that to order $0, \frac{1}{\mathrm{k}}, \frac{1}{\mathrm{k}^{2}}$, respectively, the fourth, fifth and sixth order in $\frac{1}{\mathrm{~m}}$ of the angular condition are identically satisified for the vector current. For the axial current the same result holds but only respectively to order $\frac{1}{\mathrm{k}}, \frac{1}{\mathrm{k}^{2}}, \frac{1}{\mathrm{k}^{3}}$.

So, although the proof is incomplete ${ }^{10}$ and given only to the fourth order in $\frac{1}{m}$, these results strongly suggest that any quark-antiquark potential is allowed or, in other words, that the angular condition does not lead to any restriction at all on the meson mass spectrum. Although somewhat disappointing, the
conclusion is rather natural: the angular condition is essentially a kinematical restriction and does not fix the dynamics in any way.

Adding spin-spin or spin-orbit couplings does not alter the conclusion: for example with a term in the mass operator of the form $\vec{\sigma}^{(1)} \cdot \vec{\sigma}^{(2)} \mathrm{V}_{1}$ one gets

$$
s^{(1)}=-\frac{1}{4} z V_{1}+\text { h.c. } \quad \text { etc. . . }
$$

With the help of Eqs. (7-9) it becomes relatively easy to compute magnetic moments, form factors, etc... for any "concrete" quark model. 11 The results of these calculations will be published elsewhere.

## ACKNOWLEDGEMENTS

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## REFERENCES AND FOOTNOTES

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3. J. Schwinger, Phys. Rev. Letters 3, 296 (1959).
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6. M. Gell-Mann. Revised version of the lecture notes given at the Ettore Majorana Summer School, Eryce; (1966); California Institute of Technology preprints CALT-68-102 and CALT-68-103.
7. $J_{y}$ acts only on the helicity indices.
8. This is not the only condition one gets. For further details see Ref. 1 and 6 .
9. The terms which do not depend on the potential can easily be calculated either directly or from the exact solution of the free case given in Ref. 6.
10. The validity of the power series in $\frac{1}{\mathrm{~m}}$ may be questionable. See Ref. 6 .
11. R. H. Dalitz, Rapporteur's talk at the XIIIth International Conference on High Energy Physics, Berkeley (1966). G. Morpurgo, Physics 2, 95 (1965).

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