# ANALYTIC CONTINUATION OF GROUP REPRESENTATIONS, VI* 

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#### Abstract

The Gell-Mann formula fon analytically continuing group representations is worked out explicitly for more cases than in previous work, and extended to certain pseudo-Riemannian symmetric spaces. The method of finding the asymptotic behavior of matrix elements of group representations introduced in Part $V$ is developed in more detail and it is shown how it leads to new mathematical problems in the theory of dynamical systems and Hilbert space theory.


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## I. INTRODUCTION

We continue work on the order of ideas introduced in the earlier papers of this series [4]. The main types discussed here are: Further development of the Gell-Mann formula [3], and development of the theory of asymptotic behavior of matrix elements of group representations.

## II. THE GELL-MANN FORMULA IN TERMS OF THE ENVELOPING ALGEBRA

Suppose that $\underset{\sim}{K}$ is a Lie algebra, with a basis $Z_{i}(1 \leq i, j, \ldots \leq n$; summation convention) such that:

$$
\left[Z_{i}, Z_{j}\right]=c_{i j k} Z_{k}
$$

Suppose $\underset{\sim}{P}$ is an abelian Lie algebra, with a basis $X_{a}(1 \leq a, b, \ldots \leq m)$. Suppose that ${\underset{\sim}{G}}^{\prime}=\underset{\sim}{K}+\underset{\sim}{P}$ is a Lie algebra with $\underset{\sim}{P}$ an ideal, i.e.,

$$
\left[\mathrm{Z}_{\mathrm{i}}, \mathrm{X}_{\mathrm{a}}\right]=\mathrm{c}_{\mathrm{iab}} \mathrm{X}_{\mathrm{b}}
$$

Form the elements:

$$
\mathrm{x}_{\mathrm{a}}^{\lambda}=\left[\Delta, \mathrm{x}_{\mathrm{a}}\right]+\lambda \mathrm{X}_{\mathrm{a}}
$$

of $\mathrm{U}\left(\mathrm{G}^{\prime}\right)$, the universal enveloping algebra of $\mathcal{G}^{\prime}$ ( $\Delta$ is the second or der Casimir operator of $\underset{\sim}{K}$ ). In [3] we have investigated the condition that $\left[X_{a}^{\lambda}, X_{b}^{\lambda}\right]$ be expressible in terms of the $Z^{\prime}$ 's, where $\underset{\sim}{G}$ is realized as a Lie algebra of skewHermitian operators on a Hilbert space H. Here, we will present a representationindependent version of this calculation, aiming to find conditions that $\left[x_{a}^{\lambda}, x_{b}^{\lambda}\right]$ be
expressible within the enveloping algebra of $\mathcal{G}^{\prime}$ in terms of the $Z^{\prime} s$ and the Casimir operator of ${\underset{\sim}{c}}^{\prime}$. (Such a calculation has been done in [4] for the case $\underset{\sim}{K}=\operatorname{SO}(2, R)$, ${\underset{\sim}{P}}^{\prime} 2$-dimensional, i.e., ${\underset{\sim}{G}}^{\prime}=$ Lie algebra of the group of rigid motions of the plane.)

Let us proceed to the calculation. $\Delta$ is of the form $\mathrm{g}_{\mathrm{ij}} \mathrm{Z}_{\mathrm{i}} \mathrm{Z}_{\mathrm{j}}$. For $X \in \underset{\sim}{P}, X^{0}=[\Delta, X]=f_{i}(X) Z_{i}+g_{a}(X) X_{b}$, where each $f_{i}(X)$ is a linear polynomial in the $X^{\prime} s$, and each $g_{a}$ is a real number.

For $Z \in \underset{\sim}{K}$, we have

$$
\begin{aligned}
{\left[Z, X^{o}\right] } & =[\Delta,[Z, X]]=f_{i}([Z, X]) Z_{i}+g_{a}([Z, X]) X_{a} \\
& =\left[Z, f_{i}(X)\right] Z_{i}+f_{i}(X)\left[Z, Z_{i}\right]+g_{a}(X)\left[Z, X_{a}\right] \\
& =\left[Z_{Z}, f_{i}(X)\right] Z_{i}+f_{i}(X) c_{i j}(Z) Z_{j}+g_{a}(X) c_{a b}(Z) X_{b}
\end{aligned}
$$

where $c_{i j}(Z)$ and $c_{a b}(Z)$ are defined by:

$$
\begin{aligned}
& {\left[\mathrm{Z}, \mathrm{Z}_{\mathrm{i}}\right]=\mathrm{c}_{\mathrm{ij}}(\mathrm{Z}) \mathrm{Z}_{\mathrm{j}}} \\
& {\left[\mathrm{Z}, \mathrm{X}_{\mathrm{a}}\right]=\mathrm{c}_{\mathrm{ab}}(\mathrm{Z}) \mathrm{X}_{\mathrm{b}}}
\end{aligned}
$$

Thus, $Z \rightarrow\left(c_{i j}(Z)\right) \quad$ and $\left(c_{a b}(Z)\right)$ define matrix representations of $\underset{\sim}{K}$, that are, in fact, just the matrix representations corresponding to AdK acting in $\underset{\sim}{K}$ and $\underset{\sim}{P}$, respectively. Comparing these two calculations gives the relations:

$$
\begin{align*}
& {\left[Z, f_{j}(X)\right]+c_{i j}(Z) f_{i}(X)=f_{j}([Z, X])}  \tag{2.1}\\
& g_{b}([Z, X])=g_{a}(X) c_{a b}(Z)  \tag{2.2}\\
& \quad \text { for } X \in \underset{\sim}{P}, Z \in \underset{\sim}{K}
\end{align*}
$$

Put:

$$
\begin{gathered}
X^{\prime}=\mathrm{f}_{\mathrm{i}}(\mathrm{X}) \mathrm{Z}_{\mathrm{i}} \text { for } \mathrm{X} \in \underset{\sim}{\mathrm{P}} \\
-3-
\end{gathered}
$$

For $X, Y \in \underset{\sim}{P}$,

$$
\begin{aligned}
{\left[X^{\prime}, Y^{\prime}\right]=} & f_{i}(X) Z_{i} f_{j}(Y) Z_{j}-f_{i}(Y) Z_{i} f_{j}(X) Z_{j} \\
= & f_{i}(X) f_{j}(Y) Z_{i} Z_{j}-f_{i}(Y) f_{j}(X) Z_{i} Z_{j} \\
& +f_{i}(X)\left[Z_{i}, f_{j}(Y)\right] Z_{j}-f_{i}(Y)\left[Z_{i}, f_{j}(X)\right] Z_{j} \\
= & f_{i}(X) f_{j}(Y)\left[Z_{i}, Z_{j}\right] \\
& +f_{i}(X)\left(f_{j}\left(\left[Z_{i}, Y\right]\right)-c_{k j}\left(Z_{i}\right) f_{k}(Y)\right) Z_{j} \\
& -f_{i}(Y)\left(f_{j}\left(\left[Z_{i}, X\right]\right)-c_{k j}\left(Z_{i}\right) f_{k}(X)\right) Z_{j}
\end{aligned}
$$

Set this equal to:

$$
\begin{equation*}
\mathrm{f}_{\mathrm{j}}(\mathrm{X}, \mathrm{Y}) \mathrm{X}_{\mathrm{j}} \tag{2.3}
\end{equation*}
$$

with

$$
\begin{aligned}
f_{j}(X, Y)= & f_{i}(X) f_{j}\left(\left[Z_{i}, Y\right]\right)-f_{i}(Y) f_{j}\left(\left[Z_{i}, X\right]\right) \\
& +c_{j k}\left(Z_{i}\right)\left(f_{i}(Y) f_{k}(X)-f_{i}(X) f_{k}(Y)\right)
\end{aligned}
$$

Also, we have:

$$
\left[\mathrm{Z}, \mathrm{X}^{\prime}\right]=[\mathrm{Z}, \mathrm{X}]^{\prime} \text { for } \mathrm{Z} \in \underset{\sim}{\mathrm{~K}}, \mathrm{X} \in \underset{\sim}{\mathrm{P}}
$$

Hence,

$$
\begin{align*}
& {\left[[Z, X]^{\prime}, Y^{\prime}\right]+\left[X^{\prime},[Z, Y]^{\prime}\right] } \\
&= {\left[Z, f_{j}(X, Y)\right] Z_{j}+f_{j}(X, Y)\left[Z, Z_{j}\right], \text { or } } \\
& f_{k}([Z, X], Y)+f_{k} X,([Z, Y]) \\
&= {\left[Z, f_{k}(X, Y)\right]+c_{j k}(Z) f_{j}(X, Y) } \tag{2.4}
\end{align*}
$$

Let us make explicit that $f_{i}$ is a second-degree polynomial:

$$
\left[X^{\prime}, Y^{\prime}\right]=A_{a b j}(X, Y) \quad X_{a} X_{b} Z_{j}
$$

where $\left(X_{a}\right)(1 \leq a, b, \leq m)$ is a basis for $\underset{\sim}{P}$.
Then, $A_{a b j}(X, Y)$ can be supposed symmetric in the indices $a, b$, and skewsymmetric in X and Y. Suppose

$$
\Delta^{\prime}=g_{a b} X_{a} X_{b}
$$

is a Casimir operator of $\underset{\sim}{G}$, i.e.,

$$
\left[\underset{\sim}{K}, \Delta^{\prime}\right]=0
$$

If there are constants $\left(c_{a b i}\right)$ such that:

$$
\begin{equation*}
\left[\mathrm{X}_{\mathrm{a}}^{\prime}, \mathrm{X}_{\mathrm{b}}^{\prime}\right]=\Delta^{\prime} \mathrm{c}_{\mathrm{abi}} \mathrm{Z}_{\mathrm{i}} \tag{2.5}
\end{equation*}
$$

then we have a relation of the following form:

$$
\begin{equation*}
\left[\frac{\mathrm{X}_{\mathrm{a}}^{\prime}}{\sqrt{\Delta^{\prime}}}, \frac{\mathrm{X}_{\mathrm{b}}^{\prime}}{\sqrt{\Delta^{\prime}}}\right]=\mathrm{c}_{\mathrm{abi}} \mathrm{Z}_{\mathrm{i}} \tag{2.6}
\end{equation*}
$$

Thus, at the expense of addition to $\mathrm{U}(\underset{\sim}{\mathrm{G}})$ elements that are more general than polynomial "functions" of the elements of $G$, we have constructed a new Lie algebra whose basis is $\left(Z_{i}, X_{a}^{\prime} / \sqrt{\Delta^{\prime}}\right)$.

The existence of the ( $\mathrm{c}_{\mathrm{abi}}$ ) and ( $\mathrm{g}_{\mathrm{ab}}$ ) can be approached in two ways: Either they can be constructed explicitly in the needed special cases, or one can attempt to prove by using basic principles that conditions 2.4 imply conditions in the tensor $\left(A_{a b j}\left(X_{c}, X_{d}\right)\right)$ that in turn imply it must be of the form 2.5. The latter approach involves a generalization of Kostant's results [7] on the decomposition
of the universal enveloping algebra under the action of a linear group, and will not be attempted in this paper. Note, however, that 2.6 implies that the GellMann formula holds for representations of ${\underset{\sim}{\prime}}^{\prime}$, a topic we have analyzed in [3] , at least for representations in which the operators of $P$ are "diagonalizable." Thus, the conditions presented in [3] can be regarded as necessary conditions that a Gell-Mann formula of type 2.6 hold in the enveloping algebra.

We now turn to computing some examples.
III. THE GELL-MANN FORMULA FOR ROTATION GROUPS

Let $K=S O(n, R), \underset{\sim}{P}=R^{n}$, with the representation of $K$ on $P$ just the "vector" representation of $\mathrm{SO}(\mathrm{n}, \mathrm{R})$. We shall show that the Gell-Mann formula holds within the enveloping algebra of ${\underset{\sim}{G}}^{\prime}=\underset{\sim}{K}+\underset{\sim}{P}$, in the sense described in Section II. We will not use the technique described in Section II, but another that has interesting geometric consequences.

Regard $\underset{\sim}{K}$ as a Lie algebra of differential operators on $R^{n}: Z_{i j}, 1 \leq 1, j, \ldots \leq n$ summation convention in force, are the generators of $K$, with

$$
z_{i j}=x_{i} \partial_{j}-x_{j} \partial_{i}
$$

(Notation: $x_{i}$ the Euclidean coordinates on $R^{n}, \quad \partial_{j}=\frac{\partial}{\partial_{x j}}$ )
$\underset{\sim}{P}$ is realized as the vector space generated by the $x_{i}$.
$\Delta=Z_{i j} Z_{i j}$ is the second degree Casimir operator of $\operatorname{SO}(\mathrm{n}, \mathrm{R})$.

Following the Gell-Mann formula prescription, we construct the operators:

$$
\begin{align*}
x_{k} & =\left[z_{i j}, x_{k}\right] z_{i j}  \tag{3.1}\\
& =\left(x_{i} \delta_{j k}-x_{j} \delta_{i k}\right) z_{i j} \\
& =x_{i} z_{i k}-x_{j} z_{k j} \\
& =2 x_{i} z_{i k} \\
& =2 x_{i}\left(x_{i} \partial_{k}-x_{k} \partial_{i}\right) \\
& =2\left(r^{2} \partial_{k}-x_{k} x\right) \tag{3.2}
\end{align*}
$$

with the following notations:

Now,

$$
\begin{aligned}
& r^{2}=x_{i} x_{i}, x=x_{i} \partial_{i} \\
& x\left(x_{k}\right)=\left[x, x_{k}\right]=x_{k} \\
& {\left[x, \partial_{k}\right]=-\partial_{k}} \\
& {\left[x, r^{2}\right]=2 r^{2}} \\
& {\left[\partial_{j}, r^{2}\right]=2 x_{j}}
\end{aligned}
$$

Hence,

$$
\begin{gather*}
{\left[\mathrm{X}_{\mathrm{k}}, \mathrm{r}^{2}\right]=2\left(2 \mathrm{r}^{2} \mathrm{x}_{\mathrm{k}}-2 \mathrm{r}^{2} \mathrm{x}_{\mathrm{k}}\right)=0}  \tag{3.4}\\
{\left[\mathrm{x}, \mathrm{x}_{\mathrm{k}}\right]=2\left(2 \mathrm{r}^{2} \partial_{\mathrm{k}}-\mathrm{r}^{2} \partial_{\mathrm{k}}-\mathrm{x}_{\mathrm{k}} \mathrm{x}\right)} \\
\frac{1}{2}\left[\partial_{\mathrm{j}}, \mathrm{X}_{\mathrm{k}}\right]=2 \mathrm{x}_{\mathrm{j}} \partial_{\mathrm{k}}-\delta_{\mathrm{jk}} \mathrm{X}-\mathrm{x}_{\mathrm{k}} \partial_{\mathrm{j}}
\end{gather*}
$$

Thus,

$$
\begin{align*}
\frac{1}{2}\left[x_{j}, x_{k}\right]= & {\left[x_{j}, r^{2} \partial_{k}-x_{k} x\right] } \\
= & 2 r^{2}\left(\delta_{j k} x+x_{j} \partial_{k}-2 x_{k} \partial_{j}\right) \\
& -2\left(r^{2} \delta_{j k}-x_{j} x_{k}\right) x \\
& -x_{k}\left(-x_{j}\right) \\
= & 2 r^{2} \delta_{j k} x+2 r^{2} x_{j} \partial_{k}-4 r^{2} x_{k} \partial_{j} \\
& -2 r^{2} \delta_{j k} X+2 x_{j} x_{k} x+2 x_{k} r^{2} \partial_{j}-2 x_{k} x_{j} X \\
= & 2 r^{2}\left(x_{j} \partial_{k}-x_{k} \partial_{j}\right) \\
= & 2 r^{2} x_{k j} \tag{3.5}
\end{align*}
$$

Let ${\underset{\sim}{G}}^{\prime}=\underset{\sim}{K}+{\underset{\sim}{R}}^{n}$, the Lie algebra of the group of rigid motions of $R^{n}$, although wc have calculated in terms of a realization of $U(\underset{\sim}{G})$ by differential operators on $R^{n}$, the results are also true in $U(\underset{\sim}{G})$, since the realization of $\mathrm{U}\left({\underset{\sim}{G}}^{\prime}\right)$ by differential operators on $\mathrm{R}^{\mathrm{n}}$ is faithful. 3.4 is then interpreted as a Gell-Mann formula giving the Lie algebra of $\mathrm{SO}(\mathrm{n}, 1)$ in terms of elements of $\mathrm{U}\left(\mathrm{G}^{\prime}\right):$ In fact,

$$
\begin{equation*}
\left[\frac{\mathrm{X}_{\mathrm{j}}}{2 \mathrm{r}}, \frac{\mathrm{X}_{\mathrm{k}}}{2 \mathrm{r}}\right]=\mathrm{Z}_{\mathrm{jk}} \tag{3.6}
\end{equation*}
$$

which show that the $Z_{j k}$ and $\frac{X_{j}}{2 r}$ together generate the Lie algebra of $\operatorname{SO}(n, 1)$.
3.4 has an interesting geometric interpretation. Interpret $X_{k}$ as a first order differential operator, i.e., as a vector field on $R^{n}$. 3.4 then says that this vector field is tangent to the surfaces

$$
\mathrm{r}^{2}=\text { constant }
$$

i.e., to the spheres in $R^{n}$. Of course, the $Z_{i j}$ are also tangent to these spheres; the $Z_{i j}$ and $X_{k}$ when restricted to each such sphere generate a transformation group, whose Lie algebra is isomorphic with $\operatorname{SO}(\mathrm{n}, 1)$. One knows that $\mathrm{SO}(\mathrm{n}, 1)$ is just the group of conformal transformations (of the metric constant curvature) on the sphere. Now, the group generated by the $\mathrm{Z}_{\mathrm{ij}}$ acts as a group of isometries of this metric. It is reasonable, then, to suspect that the one-parameter group generated by each $\mathrm{X}_{\mathbf{j}}$ on the spheres 3.6 is a group of conformal transformations. In fact, we will now prove that this is so, using methods of differential geometry [6]. We must calculate the Lie derivative

$$
\begin{aligned}
& x_{k}\left(d x_{i} d x_{i}\right) \\
& =2 d\left(x_{k}\left(x_{i}\right)\right) d x_{i} \\
& =4 d\left(r^{2} \delta_{k i}-x_{j} x_{i}\right) d x_{i} \\
& =4\left(2 x_{j} d x_{j} \delta_{k i}-d x_{k} k_{i}-x_{k} d x_{i}\right) d x_{i} \\
& =4\left(2 x_{j} d x_{j} d x_{k}-d x_{k} x_{i} d x_{i}-x_{k} d x_{i} d x_{i}\right)
\end{aligned}
$$

On the hypersurface 3.6,

$$
\begin{aligned}
x_{j} d x_{j} & =0, \text { hence with the relation, } \\
x_{k}\left(d x_{i} d x_{i}\right) & =-4 x_{k}\left(d x_{i} d x_{i}\right)
\end{aligned}
$$

which shows that $X_{k}$ is an infinitesimal conformal transformation on the sphere 3.6.

Thus we see that there is a close relation between the Gell-Mann formula for $\mathrm{SO}(\mathrm{n}, 1)$ and the geometric fact that the group acts as a group of conformal transformations on the plane.

## IV. DEFORMATION OF THE GELL-MANN ENVELOPING ALGEBRAIC RELATIONS

Let us return to the general setting for the Gell-Mann formula, i.e., $\underline{\sim}^{\prime}={\underset{\sim}{K}}^{\prime}+{\underset{\sim}{P}}^{\prime}$ is the semidirect sum of a Lie algebra $\underset{\sim}{K}$ and an abelian Lie algebra $\underset{\sim}{P}$. Let $Z_{u}, 1 \leq u, v, \ldots \leq m$, be a basis for $K, X_{i},\left(1 \leq 1_{j}, \ldots \leq n\right)$ for a basis for P. Let

$$
\begin{equation*}
\mathrm{x}_{\mathrm{i}}^{\mathrm{o}}=\left[\mathrm{z}_{\mathrm{u}}, \mathrm{x}_{\mathrm{i}}\right] \mathrm{z}_{\mathrm{u}} \tag{4.1}
\end{equation*}
$$

In [5] we pointed out the following fact: If $\underset{\sim}{G}$ is realized as a Lie algebra of operators on a vector space $H$, if the $Z_{u}$ and $X_{i}^{o}$ given by 4.1 span a Lie algebra of operators, then the following operators also span the same Lie algebra, i.e., the Gell-Mann formula enables us to analytically continue representations:

$$
\begin{equation*}
x_{i}^{\lambda}=\left[z_{u}, x_{i}\right] z_{u}+\lambda x_{i} \tag{4.2}
\end{equation*}
$$

Now, we would like to inquire what this relation may mean in terms of the enveloping algebra interpretation of the Gell-Mann formula given in Section II. In fact, interpret 4.2 as a formula in the enveloping algebra $U\left(G^{\prime}\right)$. Suppose that:

$$
\left[\begin{array}{ll}
x_{i}^{o}, & x_{j}^{0} \tag{4.3}
\end{array}\right]=\Delta^{\prime} z_{i j}
$$

where $\Delta^{\prime}$ is an element of the center of $U\left({\underset{\sim}{G}}^{\prime}\right)$, and $Z_{i j}$ are elements of K . Then,

$$
\begin{aligned}
{\left[x_{i}^{\lambda}, x_{j}^{\lambda}\right] } & =\left[x_{i}^{o}+\lambda x_{i}, x_{j}^{o}+\lambda x_{j}\right] \\
& =\Delta Z_{i j}+\lambda\left(\left[x_{i}, x_{j}^{0}\right]+\left[x_{i}^{o}, x_{j}\right]\right) \\
& =\Delta Z_{i j}+\lambda\left(\left[x_{i}, x_{j}^{0}\right]-\left[x_{j}, x_{i}^{0}\right]\right)
\end{aligned}
$$

But,

$$
\left[x_{i}^{o}, x_{j}\right]=\left[z_{u}, x_{i}\right]\left[z_{u}, x_{j}\right]
$$

which is clearly symmetric in $i$ and $j($ since $[\underset{\sim}{\mathrm{K}}, \underset{\sim}{\mathrm{P}}] \underset{\sim}{\mathrm{P}}$, and $[\underset{\sim}{\mathrm{P}}, \underset{\sim}{\mathrm{P}}]=0$ ), ie.,

$$
\begin{align*}
& {\left[\underset{\sim}{\underset{\sim}{x}} \underset{\sim}{\lambda}, \underset{\sim}{{\underset{\sim}{j}}_{\lambda}^{\lambda}}\right]=\Delta^{\prime} \mathrm{Z}_{\mathrm{ij}}, \text { hence: }} \\
& {\left[\frac{\mathrm{X}_{\mathrm{i}}^{\lambda}}{\sqrt{\Delta^{\prime}}}, \frac{\mathrm{X}_{\mathrm{j}}^{\lambda}}{\sqrt{\Delta^{\prime}}}\right]=\mathrm{Z}_{\mathrm{ij}}} \tag{4.4}
\end{align*}
$$

We can now prove another useful fact about the Gell-Mann formula. Let us compute:

$$
\begin{equation*}
\Delta_{K}=Z_{i j} Z_{i j} \tag{4.5}
\end{equation*}
$$

Let

$$
\begin{equation*}
\Delta_{\lambda}=\frac{\mathrm{X}_{\mathrm{i}}^{\lambda} \mathrm{X}_{\mathrm{i}}^{\lambda}}{\Delta^{\prime}} \tag{4.6}
\end{equation*}
$$

Assume that: $\quad\left[\Delta_{\lambda}, X_{i}^{\lambda}\right]=0$

$$
\begin{align*}
& \Delta_{K}=\frac{1}{\Delta^{\prime 2}}\left[X_{i}^{\lambda}, X_{j}^{\lambda}\right]\left[X_{i}^{\lambda}, X_{j}^{\lambda}\right]  \tag{4.7}\\
& =\frac{1}{\Delta^{12}}\left(x_{i}^{\lambda} x_{j}^{\lambda}-x_{j}^{\lambda} x_{i}^{\lambda}\right)\left(x_{i}^{\lambda} x_{j}^{\lambda}-x_{j}^{\lambda} x_{i}^{\lambda}\right) \\
& =\frac{1}{\Delta^{\prime 2}}\left(X_{i}^{\lambda} X_{j}^{\lambda} X_{i}^{\lambda} X_{j}^{\lambda}-X_{j}^{\lambda} X_{i}^{\lambda} X_{i}^{\lambda} X_{j}^{\lambda}-X_{i}^{\lambda} X_{j}^{\lambda} X_{j}^{\lambda} X_{i}^{\lambda}+X_{j}^{\lambda} X_{i}^{\lambda} X_{j}^{\lambda} X_{i}^{\lambda}\right) \\
& =\frac{1}{\Delta^{\prime 2}}\left(X_{i}^{\lambda} X_{i}^{\lambda} X_{j}^{\lambda} X_{j}^{\lambda}+X_{i}^{\lambda}\left[X_{j}^{\lambda}, X_{i}^{\lambda}\right] X_{j}^{\lambda}-X_{j}^{\lambda} \Delta_{\lambda} \Delta^{\prime} X_{j}^{\lambda}-X_{i}^{\lambda} \Delta_{\lambda} \Delta^{\prime} X_{i}^{\lambda}\right. \\
& \left.+x_{j}^{\lambda} x_{j}^{\lambda} x_{i}^{\lambda} x_{i}^{\lambda}+x_{j}^{\lambda}\left[x_{i}^{\lambda}, x_{j}^{\lambda}\right] x_{i}^{\lambda}\right) \\
& =\frac{1}{\Delta^{\prime 2}} 2 X_{i}^{\lambda}\left[X_{j}^{\lambda}, X_{i}^{\lambda}\right] X_{j}^{\lambda}=\frac{2}{\Delta} X_{i}^{\lambda} Z_{j i} X_{j}^{\lambda} \\
& =2\left(\frac{X_{i}^{\lambda}}{\sqrt{\Delta^{\prime}}} Z_{j i} \frac{x_{j}^{\lambda}}{\sqrt{\Delta^{\prime}}}\right) \tag{4.8}
\end{align*}
$$

Suppose now that $\underset{\sim}{P}$ admits a positive definitc quadratic form that is invariant under $A d K$. Suppose that the $X_{i}$ were originally chosen to be an orthonormal basis with respect to the quadratic form. Then, it is readily verified that $\Delta_{K}$ is a Casimir operator of $\underset{\sim}{K}$, i.e., is invariant under AdK. Hence, so is the right-hand side of 4.8. But, this involves operators of ${\underset{\sim}{G}}^{\lambda}\left({\underset{\sim}{G}}^{\lambda}\right.$ is the algebra generated by the $Z_{u}$ and $\left.X_{i}^{\lambda} / \sqrt{\Delta^{\prime}}\right)$. This is a nontrivial relation.

## V. RELATIONS BETWEEN THE CASIMIR OPERATORS FOR THE SO(n, R) GELL-MANN FORMULA

Return to the situation considered in Section III, i.e., $K=\operatorname{SO}(n, R)$, $\underset{\sim}{P}=R^{n}, X_{i}=x_{k}$,

$$
x_{k}^{\lambda}-r^{2} \partial_{k}-x_{k} x+\lambda x_{k}
$$

Thus, we know from Section IV that:

$$
\left[\frac{x_{i}^{\lambda}}{r}, \frac{x_{j}^{\lambda}}{r}\right]=z_{i j}=x_{i} \partial_{j}-x_{j} \partial_{i}
$$

For each value of $\lambda$, let us compute the Casimir operator of the $\mathrm{SO}(\mathrm{n}, 1)$-algebra generated by $Z_{i j}$, $X_{i}^{\lambda} / r$, as a function of $\lambda$ and $r$.

$$
\begin{aligned}
& X_{k}^{\lambda} X_{k}^{\lambda}=\left(r^{2} \partial_{k}-x_{k} X+x_{k}\right)\left(r^{2} \partial_{k}-x_{k} X+\lambda x_{k}\right) \\
& =r^{2} \partial_{k} r^{2} \partial_{k}-r^{2} \partial_{k} x_{k} X+\lambda r^{2} \partial_{k} x_{k} \\
& -\mathrm{x}_{\mathrm{k}} \mathrm{Xr}^{2} \partial_{\mathrm{k}}+\mathrm{x}_{\mathrm{k}} \mathrm{Xx}_{\mathrm{k}} \mathrm{X}-\lambda \mathrm{x}_{\mathrm{k}} \mathrm{Xx} \mathrm{x}_{\mathrm{k}} \\
& +\lambda\left(x_{k} r^{2} \partial_{k}-r^{2} X+\lambda r^{2}\right) \\
& =2 r^{2} x_{k} \partial_{k}+r^{4} \Delta-n r^{2} x-r^{2} x^{2} \\
& +n \lambda r^{2}+\lambda r^{2} x-2 r^{2} x_{k} \partial_{k}-x_{k} r^{2} X \partial_{k} \\
& +r^{2} X+r^{2} X^{2}-\lambda r^{2}-\lambda r^{2} X \\
& +\lambda r^{2} X-\lambda r^{2} X+\lambda^{2} r^{2} \\
& =\left(2 r^{2}-n r^{2}+\lambda r^{2}-2 r^{2}+r^{2}-\lambda r^{2}+\lambda r^{2}-\lambda r^{2}\right) X+r^{4} \Delta-r^{2} X^{2}+x_{k} r^{2} \partial_{k} \\
& +\left(-r^{2}+r^{2}\right) X^{2}+n \lambda r^{2}+r^{2}\left(\lambda^{2}-\lambda\right) \\
& =r^{4} \Delta-r^{2} X^{2}+(2-n) r^{2} X+n \lambda r^{2}+r^{2}\left(\lambda^{2}-\lambda\right),
\end{aligned}
$$

hence:

$$
\begin{aligned}
\Delta_{\mathrm{p}}^{\lambda}=\frac{\mathrm{X}_{\mathrm{k}}^{\lambda}}{\mathrm{r}} \frac{\mathrm{X}_{\mathrm{k}}^{\lambda}}{\mathrm{r}} & =\mathrm{r}^{2} \Delta-\mathrm{X}^{2}+(2-\mathrm{n}) \mathrm{X}+\left(\mathrm{n} \lambda+\lambda^{2}-\lambda\right) \\
& =\Delta_{\mathrm{K}}+\lambda(\mathrm{n}-\mathrm{l}+\lambda)
\end{aligned}
$$

Now, $\Delta_{\mathrm{K}}^{\lambda}-\Delta_{\mathrm{K}}$ is the Casimir operator of the $\mathrm{SO}(\mathrm{n}, 1)$-algebra generated by the $Z_{i j}, X_{i}^{\lambda / r}$. Hence, we have:

Theorem 5.1: The second degree Casimir operator of the representation of $\mathrm{SO}(\mathrm{n}, 1)$ defined by the Gell-Mann formula has the value:

$$
\lambda(n-1+\lambda)
$$

Finally, notice that there is a curious resemblance between this theory and that of the group-theoretic treatment of the hydr ogen atom, by means of the Runge-Lenz vector [2].
VI. COMPLEXIFICATION OF THE GELL-MANN FORMULA

Most of our work up to now has been concerned with semidirect product algebras ${\underset{\sim}{G}}^{\prime}=\underset{\sim}{K}+\underset{\sim}{P}$, with $[\underset{\sim}{P}, \underset{\sim}{P}]=0$ and with $\underset{\sim}{K}$ a compact Lie algebra. Suppose we consider an algebra of the form:

$$
{\underset{\sim}{G}}^{0}={\underset{\sim}{K}}^{0}+{\underset{\sim}{P}}^{0} \text {, with }\left[{\underset{\sim}{p}}^{0},{\underset{\sim}{p}}^{0}\right]=0 \text {, and such that }
$$

(a) $\underset{\sim}{K}$ and ${\underset{\sim}{K}}^{\mathrm{K}}$ have the same complexification, i.e., $\underset{\sim}{\mathrm{K}}+\mathrm{i} \underset{\sim}{\mathrm{K}}$ is isomorphic with ${\underset{\sim}{K}}^{\circ}+i \underset{\sim}{K}{ }^{\circ}$.
(b) ${\underset{\sim}{P}}^{o}+i{\underset{\sim}{P}}^{\mathrm{P}}$ is isomorphic to $\underset{\sim}{P}+\mathrm{i} \underset{\sim}{P}$, and the isomorphism is compatible with (a).

Suppose the Gell-Mann formula holds within the enveloping algebra of ${\underset{\sim}{G}}^{\prime}$ : Does it hold within ${\underset{\sim}{G}}^{0}$ ? For example, we have proved in the last section that the enveloping-algebra Gell-Mann formula holds for $K=S O(4, R), \underset{\sim}{P}=$ vector representation. Choose $\mathrm{K}^{\mathrm{O}}=\mathrm{SO}(3,1) .{\underset{\sim}{G}}^{\mathrm{O}}$ is then the Poincare Lie algebra. A Gell-Mann formula for this would give in a way of relating the de-Sitter Lie algebra to the Poincaré Lie algebra.*

Such a relation has been discovered by $R$. Hwa, and one of the aims of this section is to show how this relation follows from the general theory.

Suppose that $\Delta^{0}$ is a Casimir operator of ${\underset{\sim}{K}}^{0}$, and that, for $X^{0} \in{\underset{\sim}{P}}^{0}$, we form:

$$
\mathrm{X}^{\mathrm{O}}=\left[\Delta^{\mathrm{o}}, \mathrm{X}^{\mathrm{o}}\right]
$$

Suppose $\Delta^{0,}$ is a Casimir operator of ${\underset{\sim}{G}}^{0}$. Suppose $T^{o}:{\underset{\sim}{P}}^{0} \times{\underset{\sim}{P}}^{0} \rightarrow K^{0}$ is a skew-symmetric bilinear mapping that commutes with the action of Ad K . Form:

$$
\begin{equation*}
\left[X^{O_{1}}, Y^{O_{1}}\right]-\Delta^{O_{1}} T^{o}\left(X^{o}, Y^{o}\right) \text { for } X^{o}, Y^{o} \in{\underset{\sim}{p}}^{o} \tag{6.1}
\end{equation*}
$$

Notice that it vanishes if and only if its complexification vanishes. Thus, if $\Delta$ is a Casimir operator of $\underset{\sim}{K}$; if $T: \underset{\sim}{P} \times \underset{\sim}{P} \rightarrow \underset{\sim}{K}$ is a skew-symmetric bilinear mapping commuting with Ad K; if $\Delta^{\prime}$ is a Casimir operator of $G^{\prime}$, such that:

$$
\begin{equation*}
X^{\prime}=[\Delta, X], \quad\left[X^{\prime}, Y^{\prime}\right]=\Delta^{\prime} T(X, Y) \text { for } X, Y \in P \tag{1}
\end{equation*}
$$ $\Delta=\Delta^{\circ}, T=T^{\circ}, \Delta^{\prime O}=\Delta^{\prime}$ under the isomorphism of the complexification of ${\underset{\sim}{K}}^{0}$ and $\underset{\sim}{K},{\underset{\sim}{T}}^{\circ}$ and $\underset{\sim}{T}$,

then 6.1 does in fact vanish also, i.e., the Gell-Mann formula holds within the enveloping algebra.

There is, however, a new feature when $\underset{\sim}{K}{ }^{\mathrm{O}}$ is not a compact Lie algebra. The Casimir operator $\Delta^{O_{t}}$ of ${\underset{\sim}{G}}^{0}$ can have values of any sign in different representations. Thus, 6.1 is zero, we have

$$
\left[-\frac{X_{o}^{\prime}}{\sqrt{ \pm \Delta^{O_{1}}}} \frac{Y_{o}^{\prime}}{\sqrt{ \pm \Delta^{O_{1}}}}\right]= \pm \mathrm{T}^{\mathrm{O}}\left(\mathrm{X}^{\mathrm{O}}, \mathrm{Y}^{\mathrm{O}}\right)
$$

Thus, depending on the representation chosen for $\mathrm{G}^{0}$, we can realize two different Lie algebras. (For example, as is well-known, the Poincaré algebra can be approximated by $\mathrm{SO}(3,2)$ and $\mathrm{SO}(4,1))$.

## VII. GROUP REPRESENTATIONS THAT ARE LINEAR IN THE DEFORMATION PARAMETER

As we have indicated in [4], Part V, (following Nijenhuis and Richardson [8]) there is a relation between deformations of group and Lie algebra deformations. Such relations are important, for example, in problems concerning the integral representation and asymptotic behavior of matrix elements of group representations. In [4], Part V, these relations were worked out in detail for the simplest example, $\mathrm{SL}(2, \mathrm{R})$. In this section we present several further general remarks, preparing the way for applications to representations satisfying the Gell-Mann formula in the following section. Let $\underset{\sim}{G}$ be a Lie algebra, $\underset{\sim}{\mathcal{P}}$ a representation of $\underset{\sim}{G}$ by linear transformations on a vector space $H$. Let $V$ be the space of linear operators: $H \rightarrow H$, and let $\phi$ be the following representation of $G$ in $V$ :

$$
\phi(\mathrm{X})(\mathrm{A})=[\rho(\mathrm{X}), \mathrm{A}] \quad \text { for } \mathrm{A} \in \mathrm{~V}, \mathrm{X} \in \mathrm{G} .
$$

Suppose $\rho_{\lambda}$ is a one-parameter family of such representations, reducing to the given one at $\lambda=0$, of the form:

$$
\begin{equation*}
\rho_{\lambda}(\mathrm{X})=\rho(\mathrm{X})+\lambda \omega(\mathrm{X}), \tag{7.1}
\end{equation*}
$$

where $\omega$ is a linear mapping $\underset{\sim}{G} \rightarrow V$, i.e., a 1-cochain in $C^{1}(\phi)$. We know that $\omega$ must satisfy the two conditions:

$$
\begin{equation*}
\mathrm{d} \omega=0 \tag{a}
\end{equation*}
$$

$$
\begin{equation*}
[\omega(\mathrm{X}), \omega(\mathrm{Y})]=0 \text { for } \mathrm{X}, \mathrm{Y} \in \underset{\sim}{\mathrm{G}} . \tag{b}
\end{equation*}
$$

Theorem 7.1 Suppose $X$ is a fixed element of $\underset{\sim}{G}$, and $A$ is an element of $V$ such that:

$$
\begin{align*}
{[\rho(\mathrm{X}), \mathrm{A}] } & =\omega(\mathrm{X})  \tag{7.2}\\
{[\mathrm{A}, \omega(\mathrm{X})] } & =0 \tag{7.3}
\end{align*}
$$

The operator

$$
B^{\lambda}=\exp (\lambda A) \text {, exists, i.e., through the usual }
$$

power series expansion. Then,

$$
\begin{equation*}
\mathrm{B}^{\lambda} \rho_{\lambda}(\mathrm{X})=\rho(\mathrm{X}) \mathrm{B}^{\lambda} \tag{7.4}
\end{equation*}
$$

Proof: If the usual power sense expansion for $B^{\lambda}$ holds, then

$$
\begin{aligned}
{\left[\rho(\mathrm{X}), \mathrm{B}^{\lambda}\right] } & =\lambda \mathrm{B}^{\lambda-1}[\rho(\mathrm{X}), \mathrm{B}] \\
& =\lambda \mathrm{B}^{\lambda-1}[\rho(\mathrm{X}), \mathrm{A}] \mathrm{B} \\
& =\lambda \mathrm{B} \omega(\mathrm{X})
\end{aligned}
$$

This proves 7.4.
Note that $B^{\lambda}$ is an intertwining operator between $\rho_{\lambda}(X)$ and $\rho(X)$. The physical interpretation of $\mathrm{B}^{\lambda}$ is then that it is the "S-matrix" relating $\rho(\mathrm{X})$ to $\rho_{\lambda}(\mathrm{X})$. For if $\rho_{\lambda}(\mathrm{X})$ and $\rho(\mathrm{X})$ were Hamilton operators on the Hilbert space H that were the Hamiltonians of physical systems, then 7.4 is the characteristic property of the "S-matrix."

Note also that 7.4 implies (at least formally) that

$$
\begin{equation*}
\mathrm{B}^{\lambda} \exp \left(\mathrm{t} \rho_{\lambda}(\mathrm{X})\right)=\exp (\mathrm{t} \rho(\mathrm{X})) \mathrm{B}^{\lambda} \tag{7.5}
\end{equation*}
$$

This relation was our starting point in [4], Part V, and we saw there how it could be used (in the case $\mathrm{G}=\mathrm{SL}(2, \mathrm{R})$ ) to derive results about the asymptotic behavior of the matrix elements of its representations.

Now, we turn to consideration of a class of representations for which one can find this intertwining operator $B^{\lambda}$ explicitly. However, we must change our emphasis from algebra to geometry.
VIII. CONTINUATIONS AND COCYCLES DETERMINED BY TENSOR FIELDS

In this and the following sections, we will need the theory of differentiable manifolds and transformation groups, for which we refer to [1] and [6] .

Let $M$ be a manifold, with $F(M)$ its ring of real-valued, $C^{\infty}$ functions. (All manifolds, maps, tensor fields, etc. will be of differentiability class $C^{\infty}$ unless mentioned otherwise.)

A vector field, $X$, is a derivation of the ring $F(M)$, i.e., a linear map $\mathrm{f} \rightarrow \mathrm{X}(\mathrm{f})$ such that

$$
\mathrm{X}\left(\mathrm{f}_{1} \mathrm{f}_{2}\right)=\mathrm{X}\left(\mathrm{f}_{1}\right) \mathrm{f}_{2}+\mathrm{f}_{2} \mathrm{f}_{1} \mathrm{X}\left(\mathrm{f}_{2}\right) \text { for } \mathrm{f}_{1}, \mathrm{f}_{2} \in \mathrm{~F}(\mathrm{M})
$$

$\mathrm{V}(\mathrm{M})$ denotes the set of vector fields. It is a Lie algebra, under the Jacobi bracket operation:

$$
\mathrm{f} \rightarrow[\mathrm{X}, \mathrm{Y}] \quad(\mathrm{f})=\mathrm{X}(\mathrm{Y}(\mathrm{f}))-\mathrm{Y}(\mathrm{X}(\mathrm{f}))
$$

If $T$ is a tensor-field on $M, X \in V(M), X(T)$ denotes the Lie-derivative of T by X , a tensor-field of the same algebraic type as $\omega$. For example, if T is an $r$-fold covariant tensor field, i.e., an $F(M)$ multilinear map

$$
\begin{aligned}
&\left(X_{1}, \ldots, X_{r}\right) \rightarrow T\left(X_{1}, \ldots ., X_{r}\right), \epsilon F(M), \text { for } X_{1}, \ldots, X_{r} \in V(M) \\
& X(T)\left(X_{1}, \ldots, X_{r}\right)= X\left(T\left(X_{1}, \ldots, X_{r}\right)\right)-T\left(\left[X_{1}, X_{1}\right], X_{2}, \ldots, X_{r}\right)-\ldots \\
&-T\left(X_{1}, \ldots,\left[X_{r}, X_{r}\right]\right)
\end{aligned}
$$

Lie derivative acts as a derivation on tensor-products of tensor fields.

$$
X\left(T_{1} \otimes T_{2}\right)=X\left(T_{1}\right) \otimes T_{2}+T_{1} \otimes X\left(T_{2}\right)
$$

Suppose that $\underset{\sim}{G}$ is a Lie algebra of vector fields on $M$, and $T$ is a tensorfield such that:

$$
X(T)=\omega(X) T \text { for each } X \in \underset{\sim}{G} .
$$

$\omega(\mathrm{X})$ is to be an element of $\mathrm{F}(\mathrm{M})$.
Now, for $X, Y \in G$,

$$
\begin{align*}
& {[\mathrm{X}, \mathrm{Y}](\mathrm{T}) }=\omega([\mathrm{X}, \mathrm{Y}]) \mathrm{T} \\
&=\mathrm{X}(\omega(\mathrm{Y}) \mathrm{T})-\mathrm{Y}(\omega(\mathrm{X}) \mathrm{T}) \\
&=\mathrm{X}(\omega(\mathrm{Y})) \mathrm{T}+\omega(\mathrm{Y}) \omega(\mathrm{X}) \mathrm{T}-\mathrm{Y}(\omega(\mathrm{X})) \mathrm{T} \\
&=\mathrm{X}(\omega(\mathrm{Y})) \mathrm{T}-\mathrm{Y}(\omega(\mathrm{X})) \mathrm{T}, \text { or } \\
& \mathrm{X}(\omega(\mathrm{Y}))-\mathrm{Y}(\omega(\mathrm{X}))-\omega([\mathrm{X}, \mathrm{Y}])=0 \tag{8.1}
\end{align*}
$$

Let $V$ be the space of linear mapping: $V(M) \rightarrow V(M)$. For each $A \in V(M)$, $X \in G$, define $\phi(X)(A)$ as the commutator $[X, A]: f \rightarrow X A(f)-A X(f)$. Interpret each $\omega(\mathrm{X})$ as an element of V :

$$
\mathrm{f} \rightarrow \omega(\mathrm{X}) \mathrm{f}
$$

Then, $\omega$ can be interpreted as 1-cochain of $G$ with coefficients on V, i.e., an element of $\mathrm{C}^{1}(\phi) .8 .1$ then says that this is a cocycle, since:

$$
[\mathrm{X}, \omega(\mathrm{Y})] \quad(\mathrm{f})=\mathrm{X}(\omega(\mathrm{Y}))(\mathrm{f})
$$

Since further $[\omega(\mathrm{X}), \omega(\mathrm{Y})]=0$, we know from our earlier work that defining

$$
\rho_{\lambda}(\mathrm{X})=\mathrm{X}+\lambda \omega(\mathrm{X}) \quad \text { for } \mathrm{X} \in \mathrm{G}
$$

gives a one-parameter family of representations of $G$ by operators on $F(M)$.

Let us see how $\omega$ changes when $T$ is changed in the following way:

$$
\mathrm{T}^{\prime}=\mathrm{f} T \text {, for a function } \mathrm{f} \in \mathrm{~F}(\mathrm{M})
$$

Then,

$$
\begin{align*}
X\left(T^{\prime}\right) & =\omega^{\prime}(X) T^{\prime} \\
& =X(f) T+f \omega(X) T, \text { or } \\
\omega^{\prime}(X) & =\left(\frac{X(f)}{f}+\omega(X)\right), \text { or } \\
\omega^{\prime}(X) & =X(\log f)+\omega(X) \tag{8.2}
\end{align*}
$$

Thus, if $\log f \in F(M), \omega^{\prime}$ differs from $\omega$ by a coboundary.
Suppose that X is a fixed element of G , and we want to satisfy the hypotheses of Theorem 7.1, i.e., we want to find an $A \epsilon V$ such that:

$$
\begin{aligned}
{[\mathrm{X}, \mathrm{~A}] } & =\omega(\mathrm{X}) \\
{[\mathrm{A}, \omega(\mathrm{X})] } & =0
\end{aligned}
$$

We can satisfy the second of these conditions by demanding that A result from multiplication by a fixed function $\mathrm{g}_{\mathrm{X}}$. Then, the first condition requires that:

$$
\begin{equation*}
\mathrm{X}\left(\mathrm{~g}_{\mathrm{X}}\right)=\omega(\mathrm{X}) \tag{8.3}
\end{equation*}
$$

Suppose, in local coordinates $\left(x_{1}, \ldots, x_{n}\right)$ for $M, X=A_{1} \quad \partial / \partial x_{1}+\ldots$ $+A_{n} \partial / \partial x_{n}$. Then, $g_{X}$ is a solution of the differential equation:

$$
A_{1} \frac{\partial g_{X}}{\partial x_{1}}+\ldots+A_{n} \frac{\partial g_{X}}{\partial x_{n}}=\omega(X)
$$

Let us examine the case where $T$ is a differential form of the same degree as the dimension of M , i.e., a volume-element differential form for M .

Suppose local coordinates ( $\mathrm{x}_{1}, \ldots, \mathrm{x}_{\mathrm{n}}$ ) chosen so that:

$$
\begin{gathered}
T=d x_{1} \wedge \ldots \wedge d x_{n} \\
\text { and } X=A_{1} \partial / \partial x_{1}+\ldots+A_{n} \partial / \partial x_{n} .
\end{gathered}
$$

Then,

$$
\begin{align*}
X(T)= & X\left(d x_{1}\right) \wedge d x_{2} \wedge \ldots \wedge d x_{n}+d x_{1} \wedge X\left(d x_{2}\right) \wedge \ldots \wedge d x_{n} \\
& +\ldots d x_{1} \wedge \ldots \wedge X\left(d x_{n}\right) \\
= & \left(\frac{\partial A_{1}}{\partial x_{1}}+\ldots+\frac{\partial A_{n}}{\partial x_{n}}\right) d x_{1} \wedge \ldots \wedge d x_{n}, \text { or } \\
& \omega(X)=\frac{\partial A_{1}}{\partial x_{1}}+\ldots+\frac{\partial A_{n}}{\partial x_{n}} . \tag{8.4}
\end{align*}
$$

Solving 8.3 means solving

$$
\begin{equation*}
A_{1} \frac{\partial g}{\partial x_{1}}+\ldots+A_{n} \frac{\partial g}{\partial x_{n}}=\left(\frac{\partial A_{1}}{\partial x_{1}}+\ldots+\frac{\partial A_{n}}{\partial x_{n}}\right) \tag{8.5}
\end{equation*}
$$

One case where the solution can be written down can be immediately suggested. Suppose $A_{1}$ is a function $A_{1}\left(x_{1}\right)$ of $x_{1}$ alone, $A_{2}\left(x_{2}\right)$, etc. Then, $g$ can be taken as:

$$
\begin{equation*}
g=\log A_{1}+\log A_{2}+\ldots+\log A_{n} \tag{8.6}
\end{equation*}
$$

IX. CALCULATION OF THE INTERTWINING OPERATOR $B^{\lambda}$ FOR GERTAIN REPRESENTATIONS

Let $G$ be a non-compact, connected semisimple Lie group with finite center, $K$ be its maximal compact subgroup, $\underset{\sim}{G}=\underset{\sim}{K}+\underset{\sim}{P}$ its Cartan
decomposition, i.e.,

$$
[\underset{\sim}{\mathrm{K}}, \underset{\sim}{\mathrm{P}}] \subset \underset{\sim}{\mathrm{P}},[\underset{\sim}{\mathrm{P}}, \underset{\sim}{\mathrm{P}}] \subset \underset{\sim}{\mathrm{K}}
$$

Let $X_{o}$ be an element of $\underset{\sim}{p}$. Then, Ad $X_{o}$ has real eigenvalues and is completely reducible [2]. Let $\underset{\sim}{H}\left(X_{o}\right)$ be the subspace of $G$ spanned by the eigenvectors of $\mathrm{Ad} \mathrm{X}_{\mathrm{o}}$ with non-negative eigenvalues. $\underset{\sim}{\underset{\sim}{( }} \underset{\mathrm{o}}{ }$ ) is a subalgebra of $\underset{\sim}{G}$ : Let $H\left(X_{o}\right)$ be the connected subgroup of $G$ generated by $\underset{\sim}{H}\left(X_{o}\right)$. Let $M^{\prime}$ be the coset space $G / H$, and let $p_{o}$ be the coset of the identity elements. Let ${\underset{\sim}{N}}^{-}\left(\mathrm{X}_{\mathrm{O}}\right)$ be the subalgebra of $\underset{\sim}{G}$ spanned by the eigenvectors of $\operatorname{Ad} \mathrm{X}_{\mathrm{o}}$ for negative eigenvalues. Thus $\underset{\sim}{G}$, as a vector space, is the direct sum $\underset{\sim}{H}\left(X_{0}\right)+{\underset{\sim}{N}}^{-}\left(X_{0}\right)$. Let $N^{-}\left(X_{0}\right)$ be the connected subgroup of $G$ generated by the subalgebra $\underset{\sim}{N}\left(X_{o}\right)$. Let $M$ be the orbit $N^{-}\left(X_{o}\right) \cdot p_{o}$. It is known that it is an open subset of $M$, and the complement of $M$ in $M^{\prime}$ is a set of measure zero. (Typically, it is these spaces $M$ that are used by Gelfand and Neumark to construct unitary representations of the classical groups [2]). Now, G acts as a global transformation group on $M^{\prime}=G / I I$. Hence, the Lie algebra G acts on $F\left(M^{\prime}\right)$ as a subalgebra of $V\left(M^{\prime}\right)$ :

$$
X(f)(p)=\partial|\partial t(f(\exp (-X t) p))|_{t=0}
$$

$$
\text { for } \mathrm{X} \in \mathrm{G}, \mathrm{f}=\mathrm{F}\left(\mathrm{M}^{\prime}\right), \mathrm{p} \in \mathrm{M}^{\prime}
$$

Since $M$ is an open subset of $M^{\prime}$, $G$ also acts as a Lie algebra of vector fields on $M$, i.e., $\underset{\sim}{G}$ can be identified with a subalgebra of $V(M)$.

In this section we will use a volume element-differential form $d x$ on $M$ that is invariant under $\mathrm{N}^{-}\left(\mathrm{X}_{\mathrm{o}}\right)$. Using this, we will, following the pattern
described in the last section, define $\omega(\mathrm{X})$, for $\mathrm{X} \in \underset{\sim}{G}$, as the function in $F(M)$ such that:

$$
X(\mathrm{dx})=\omega(\mathrm{X}) \mathrm{dx} \text { for } \mathrm{X} \in \underset{\sim}{G}
$$

and define

$$
\rho_{\lambda}(\mathrm{X})(\mathrm{f})=\mathrm{X}(\mathrm{f})+\lambda \omega(\mathrm{X}) \mathrm{f} \text { for } \mathrm{f} \epsilon \mathrm{~F}(\mathrm{M}), \mathrm{X} \in \underset{\sim}{\mathrm{G}} .
$$

Notice then that:

$$
\omega\left({\underset{\sim}{N}}^{-}\left(\mathrm{X}_{\mathrm{o}}\right)\right)=0
$$

Given $X \in \underset{\sim}{G}$, our problem is to find the intertwining operator $B^{\lambda}$ such that:

$$
\rho_{\lambda}(\mathrm{X})=\mathrm{B}^{-\lambda} \rho_{\mathrm{o}}(\mathrm{X}) \mathrm{B}^{\lambda}
$$

We shall first deal with the following case:
X belongs to $\underset{\sim}{A}$, a maximal abelian subalgebra of $\underset{\sim}{P}$ which contains $X_{o}$ also.
Now, the elements of Ad A can be simultaneously diagonalized, and have real eigenvalues. Let $\sigma_{1}, \ldots, \sigma_{\mathrm{n}}$ be the non-zero, real-valued forms on A resulting from this diagonalization. (The $\sigma_{1}, \ldots, \sigma_{\mathrm{n}}$ are not necessarily distinct as linear forms on $\underset{\sim}{A}$.) For each $\sigma_{i}, 1 \leq i \leq n$, there are elements $\mathrm{W}_{\mathrm{i}}, \mathrm{W}_{-\mathrm{i}} \in \underset{\sim}{G}$ such that:

$$
\begin{aligned}
& {\left[\mathrm{X}, \mathrm{~W}_{\mathrm{i}}\right]=\sigma_{\mathrm{i}}(\mathrm{X}) \mathrm{W}} \\
& {\left[\mathrm{X}, \mathrm{~W}_{-\mathrm{i}}\right]=-\sigma_{\mathrm{i}}(\mathrm{X}) \mathrm{W}_{\mathrm{i}} \text { for } \mathrm{X} \in \underset{\sim}{\mathrm{~A}}}
\end{aligned}
$$

For each i, there is a decomposition:

$$
\begin{aligned}
& \mathrm{W}_{\mathrm{i}}=\mathrm{Z}_{\mathrm{i}}+\mathrm{Y}_{\mathrm{i}} \\
& \mathrm{~W}_{-\mathrm{i}}=\mathrm{Z}_{\mathrm{i}}-\mathrm{Y}_{\mathrm{i}}
\end{aligned}
$$

with $Z_{i} \in \underset{\sim}{K}, Y_{i} \in \underset{\sim}{P}, \underset{i}{B}\left(Z_{i}, Z_{i}\right)=-1, B\left(Y_{i} Y_{i}\right)=1$,

$$
\left[\mathrm{X}, \mathrm{Z}_{\mathrm{i}}\right]=\sigma_{\mathrm{i}}(\mathrm{X}) \mathrm{Y}_{\mathrm{i}}, \quad\left[\mathrm{X}, \mathrm{Y}_{\mathrm{i}}\right]=\sigma_{\mathrm{i}}(\mathrm{X}) \mathrm{Z}_{\mathrm{i}} \quad \text { for } \mathrm{X} \in \mathrm{~A} .
$$

( $B($,$) is the Kelling form on \underset{\sim}{G}$ : it is negative definite on $\underset{\sim}{K}$, positive definite on $\underset{\sim}{\mathrm{P}}$. )

Suppose the ordering of the $\sigma^{\prime}$ s is chosen so that: $\sigma_{1}, \ldots, \sigma_{m}$ are the forms that are non-zero on $\mathrm{X}_{\mathrm{o}}$, while $\sigma_{\mathrm{m}+1}\left(\mathrm{X}_{\mathrm{o}}\right)=0=\ldots \sigma_{\mathrm{n}}\left(\mathrm{X}_{\mathrm{o}}\right)$. Then, ${\underset{\sim}{N}}^{-}\left(X_{o}\right)$ is spanned by $W_{-1}, \ldots, W_{-m}$. Hence the bracket $\left[W_{-i} \cdot W_{-j}\right]$ is, if non-zero, an eigenvector of Ad $\underset{\sim}{A}$. We see that ${\underset{\sim}{N}}^{-}\left(X_{0}\right)$ is a nilpotent subalgebra of $G$. $M$ then admits a coordinate system $\left(x_{1}, \ldots, x_{m}\right)$ such that:

$$
x_{i}\left(\exp \left(t_{1} W_{-1}\right) \exp \left(t_{2} W_{-2}\right) \cdots \exp \left(t_{m} W_{-m}\right) \cdot p_{o}\right)=t_{i}
$$

```
for 1\leqi\leqm.
```

In terms of this coordinate system for $M$, the vector field on $M$ generated by an element $\mathrm{X} \in \mathrm{A}$ takes the form

$$
\begin{equation*}
X=\sum_{i=1}^{m}-\sigma_{i}(X) x_{i} \frac{\partial}{\partial x_{i}} \tag{9.1}
\end{equation*}
$$

The volume element-differential form $d x$ on $M$ that is invariant under $N^{-}\left(X_{0}\right)$ takes the form:

$$
d x=d x_{1} \wedge d x_{2} \wedge \ldots \wedge d x_{m}
$$

Hence, if $\omega(\mathrm{X})$ is defined by 8.4 , we have:

$$
\begin{aligned}
\omega(\mathrm{X})= & -\left(\sigma_{1}(\mathrm{X})+\ldots+\sigma_{\mathrm{m}}(\mathrm{X})\right) \\
& -24-
\end{aligned}
$$

We see that everything is set up so that 8.6 applies:

$$
\mathrm{X}(\mathrm{~g})=\omega(\mathrm{X}), \text { where } \mathrm{g}=\log \left((-1)^{\mathrm{m}} \sigma_{1}(\mathrm{X}) \ldots \sigma_{\mathrm{m}}(\mathrm{X}) \mathrm{x}_{1} \ldots \mathrm{x}_{\mathrm{m}}\right)
$$

This, combined with our previous remarks, proves the following
Theorem 9.1 For $\mathrm{X} \in \mathrm{A}$, the following intertwining operator links $\rho_{\lambda}(\mathrm{X})$ and $\rho_{\mathrm{o}}(\mathrm{X}):$

$$
\begin{gather*}
\rho_{\lambda}(\mathrm{X})\left(\sigma_{1}(\mathrm{X}) \mathrm{x}_{1} \cdots \sigma_{\mathrm{m}}(\mathrm{X}) \mathrm{x}_{\mathrm{m}}\right)^{\lambda_{\mathrm{f}}}  \tag{9.2}\\
=\left(\sigma_{1}(\mathrm{X}) \mathrm{x}_{1} \cdots \sigma_{\mathrm{m}}(\mathrm{X}) \mathrm{x}_{\mathrm{m}}\right)^{\lambda} \rho_{\mathrm{o}}(\mathrm{X})(\mathrm{f}) \quad \text { for } \quad \mathrm{f} \in \mathrm{~F}(\mathrm{M})
\end{gather*}
$$

The important qualitative point to keep in mind is that the coordinate system $\left(x_{1}, \ldots, x_{m}\right)$ valid in $M^{\prime}$ is that for which the vector fields of $\underset{\sim}{A}$ take the form 9.1.

So far, we have used one property of the space $M$, namely that the orbit $\mathrm{N}^{-}\left(\mathrm{X}_{\mathrm{o}}\right) \cdot \mathrm{p}$ is as open, dense subset which admits a coordinate system having nice properties.

Now, this method of writing down the operators is not the most convenient for the purposes of physics: One wants the decomposition under $K$ to be more explicit. This can be remedied by using a volume element that is invariant under K. In fact, one knows that $K$ acts transitively on $M$. Let dp be a volume element-differential form for $M$ that is invariant under the action of $K$. Suppose:

$$
X(d p)=\omega^{\prime}(X) d p \quad \text { for } X \in G
$$

Then,

$$
\rho_{\lambda}^{\prime}(\mathrm{X})(\mathrm{f})=\mathrm{X}(\mathrm{f})+\lambda \omega^{\prime}(\mathrm{X}) \mathrm{f} \text { for } \mathrm{f} \boldsymbol{\epsilon} \mathrm{~F}\left(\mathrm{M}^{\prime}\right)
$$

defines another deformation of the representation $\rho_{0}$. Let us calculate $\rho_{\lambda}{ }^{\prime}$ in terms of $\rho_{\lambda}$ : Suppose that

$$
\mathrm{dp}=\mathrm{h} d \mathrm{x}, \text { where } \mathrm{h} \in \mathrm{~F}\left(\mathrm{M}^{\prime}\right)
$$

Then, for $X \in \underset{\sim}{G}$

$$
\begin{aligned}
\mathrm{X}(\mathrm{dp}) & =\omega^{\prime}(\mathrm{X}) \mathrm{dp} \\
& =\mathrm{X}(\mathrm{~h}) \mathrm{dx}+\mathrm{h} \omega(\mathrm{X}) \mathrm{dx} \\
& =\mathrm{X}(\log \mathrm{~h}) \mathrm{dp}+\omega(\mathrm{X}) \mathrm{dp}, \text { or } \\
\omega^{\prime}(\mathrm{X}) & =\mathrm{X}(\log \mathrm{~h})+\omega(\mathrm{X}), \text { hence: } \\
\rho_{\lambda}^{\prime}(\mathrm{X}) & =\rho_{\lambda}(\mathrm{X})+\lambda \mathrm{X}(\log \mathrm{~h})
\end{aligned}
$$

The key fact is that $\mathrm{X}(\log \mathrm{h})$, as an operator, commutes with the intertwining operator between $\rho_{\lambda}(\mathrm{X})$ and $\rho_{\mathrm{o}}(\mathrm{X})$. Thus, we have:

$$
\begin{aligned}
& \left(\sigma_{1}(\mathrm{X}) \mathrm{x}_{1} \cdots \sigma_{\mathrm{m}}(\mathrm{X}) \mathrm{x}_{\mathrm{m}}\right)^{-\lambda} \rho_{\lambda}^{\prime}(\mathrm{X})(\mathrm{f})= \\
& \quad=\left(\sigma_{1}(\mathrm{X}) \mathrm{x}_{1} \cdots \sigma_{\mathrm{m}}(\mathrm{X}) \mathrm{x}_{\mathrm{m}}\right)^{-\lambda}(\mathrm{X}(\mathrm{f})+\lambda \mathrm{X}(\log \mathrm{~h})) \text { for } \mathrm{f} \epsilon \mathrm{~F}\left(\mathrm{M}^{\prime}\right)
\end{aligned}
$$

Also,

$$
\begin{equation*}
h^{\lambda}(X+\lambda X(\log h) h)^{-\lambda}=x \tag{9.3}
\end{equation*}
$$

hence,

$$
\begin{align*}
\rho^{\prime}(\mathrm{X})= & \left(\sigma_{1}(\mathrm{X}) \mathrm{x}_{1} \cdots \sigma_{\mathrm{m}}(\mathrm{X}) \mathrm{x}_{\mathrm{m}}\right)^{\lambda} \rho_{\mathrm{o}}(\mathrm{X})\left(\sigma_{1}(\mathrm{X}) \cdots \mathrm{x}_{\mathrm{m}}\right)^{-\lambda} \\
= & \left(\sigma_{1}(\mathrm{X}) \cdots \mathrm{x}_{\mathrm{m}}\right)^{\lambda}\left(\rho_{\mathrm{o}}(\mathrm{X})+\lambda \mathrm{X}(\operatorname{logh})\right)\left(\sigma_{1}(\mathrm{X}) \ldots \mathrm{x}_{\mathrm{m}}\right)^{-\lambda} \\
= & \left(\sigma_{1}(\mathrm{X}) \mathrm{x}_{1} \ldots \mathrm{x}_{\mathrm{m}}^{\mathrm{h}}\right)^{\lambda}\left(\rho_{\mathrm{o}}(\mathrm{X})\right)\left(\sigma_{1}(\mathrm{X}) \mathrm{x}_{1} \ldots \mathrm{x}_{\mathrm{m}} \mathrm{~h}\right)^{-\lambda}  \tag{9.4}\\
& -26-
\end{align*}
$$

This is the most useful form of the indentity for the application to group representation theory. Let $F\left(M^{\prime}, C\right)$ be the complex-valued, $C^{\infty}$ functions on $M^{\prime}$, i.e., $F\left(M^{\prime} C\right)=F\left(M^{\prime}\right)+i F\left(M^{\prime}\right)$. Let us make $F\left(M^{\prime}, C\right)$ into an (incomplete) Hilbert space by adopting the following inner product:

$$
\left\langle\psi \mid \psi^{\prime}\right\rangle=\int_{\mathbf{M}^{\prime}} \psi(p)^{*} \psi^{\prime}(p) d p
$$

## for $\psi, \psi^{\prime} \in \mathrm{F}(\mathrm{M}, \mathrm{C})$

Consider $F(M, C)=F(M)+i F(M)$ as a subspace of $F\left(M^{\prime}, C\right)$, i.e., $F(M, C)$ consists of those $\mathrm{C}^{\infty}$ functions that can be extended smoothly from $\mathrm{M}^{1}$ to M . Since the complement of $\mathrm{M}^{\prime}$ in M is (relative to the measure defined by dp) a set of measure zero, $F(M, C)$ is (relative to the Hilbert space topology) dense in $\mathrm{F}\left(\mathrm{M}^{\prime}, \mathrm{C}\right)$, and

$$
\left\langle\psi \mid \psi^{\prime}\right\rangle=\int_{\mathrm{M}} \psi(\mathrm{p})^{*} \psi^{\prime}(\mathrm{p}) \mathrm{dp} \quad \text { for } \psi, \psi^{\prime} \in \mathrm{F}(\mathrm{M}, \mathrm{C})
$$

Thus, for $\underset{\sim}{X} \in G, \psi, \psi^{\prime} \in \operatorname{F}(M, C)$,

$$
\begin{aligned}
& \left\langle\rho_{\lambda}^{\prime}(\mathrm{X}) \psi \mid \psi^{\prime}\right\rangle=\int_{\mathrm{M}}\left(\mathrm{X}(\psi)^{*}+\lambda^{*} \omega^{\prime}(\mathrm{X}) \psi^{*}\right) \psi^{\prime} \mathrm{dp} \\
& \quad=\int_{\mathrm{M}}\left(-\psi^{*} \mathrm{X}\left(\psi^{\prime}\right)-\psi^{*} \psi^{\prime} \omega^{\prime}(\mathrm{X})+\lambda^{*} \omega^{\prime}(\mathrm{X}) \psi^{*} \psi^{\prime}\right) \mathrm{dp} \\
& \quad=\left\langle\psi \mid\left(-\mathrm{X}+\left(\lambda^{*}-1\right) \omega^{\prime}(\mathrm{X})\right) \psi^{\prime}\right\rangle
\end{aligned}
$$

We see that: $\quad \rho_{\lambda}^{\prime}(G)$, acting the domain $F(M, C)$, is skew-Hermitian if and only if

$$
\begin{equation*}
\lambda^{*}+\lambda=1, \text { or } \tag{9.5}
\end{equation*}
$$

$\lambda$ is the form $1 / 2+i b$, with real $b$.
We are now in position to show how to calculate the asymptotic behavior of matrix elements:

$$
\left\langle\psi^{\prime} \mid \rho_{\lambda}(\exp (\mathrm{t} \mathrm{X})) \psi^{\prime}\right\rangle \text { as } t \rightarrow \infty
$$

## X. ASYMPTOTIC BEHA VIOR OF MATRIX ELEMENTS

Suppose the group $G$ acts on a manifold $M^{\prime}$ as a transformation group. Let $\rho_{\lambda}^{\prime}$ be a representation of $G$ by operators in an (incomplete) Hilbert space $H$. Suppose, in fact, that $H$ is just $F\left(M^{\prime}, C\right)$, the space of complex-valued, $C^{\infty}$ functions on $\mathrm{M}^{\mathbf{\prime}}$, with the inner product given by:

$$
\left\langle\psi \mid \psi^{\prime}\right\rangle=\int_{\mathrm{M}} \psi(\mathrm{p})^{*} \psi^{\prime}(\mathrm{p}) \mathrm{dp}
$$

where $d p$ is a volume element-differential form on $M^{\prime}$. Suppose $\rho_{o}^{\prime}(X)$ is just the action $\psi \rightarrow X(\psi)$ of $X \in G$ by derivations of $F(M, C)$, describing the infinitesimal action of the one-parameter group $t \rightarrow \exp (t X)$ on $M$.

Let $X$ be a fixed element of $G$. Suppose $h_{X}$ is a function on $M$ (possibly with singularities lying on submanifolds of $\mathrm{M}^{+}$) such that:

$$
\begin{equation*}
\rho_{\lambda}(\mathrm{X})(\psi)=\mathrm{h}_{\mathrm{X}}^{\lambda} \rho_{\mathrm{o}}(\mathrm{X})\left(\mathrm{h}_{\mathrm{X}}^{-\lambda} \psi\right) \tag{10.1}
\end{equation*}
$$

(In terms of 9.4, $\mathrm{h}_{\mathrm{X}}$ can be identified with

$$
\sigma_{1}(\mathrm{X}) \mathrm{x}_{1} \cdots \sigma_{\mathrm{m}}(\mathrm{X}) \mathrm{x}_{\mathrm{m}}^{\mathrm{h}}
$$

Then,

$$
\exp \left(\mathrm{t} \rho_{\lambda}(\mathrm{X})\right)=\mathrm{h}_{\mathrm{X}}^{\lambda} \exp \left(\mathrm{t} \rho_{\mathrm{o}}(\mathrm{X})\right) \mathrm{h}_{\mathrm{X}}^{-\lambda}
$$

Now, for $\psi \epsilon H, p \in M^{\prime}$,

$$
\exp \left(\mathrm{t} \rho_{\mathrm{o}}(\mathrm{X})\right)(\psi)(\mathrm{p})=\psi(\exp (-\mathrm{tX}) \cdot \mathrm{p})
$$

where $p \rightarrow \exp (t X) \cdot p$ is the given action of the one-parameter subgroup $t \rightarrow \exp (\mathrm{tX})$ on M .

Thus, for $\psi, \psi^{\prime} \epsilon \mathrm{H}$,

$$
\begin{aligned}
& \alpha(\mathrm{t}, \lambda)=\left\langle\psi \mid \exp \left(\mathrm{t} \rho_{\lambda}(\mathrm{X})\right) \psi^{\prime}\right\rangle \\
& =\quad \int_{M} \psi(\mathrm{p})^{*} \mathrm{~h}_{\mathrm{X}}{(\mathrm{p})^{\lambda} \mathrm{h}(\exp (-\mathrm{tX}) \mathrm{p})^{-\lambda} \psi^{\prime}(\exp (-\mathrm{tX}) \mathrm{p}) \mathrm{dp} .}^{=} .
\end{aligned}
$$

As we have seen in [4], Pt. V, for the case $G=S L(2, R)$, there are two immediate interesting asymptotic problems:
(a) Asymptotic behavior as $t \rightarrow \infty$, with $\lambda$ held fixed.
(b) Asymptotic behavior as $\lambda$ goes to infinity. (For example, for the case $G=\operatorname{SL}(2, R), \lim _{\lambda \rightarrow \infty} \alpha(t / \lambda, \lambda)$ exists.)

In turn, 10.1 shows that these are reduced to various geometric questions concerning the asymptotic behaviour of the orbits $\exp (-t \mathrm{X}) \cdot \mathrm{p}$ as $\mathrm{t} \rightarrow \infty$ hence
are closely related to the problems of the modern theory of dynamical systems. We will deal with these problems in full technical details in a paper that will be published in a mathematics journal. We will present here various heuristic remarks.

The general problem we face can be described as follows: Suppose $M^{1}$ is a space with a measure $d p$, such that the total measure of $M^{\prime}$ is finite. Suppose $t \rightarrow g(t)$, defined for $t \geq 0$, as a one-parameter semigroup acting on M: Suppose $\mathrm{f}_{1}$ and $\mathrm{f}_{2}$ are measurable functions on M. Does there exist a number a such that:

$$
\lim _{t \rightarrow \infty} t^{a} \int_{M^{\prime}} f_{1}(p) f_{2}(g(t) p) d p \text { exists? }
$$

For example, suppose $f_{1}, f_{2}$ are bounded, continuous functions on $M^{\prime}$, space such that continuous functions are measurable. Suppose the following condition is satisfied:

There is a point $\mathrm{p}^{\prime} \boldsymbol{\epsilon} \mathrm{M}^{\boldsymbol{\prime}}$ such that

$$
\lim _{t \rightarrow \infty} g(t) \cdot p=p^{\prime} \text { for all } p \in M^{\prime}
$$

except possibly for a set of points of zero measure.
Then, the sequence of functions,

$$
t \rightarrow f_{3}^{\mathrm{t}}(\mathrm{p})
$$

with

$$
\mathrm{f}_{3}^{\mathrm{t}}(\mathrm{p})=\mathrm{f}_{1}(\mathrm{p}) \mathrm{f}_{2}(\mathrm{~g}(\mathrm{t}) \mathrm{p})
$$

converges as $t \rightarrow \infty$ to the function

$$
\mathrm{f}_{4}(\mathrm{p})=\mathrm{f}_{1}(\mathrm{p}) \mathrm{f}_{2}\left(\mathrm{p}^{\prime}\right)
$$

with the convergence taking place for all but a set of measure zero of points p . Thus, by the Lesbesque bounded convergence theorem,

$$
\begin{gathered}
\lim _{t \rightarrow \infty} \int_{M^{\prime}} f_{3}^{\mathrm{t}}(\mathrm{p}) \mathrm{dp}=\int_{\mathrm{M}^{\prime}} \mathrm{f}_{4}(\mathrm{p}) \mathrm{dp}, \text { or } \\
\lim _{\mathrm{t} \rightarrow \infty} \int_{M^{\prime}} \mathrm{f}_{1}(\mathrm{p}) \mathrm{f}_{2}(\mathrm{~g}(\mathrm{t}) \mathrm{p}) \mathrm{dp}=\mathrm{f}_{2}\left(\mathrm{p}^{\prime}\right) \int_{M^{\prime}} \mathrm{f}_{1}(\mathrm{p}) \mathrm{dp}
\end{gathered}
$$

Let us now consider a more general such problem. Suppose that 10.2 continues to be satisfied, and the problem is to find the limit as $t \rightarrow \infty$ of:

$$
\int_{\mathrm{M}^{+}} \mathrm{f}_{1}(\mathrm{p}) \mathrm{f}_{2}(\mathrm{~g}(\mathrm{t}) \mathrm{p}) \mathrm{dp}
$$

as before. However, we do not assume that $f_{2}(p)$ is everywhere continuous, but assume that it has poles. For example, suppose it has a pole at $p=p^{\prime}$. To have no trouble with the convergence of the integral, let us suppose that:

$$
g(t) \cdot p^{\prime}=p^{\prime} \text { for all } t \geq 0, \text { and } p \rightarrow f_{1}(p) f_{2}(g(t) p)
$$

is continuous in a neighborhood of $p^{\prime}$, i.e., $f_{1}(p)$ has a zero at $p^{\prime}$
sufficiently strong to cancel out the pole at $f_{2}\left(p^{\prime}\right)$.
Now, our assumption 10.2 is that, for all $p$ except possibly for a set of measure zero, $\lim _{t \rightarrow \infty} g(t) p=p^{\prime}$.

Let us assume that

$$
\mathrm{f}_{2}(\mathrm{~g}(\mathrm{t}) \mathrm{p}) \sim c \mathrm{e}^{\text {at }} \quad \text { as } \mathrm{t} \longrightarrow \infty
$$

Then, using the Lesbesque bounded convergence theorem as before, we see that:

$$
e^{-a t} \int_{M^{\prime}} f_{1}(p) f_{2}(g(t) p) d p \rightarrow c \quad \int_{M^{\prime}} f_{1}(p) d p \quad \text { at } t \rightarrow \infty
$$

This, then, is a sketch of our "geometric" method for finding the asymptotic behavior of matrix elements of certain types of group representations.

## XI. AN ABSTRACT APPROACH TO THE PROBLEM OF ASYMPTOTIC BEHAVIOR OF MATRIX ELEMENTS OF REPRESENTATIONS

There is an abstract pattern to the preceding work that is worth discussing separately. Suppose G is a Lie group, realized or a group of operators on a Hilbert space $H$. Let $t \rightarrow g(t)$ be a one-parameter subgroup of $G$, and let $\psi$ be an element of $H$. Let $t \rightarrow c(t)$ be a curve in $C$, the complex numbers. Suppose that $\psi$ is an element of H. Suppose that:

$$
\frac{\mathrm{g}(\mathrm{t}) \psi}{\mathrm{c}(\mathrm{t})}
$$

approaches $\psi_{\infty}$ via weak convergences as $\mathrm{c} \rightarrow \infty$. (Recall this means that:

$$
\left.\lim _{\mathrm{t} \rightarrow \infty} \frac{\left\langle\psi^{\prime} \mid \mathrm{g}(\mathrm{t}) \psi\right\rangle}{\mathrm{c}(\mathrm{t})}=\left\langle\psi^{\prime} \mid \psi_{\infty}\right\rangle \quad \text { for each } \psi^{\prime} \in \quad \mathrm{H} .\right)
$$

We will symbolize this relation as follows:

$$
\mathrm{g}(\mathrm{t})(\psi) \sim \mathrm{c}(\mathrm{t}) \psi_{\infty}
$$

We will understand that $H$ is not necessarily a complete Hilbert space: In fact, much of the same set of ideas can be applied to the case where H is a general set of topological vector space, and the elements $\psi^{\prime}$ are taken from a given family of complex-valued continuous linear functions on H .

Our main concern in this section will be to consider (as far as possible without making further specific assumptions) what one can say about the action of the elements of $G$ on limiting element $\psi_{\infty}$.

First, suppose that $g$ is an element of $G$. Let $g *$ be the adjoint transformation of g , i.e.,

$$
\left\langle\mathrm{g}^{*} \psi^{\prime} \mid \psi\right\rangle=\left\langle\psi^{\prime} \mid \mathrm{g} \psi\right\rangle \text { for } \psi, \psi^{\prime} \in \mathrm{H}
$$

We will, in fact, suppose that $\mathrm{g}^{*}$ is defined on H also. (This is why we want H to be non-complete.) Then:

$$
\begin{aligned}
& \left\langle\psi^{\prime} \mid g \psi_{\infty}\right\rangle=\left\langle\mathrm{g}^{*} \psi^{\prime} \mid \psi_{\infty}\right\rangle \\
= & \lim _{\mathrm{t}} \frac{\left\langle\mathrm{~g}^{*} \psi^{\prime} \mid g(\mathrm{t}) \psi\right\rangle}{\mathrm{c}(\mathrm{t})} \\
= & \lim _{\mathrm{t}}\left\langle\psi^{\prime} \mid \mathrm{gg}(\mathrm{t}) \psi\right\rangle \mathrm{c}(\mathrm{t}) .
\end{aligned}
$$

We can read off immediately the following facts:
Theorem 11.1 If $g$ commutes with each $g(t)$, then

$$
\mathrm{g}(\mathrm{t})(\mathrm{g} \psi) \sim \mathrm{c}(\mathrm{t}) \mathrm{g} \psi_{\infty}
$$

Suppose now that $\mathrm{g}=\mathrm{g}(\mathrm{s})$ for some real s .

Then:

$$
\begin{aligned}
\left\langle\psi^{\prime} \mid \psi_{\infty}\right\rangle & =\lim _{t}\left\langle\psi^{\prime} \mid g(t+s) \psi\right\rangle / c(t) \\
& =\lim _{t}\left\langle\psi^{\prime} \mid g(t) \psi\right\rangle / c(t-s) \\
& =\lim _{t} \frac{\left\langle\psi^{\prime} \mid g(t) \psi\right\rangle}{c(t)} \frac{c(t)}{c(t-s)}
\end{aligned}
$$

Suppose that:

$$
\lim _{t \rightarrow \infty} \frac{c(t)}{c(t-s)}=b(s)
$$

11.4

Then, we have:
Theorem 11.2 If 11.4 is satisfied, then

$$
g(t) \psi_{\infty}=b(t) \psi_{\infty}
$$

i.e., $\psi_{\infty}$ is an eigenvector for each $g(t)$.

Now, suppose X is the infinitesimal generator for the one-parameter group $g(t)$, i.e.,

$$
g(t)=\exp (t X)
$$

Suppose $Y$ is an operator on $H$ such that:

$$
[\mathrm{X}, \mathrm{Y}]=\sigma \mathrm{Y}
$$

Then:

$$
\begin{aligned}
& \mathrm{Ad} \exp (\mathrm{tX})(\mathrm{y})=\mathrm{e}^{\sigma \mathrm{t}} \mathrm{Y}, \text { or } \\
& \begin{aligned}
& \exp (\mathrm{tX}) \mathrm{Y} \exp (-\mathrm{tX})=\mathrm{e}^{\sigma \mathrm{t}} \mathrm{Y}, \text { or } \\
& \mathrm{g}(\mathrm{t}) \mathrm{Yg}(\mathrm{t})^{-1}=\mathrm{e}^{\sigma \mathrm{t}} \mathrm{Y} \\
&-34-
\end{aligned}
\end{aligned}
$$

Then,

$$
\mathrm{g}(\mathrm{t}) \mathrm{e}^{-\sigma \mathrm{t}} \mathrm{Y} \quad \mathrm{~g}(\mathrm{t})^{-1}=\mathrm{Y}
$$

Suppose:

$$
h(s)=\exp (S Y)
$$

Then,

$$
\mathrm{g}(\mathrm{t}) \mathrm{h}\left(\mathrm{e}^{-\sigma \mathrm{t}} \mathrm{~s}\right)=\mathrm{h}(\mathrm{~s}) \mathrm{g}(\mathrm{t}) \quad 11.7
$$

We will, in fact, use 11.7 the "global" form of 11.6. Then, using 11.3, we have:

$$
\begin{aligned}
\left\langle\psi^{2} \mid h(s) \psi_{\infty}\right\rangle & =\lim _{t} \frac{\left\langle\psi^{\prime} \mid \mathrm{h}(\mathrm{~s}) \mathrm{g}(\mathrm{t}) \psi\right\rangle}{c(\mathrm{t})} \\
& =\lim _{\mathrm{t}} \frac{\left\langle\psi^{\prime} \mid g(\mathrm{t}) \mathrm{h}\left(\mathrm{e}^{-\sigma t} \mathrm{~s}\right) \psi\right\rangle}{c(\mathrm{t})}
\end{aligned}
$$

Now,

$$
\mathrm{g}(\mathrm{t}) \mathrm{h}\left(\mathrm{e}^{-\sigma \mathrm{t}} \mathrm{~s}\right) \psi-\mathrm{g}(\mathrm{t}) \psi=\mathrm{g}(\mathrm{t})\left(\mathrm{h}\left(\mathrm{e}^{-\sigma \mathrm{t}} \mathrm{~s}\right) \psi-\psi\right)
$$

Theorem 11.3 Suppose that
(a) $\sigma>0$.
(b) The representation of G by operators on H is continuous.
(c) The operators $g(t) / c(t)$ on $H$ have a common bound $B$.

Then,

$$
\mathrm{h}(\mathrm{~s}) \psi_{\infty}=\psi_{\infty} \text { for all s. }
$$

Proof
Hypotheses (a) and (b) tell us that:

$$
\left\|\mathrm{h}\left(\mathrm{e}^{-\sigma \mathrm{t}} \mathrm{~s}\right) \psi-\psi\right\| \rightarrow 0 \text { as } t \rightarrow \infty
$$

Hence,

$$
\begin{aligned}
& \left\|(\mathrm{g}(\mathrm{t}) / \mathrm{c}(\mathrm{t})) \mathrm{h}\left(\mathrm{e}^{-\sigma \mathrm{t}} \mathrm{~s}\right) \psi-\mathrm{g}(\mathrm{t}) / \mathrm{c}(\mathrm{t}) \psi\right\| \\
& \leq \mathrm{B}\left\|\mathrm{~h}\left(\mathrm{e}^{-\sigma \mathrm{t}} \mathrm{~s}\right) \psi-\psi\right\|
\end{aligned}
$$

In particular, we see that it converges strongly to zero as $t \rightarrow \infty$, hence also converges weakly to zero.

We know that $\lim _{\mathrm{t}} \frac{\left\langle\psi^{\eta}\right| \mathrm{g}(\mathrm{t}) \mathrm{h}\left(\mathrm{e}^{\left.\left.-\sigma \mathrm{t}_{\mathrm{S}}\right) \psi\right\rangle}\right.}{\mathrm{c}(\mathrm{t})} \quad$ exists, hence, by the above argument, it equals

$$
\lim _{t} \frac{\left\langle\psi^{t} \mid g(t) \psi\right\rangle}{c(t)} \text {, i.e., }
$$

$\mathrm{h}(\mathrm{s}) \psi_{\infty}=\psi_{\infty}$, since $\mathrm{h}(\mathrm{s}) \psi_{\infty}-\psi_{\infty}$ is perpendicular to all of H .
This simple argument enables us to say that $\psi_{\infty}$ is left fixed by a whole subgroup of $G$ determined by $X$. Let ${\underset{\sim}{N}}^{+}(X)$ be the subalgebra of $G$ spanned by the eigenvector of $A d X$ for positive eigenvalues. Let $N^{+}(X)$ be the connected subgroup of G. It is nilpotent, hence every element is a product of exponentials of the AdX-eigenvector generators of ${\underset{\sim}{N}}^{+}(\mathrm{X})$, hence, using Theorem 11.3, we have:

$$
\mathrm{N}^{+}(\mathrm{X}) \cdot \psi_{\infty}=\psi_{\infty}
$$

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