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INELASTIC UNITARITY IN THE BETHE-SALPETER EQUATION AND THE SELF-ENERGY PROBLEM*

by

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ABSTRACT

The Bethe-Salpeter equation in the ladder approximation for spinless particles is treated for $2\mu > KE > \mu$ where an explicit expression is derived for $\frac{4\pi}{k}$ Im f(0). In order to achieve inelastic unitarity and maintain a precise correspondence to Feynman graphs, a nonlinear propagator equation is formulated which approximates Dyson's equation. It is shown that this equation can be solved for any coupling strength and that the resulting propagators exhibit the structure of a Lehmann spectral representation without ghost poles.

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I. INTRODUCTION

The Bethe-Salpeter equation offers a simplified mathematical treatment of two-body interactions within the framework of quantum field theory.[1] However, formidable difficulties in solving the equation result from the fact that its interaction kernel is an infinite sum over irreducible kernels and, similarly, the Green's function, which describes the travel of the two particles between interactions, contains all the self-energy graphs of each particle separately. A standard approach to these infinite sums is to "approximate" them by their first terms. One then writes an integral equation in which the interaction kernel, denoted by V below, is represented by the exchange of one bare quantum of mass μ, and in which the Green's function, denoted by 1/D, is the product of two bare propagators. We will refer to this as the "bare ladder-exchange equation" (see Eq. (2.1) below).

Recent work has shown that numerical values can be obtained for the elastic scattering amplitudes given by this simplified equation in the center-of-mass kinetic energy range,[2] $KE < \mu$, and even in the range of one quantum production,[3] $\mu < KE < 2\mu$. In this limited inelastic range a new difficulty has been treated by M. Levine, J. Wright and J. Tjon: the scattering amplitude given by the bare ladder-exchange equation does not satisfy unitarity. By "unitarity" we mean specifically the relationship of the optical theorem:

$$\frac{4\pi}{k} \operatorname{Im} f(0) = \sigma_{total} = \sigma_{elastic} + \sigma_{inelastic} \tag{1.1}$$

All these quantities are "predicted" separately by the equation.

In this paper we shall be concerned with the mathematical problem of unitarity in the bare ladder-exchange equation and several more complete ladder-exchange equations. The essential difference between our treatment and

the treatment of Levine, Wright, and Tjon is that we shall not use a perturbation analysis to verify unitarity. As a result, we can treat some consequences of self-energy which are not so easily deduced from a perturbation expansion alone. But before entering into a detailed analysis of these matters, it will be helpful to discuss in diagrammatic terms now unitarity relates to the various simplified forms of the Bethe-Salpeter equation which we shall consider.

The bare ladder-exchange equation corresponds to summing all bare ladder-diagram Feynman graphs for the two-body amplitude $\psi(x_1,x_2)$ which describes the scattering of incident particles "1" and "2" (see Fig. 1a). These graphs are "bare" in the sense that the lines of particles 1 and 2 are represented by bare propagators.

Since $\psi(x_1,x_2)$ is only a function of two points, it cannot directly describe final states in which a quantum of mass μ is produced. However, within the context of the Bethe-Salpeter equation, it is natural to take the view that such radiated quanta can be absorbed by one of the scattering particles at some point in the very distant future, combining with that real particle to form a virtual particle (see Fig. 1b). Thus, for KE $> \mu$, we can have an inelastic production process <u>inside</u> a ladder diagram and such processes will effect Im f(0).

The topology of ladder diagrams automatically limits the sort of inelastic graphs which can be contained within them. For example, the one quantum production graph in Fig. 1c cannot occur inside a ladder diagram since, if the radiated quantum in Fig. 1c is absorbed by either scattering particle, a graph results which is not of the ladder-diagram topology. Only one-quantum production graphs of the sort shown in Fig. 1b (with radiation by either particle 1 or 2) can occur inside ladder diagrams. But it is clear that the free quantum in Fig. 1b can also be absorbed by the same particle which radiated it. If so absorbed, the

quantum forms a self-energy graph of particle 1. Since this possibility is not included within the bare ladder diagrams, it is natural to expect that $\frac{4\pi}{k}\mathrm{Im}f(0)$, calculated from bare ladder diagrams, will not fully represent the inelastic processes which can occur within these graphs.

One has here an intuitive criterion for judging whether a given set of elastic and inelastic scattering graphs will satisfy a unitarity relationship in a given energy range: the inelastic production graphs must be just those which can occur, subject to energy restrictions, if quanta absorbed last in any graph escape, and all graphs must be included which can occur when quanta produced in any of the inelastic graphs are re-absorbed. The system of bare ladder diagrams and those one-quantum production graphs (Fig. 1b) which occur within them for KE $> \mu$ does not meet this criterion. However, for KE $> 2\mu$, there is the important simplification that no more than one exchanged quantum can escape. Thus, for $2\mu >$ KE $> \mu$, a simple remedy for the failing of bare ladder diagrams is suggested: one should include for each scattering particle (1 or 2) a bubble self-energy graph (see Fig. 2).

To incorporate such self-energy graphs into the bare ladder-exchange equation one merely replaces the bare propagator of each scattering particle by the sum of that bare propagator and whatever self-energy graphs one chooses. The Bethe-Salpeter Green's function then becomes the product of two (partially) dressed propagators. (Note that we retain the ladder exchange interaction kernel.) But by including a bubble self-energy graph for each particle in the Green's function, we are including interaction graphs such as the graph in Fig. 3a. Now if the last exchanged quantum in Fig. 3a is radiated by particle 1 (Fig. 3b), we should also include the graph in which this quantum is absorbed again by particle 1, as shown in Fig. 3c. In this way, we are forced to include all finite iterations of bubble graphs.

But even when these iterated bubble propagators are used in the ladder-exchange equation, an obstacle to unitarity in the range of one quantum production remains: If now a quantum of mass μ escapes from a dressed ladder diagram graph and is finally absorbed at infinity by the particle from which it was radiated, the graph shown in Fig. 4b can be formed. In Fig. 4b one finds a self-energy graph of particle 1 that is not an iterated bubble graph and is not included in the augmented equation we just suggested. Therefore, one is required to include <u>not only</u> iterated self-energy bubbles, but also all possible finite arrangements of iterated bubbles within iterated bubbles, within iterated bubbles, ..., etc. We will refer to these as "nested bubble graphs." They are characterized by the property that in such self-energy graphs any two bubbles are either disjoint or one contains the other (see Fig. 4c).

If all these nested graphs are included in the propagators for the ladder-exchange equation, an appealing property results: re-absorption, such as that shown in Figs. 3 and 4, can no longer produce a graph which is not already included. The system of graphs consisting of all the resulting dressed ladder diagrams and all the one-quantum production graphs contained within them (graphs in which a quantum absorbed last by either particle "escapes") satisfies our criterion for unitarity when KE $< 2\mu$.

It should be emphasized that one only expects unitarity to be satisfied in a restrictive sense. Inclusion of production graphs, such as Fig. 1c in $\sigma_{\text{inelastic}}$, would require crossed diagrams in the interaction kernel, over-lapping self-energy bubbles, and vertex corrections, in order to achieve a closed system. In fact, for KE $> 2\mu$, our scheme for achieving a unitary system leads to the inclusion of all Feynman graphs for two-body scattering; no subset will suffice. It is possible for KE $> 2\mu$ that the last <u>two</u> absorbed quanta of an elastic diagram

be radiated. Then, in order to close the system, one must allow all possible absorptions of the two radiated quanta (in any order) by either scattered particle. The net effect of including such possibilities is that for <u>any</u> two quanta absorbed successively by one of the scattering particles in any elastic graph, it is necessary to include the corresponding graph in which those quanta are absorbed successively in the reverse order. But by such interchanges of order, all elastic graphs for the scattering of particles 1 and 2 can be formed from ladder graphs.

The indication of our graphical analysis is that unitarity requires one to consider progressively more complex approximations to the full Bethe-Salpeter equation. However, it appears that in the range $KE < 2\mu$, we can specify partial forms of the equation which do satisfy a restricted form of unitarity. The main part of this paper will consist of giving more rigorous form to these arguments as they apply to the ladder-exchange equation.

It should be commented in advance that the necessity of including all nested bubble self-energy graphs to treat this case is not a handicap but an advantage. These nested graphs can be handled by established techniques of propagator renormalization. It is possible to sum them by a non-linear equation which approximates Dyson's equation and can be solved precisely.

II. THE UNITARITY RELATIONSHIP IN THE LADDER EXCHANGE EQUATION

In this section we shall derive expressions for $\frac{4\pi}{k}$ Im f(0) given by the various ladder-exchange equations described above.

First consider the bare ladder exchange equation for two spin zero particles, "1" and "2", of equal mass $m_1 = m_2 = m$, scattering via the exchange of a spinless quantum of mass μ:

$$\varphi(q) = \varphi_o(q) - i/D(q) \int V(q - q')\, \varphi(q')\, d^4q' \tag{2.1}$$

where:

$q = (\underset{\sim}{q}, q_o)$ is the relative c.m. 4-momentum

$1/D(q) = \Delta_1(q_1) \cdot \Delta_2(q_2)$

$q_1 = (\underset{\sim}{q}, E/2 + q_o)$; $q_2 = (-q, E/2 - q_o)$

E is the total c.m. energy. The propogators are bare,

$$\Delta_i(q_i) = 1/(q_i^2 - m_i^2) \ ,$$

$$\varphi_o(q) = \delta^4(q - q_{in}) = \delta^3(\underset{\sim}{q} - \underset{\sim}{q}_{in})\, \delta(q_o) \ ,$$

and the ladder-exchange kernel is

$$V(q - q') = \frac{\beta}{(\underset{\sim}{q} - \underset{\sim}{q}')^2 + \mu^2 - (q_o - q_o')^2}$$

Conventionally, $\beta = \lambda/\pi^2$. In order to maintain a closer correspondence to the Feynman-diagram formalism, we will set $\beta = e_1 \cdot e_2/(2\pi)^4$.

One can deform the q_o and q_o' contours in Eq. (2.1) from the real axis to the contours $C_{\underset{\sim}{q}^2}$ and $C_{\underset{\sim}{q}'^2}$ which depend on $\underset{\sim}{q}^2$ and $\underset{\sim}{q}'^2$, respectively.[4] For $\underset{\sim}{q}^2 > k^2 = (E/2)^2 - in^2$, $C_{\underset{\sim}{q}^2}$ is just the imaginary axis. For $\underset{\sim}{q}^2 < k^2$, $C_{\underset{\sim}{q}^2}$ detours around the poles in $1/D(q)$. (See Fig. 5.)

In order to more clearly identify real and imaginary quantities, we will often use rotated coordinates $p = (\underset{\sim}{p}, p_o) = (\underset{\sim}{q}, q_o/i)$ and consider the

equation for $\psi(p) = i\,\varphi(q)$

$$\psi(p) = \psi_o(p) + 1/D(p) \iint_{C'^2_{\underset{\sim}{p}}} V(p - p')\,\psi(p') \tag{2.2}$$

with

$$\psi_o(p) = \delta^3(\underset{\sim}{p} - \underset{\sim}{q}_{in})\,\delta(p_o).$$

Symbolically,

$$\psi = \psi_o + 1/D\, V\psi \tag{2.3}$$

where

$$V\psi\ (p) \equiv \iint_{C_{\underset{\sim}{p}'^2}} V(p - p')\,\psi\,(p') \equiv T(p) \tag{2.4}$$

To construct an expression for $\frac{4\pi}{k}\,\mathrm{Im}\,f(0)$ we will need an analytic function of p_o which yields the complex conjugate of T. For this purpose we define

$$T^\dagger(p) = T(\underset{\sim}{q}, q_o^*)^* \tag{2.5}$$

so that

$$T^\dagger(p) = \left(T(\underset{\sim}{p}, -p_o^*)\right)^*$$

$$= \left(\iint_{C_{\underset{\sim}{p}'^2}} V(\underset{\sim}{p} - \underset{\sim}{p}', -p_o^* - p_o')\,\psi(\underset{\sim}{p}', p_o')\,dp_o'\,d^3\underset{\sim}{p}'\right)^*$$

setting $p_o'' = -p_o'^* \equiv p_o'^\dagger$

$$T^\dagger(p) = \iint_{C^\dagger_{\underset{\sim}{p}'^2}} \psi^*(\underset{\sim}{p}'', -p_o''^*)\,V^*(\underset{\sim}{p} - \underset{\sim}{p}'', p_o''^* - p_o^*) \tag{2.6}$$

It is a great simplification to be able to say that $C^\dagger_{\underset{\sim}{p}^2}$ is the same path as $C_{\underset{\sim}{p}^2}$, but followed in the opposite direction, This is clearly the case for $\underset{\sim}{p}^2 > k^2$, since the path $C_{\underset{\sim}{p}^2}$ then has no detours. To arrange the same relation for $\underset{\sim}{p}^2 < k^2$ one can put the poles in $1/D(\underset{\sim}{p}, p_o)$ precisely on the imaginary p_o axis as shown in Fig. 5. (This construction will not be entirely adequate for $KE > 2\mu$ when cuts in T cross the real p_o axis. We therefore are assuming $KE \lesssim 2\mu$.)

Then $1/D$ is real on the imaginary p_o axis away from its poles and away from $\underset{\sim}{p}^2 = k^2$. One handles the special case $\underset{\sim}{p}^2 = k^2$ via the prescription

$$1/D\,(\underset{\sim}{p}\,,\,p_o) = \frac{1}{\underset{\sim}{p}^2 - k^2}\; d\,(\underset{\sim}{p}^2\,,\,p_o)$$

$$= P\,\frac{1}{\underset{\sim}{p}^2 - k^2}\; d\,(\underset{\sim}{p}^2\,,\,p_o) + i\pi\delta\,(\underset{\sim}{p}^2 - k^2)\,d\,(k^2,\,p_o)$$

$$= P\,\frac{1}{\underset{\sim}{p}^2 - k^2}\; d\,(\underset{\sim}{p}^2,\,p_o) + \frac{i\pi^2}{E}\,\delta\,(\underset{\sim}{p}^2 - k^2)\,\delta(p_o) \qquad (2.7)$$

All further singularities involving V can be located by the prescription $\mu \to \mu - i\epsilon$. Returning to Eq. (2.6) we now have

$$T^\dagger(p) = \iint_{C_{\underset{\sim}{p}''^2}} \psi^*(\underset{\sim}{p}'',\,-p_o''^*)\;\tilde{V}\,(p'' - p)$$

$$\equiv [\psi^\dagger\,\tilde{V}]\,(p) \qquad (2.8)$$

where

$$\tilde{V}\,(p'' - p) \equiv V^*(\underset{\sim}{p} - \underset{\sim}{p}'',\,p_o''^* - p_o^*)$$

Note that V is even, $V(q) = V(-q)$, so

$$\tilde{V}(p' - p) = V^*\left(\underset{\sim}{p}' - \underset{\sim}{p}\,,\;-(p_o' - p_o)^*\right)$$

$$\equiv V^\dagger(p' - p) \qquad (2.9)$$

Formally multiplying Eq. (2.3) on the left by $[\psi^\dagger\tilde{V}]\,(p)$ and integrating p over $\underset{\sim}{p}$ and p_o on $C_{\underset{\sim}{p}^2}$,

$$\psi^\dagger\,\tilde{V}\,\psi = \psi^\dagger\,\tilde{V}\psi_o + \psi^\dagger\,\tilde{V}\;\frac{1}{D}\;V\psi \qquad (2.10)$$

or,

$$-T^\dagger\psi_o = T^\dagger\,\frac{1}{D}\,T - \psi^\dagger\,\tilde{V}\psi \qquad (2.11)$$

But T evaluated at $\underset{\sim}{p}^2 = k^2$, $p_o = 0$ yields a simple multiple of the elastic scattering amplitude, f,

$$f(\Omega_{out}) = \frac{2\pi^3}{E} T(\underset{\sim}{p}_{out}, 0) \quad ,$$

so that

$$T^\dagger \psi_o = T^*(\underset{\sim}{q}_{in}, 0) = E/2\pi^3 f^*(0) \tag{2.12}$$

Then, taking the complex conjugate of Eq. (2.11) and subtracting,

$$\frac{E}{\pi^3} i \operatorname{Im} f(0) = \left[(T^\dagger \frac{1}{D} T) - (T^\dagger \frac{1}{D} T)^*\right] + \left[(\psi^\dagger \tilde{V} \psi)^* - (\psi^\dagger V^\dagger \psi)\right] \tag{2.13}$$

Analyzing the terms in Eq. (2.13), one finds

$$\begin{aligned}(T^\dagger \frac{1}{D} T)^* &= \iint_{C_{\underset{\sim}{p}^2}} T^*(\underset{\sim}{p}, p_o^*) \frac{1}{D^*(\underset{\sim}{p}, -p_o^*)} T(\underset{\sim}{p}, p_o^*) \\ &\equiv T^\dagger \frac{1}{D^\dagger} T \end{aligned} \tag{2.14}$$

and

$$\begin{aligned}(\psi^\dagger \tilde{V} \psi)^* &= \iint_{C_{\underset{\sim}{p}'^2}} \iint_{C_{\underset{\sim}{p}^2}} \psi(\underset{\sim}{p}', -p_o'^*) V(\underset{\sim}{p}' - \underset{\sim}{p}, p_o' - p_o) \psi(\underset{\sim}{p}, p_o) \\ &\equiv \psi^\dagger V \psi \end{aligned} \tag{2.15}$$

Thus from Eq. (2.13) we have the simple formula

$$\frac{E}{\pi^3} i \operatorname{Im} f(0) = T^\dagger \left(\frac{1}{D} - \frac{1}{D^\dagger}\right) T + \psi^\dagger (V - \tilde{V}) \psi \tag{2.16}$$

or, using $\tilde{V} = V^\dagger$,

$$\frac{E}{\pi^3} i \operatorname{Im} f(0) = T^\dagger \left(\frac{1}{D} - \frac{1}{D^\dagger}\right) T + \psi^\dagger (V - V^\dagger) \psi \tag{2.17}$$

Now, since $V(p' - p)$ is real for all imaginary $p_o' - p_o$ away from its poles, the term $V - V^\dagger$ vanishes except near poles in $V(p' - p)$. Specifically, for elastic scattering, $E < 2m + \mu$, $V - V^\dagger$ vanishes for all p_o and p_o' on $C_{\underset{\sim}{p}^2}$ and $C_{\underset{\sim}{p}'^2}$, respectively. Thus, for elastic energies, $KE < \mu$,

Eqs. (2.17) and (2.7) give

$$\frac{E}{\pi^3}\, i\, \mathrm{Im}\, f(0) = \frac{2 i \pi^2}{E} \iint T^{\dagger}(p) \left[\delta(\underset{\sim}{p}^2 - k^2)\, \delta(p_o) \right] T(p)$$

$$= \frac{2 i \pi^2}{E} \left(\frac{E}{2\pi^3} \right)^2 \frac{k}{2}\, \sigma_{el}$$

where (2.18)

$$\sigma_{el} = \int \left| f(\Omega_{\underset{\sim}{p}}) \right|^2_{|\underset{\sim}{p}|=k} d\Omega_{\underset{\sim}{p}} .$$

So,

$$\frac{4\pi}{k}\, \mathrm{Im}\, f(0) = \sigma_{el} ,$$

verifying elastic unitarity.

For inelastic energies, $E > 2m + \mu$, there will be a new contribution to Eq. (2.17) from $(V - V^{\dagger})$, but, if we retain the bare Green's function, no new contribution from $\left(\frac{1}{D} - \frac{1}{D^{\dagger}} \right)$.

Now, for p'_o, p_o imaginary

$$V(p' - p) - V^{\dagger}(p' - p) = \frac{\beta}{(p'_o - p_o)^2 + (\underset{\sim}{p}' - \underset{\sim}{p})^2 + \mu^2 - i\epsilon} - \frac{\beta}{(p'_o - p_o)^2 + (\underset{\sim}{p}' - \underset{\sim}{p})^2 + \mu^2 + i\epsilon}$$

$$= 2\pi i\, \beta\, \delta\left[(p'_o - p)^2 + (\underset{\sim}{p} - \underset{\sim}{p}')^2 + \mu^2 \right] \qquad (2.19)$$

The occurrence of non-zero contributions from this term for p_o, p'_o on $C_{\underset{\sim}{p}2}$, $C'_{\underset{\sim}{p}2}$ (see Fig. 6) is restricted to opposite imaginary loops of $C_{\underset{\sim}{p}'2}$, $C_{\underset{\sim}{p}2}$.

For $KE < 2\mu$, there are only poles of $\psi^{\dagger}(p')$, $\psi(p)$ contained in these loops (branch cuts do not enter until $KE > 2\mu$). These poles are associated with zeros in $D^*(\underset{\sim}{p}', -p'^*_o)$ at $\pm\left(E/2 - (\underset{\sim}{p}'^2 + m^2)^{1/2} \right)$, and zeros in $D(\underset{\sim}{p}, p_o)$ at $\mp\left(E/2 - (\underset{\sim}{p}^2 + m^2)^{1/2} \right)$, respectively. Thus the new contribution to Eq. (2.17) can be found for $KE < 2\mu$ simply by taking the residues in $\iiiint \psi^{\dagger}(p')\,(V - V^{\dagger})\,\psi(p)$ at these poles. The residues in the p_o integral

come from $\left(\frac{1}{D}\,T\right)$ in $\psi = \psi_o + \frac{1}{D}\,T$,

where

$$\frac{1}{D(\underset{\sim}{p}\,,\,p_o)} = \frac{-i}{2(\underset{\sim}{p}^2+m_1^2)^{1/2}}\;\frac{1}{\left(p_o - i\left[E/2 - (\underset{\sim}{p}^2+m_1^2)^{1/2}\right]\right)}\;\frac{1}{\left[\left(E-(\underset{\sim}{p}^2+m_1^2)^{1/2}\right)^2-(-\underset{\sim}{p})^2-m_2^2\right]}$$

$$+\;\frac{i}{2\left((-\underset{\sim}{p})^2+m_2^2\right)^{1/2}}\;\frac{1}{\left(p_o + i\left[E/2 - \left((-\underset{\sim}{p})^2+m_1^2\right)^{1/2}\right]\right)}\;\frac{1}{\left[\left(E-\left((-\underset{\sim}{p})^2+m_2^2\right)^{1/2}\right)^2-p^2-m_1^2\right]}$$

$+$ terms with poles not contained inside the loops of $C_{\underset{\sim}{p}^2}$. (2.20)

(Although $m_1 = m_2 = m$ the terms here are associated with one particle or the other, so it is helpful to note the possible distinction in masses.) The poles in $\psi^\dagger(p')$ from $(1/D^\dagger\;T^\dagger)$ can be similarly analyzed. Remembering that only opposite poles enter, we then get

$$\psi^\dagger(V-V^\dagger)\,\psi = \pi^3 i\,\frac{e_1\cdot e_2}{(2\pi)^4}\int\frac{d^3\underset{\sim}{p}}{(\underset{\sim}{p}^2+m_1^2)^{1/2}}\int\frac{d^3\underset{\sim}{p}}{\left((-\underset{\sim}{p}')^2+m_2^2\right)^{1/2}}$$

$$\left(\left\{\frac{T\left(\underset{\sim}{p}\,,\,i\left[E/2-(\underset{\sim}{p}^2+m_1^2)^{1/2}\right]\right)}{\left[\left(E-(\underset{\sim}{p}^2+m_1^2)^{1/2}\right)^2-(-\underset{\sim}{p})^2-m_2^2\right]}\;\frac{T^*\left(\underset{\sim}{p}',-i\left[E/2-\left((-\underset{\sim}{p}')^2+m_2^2\right)^{1/2}\right]\right)}{\left[\left(E-\left((-\underset{\sim}{p}')^2+m_2^2\right)^{1/2}\right)^2-\underset{\sim}{p}'^2-m_1^2\right]}\right.\right.$$

$+$, (after switching $\underset{\sim}{p} \longleftrightarrow \underset{\sim}{p}'$),

$$\left.\left.\frac{T^*\left(\underset{\sim}{p}\,,\,i\left[E/2-(\underset{\sim}{p}^2+m_1^2)^{1/2}\right]\right)}{\left[\left(E-(\underset{\sim}{p}^2+m_1^2)^{1/2}\right)^2-(-\underset{\sim}{p})^2-m_2^2\right]}\;\frac{T\left(\underset{\sim}{p}'\,,\,-i\left[E/2-\left((-\underset{\sim}{p}')^2+m_2^2\right)^{1/2}\right]\right)}{\left[\left(E-\left((-\underset{\sim}{p}')^2+m_2^2\right)^{1/2}\right)^2-\underset{\sim}{p}'^2-m_1^2\right]}\right\}\right.$$

$$\left.\frac{\delta\left((\underset{\sim}{p}^2+m_1^2)^{1/2}+\left((-\underset{\sim}{p}')^2+m_2^2\right)^{1/2}+\left((\underset{\sim}{p}-\underset{\sim}{p}')^2+\mu^2\right)^{1/2}-E\right)}{\left((\underset{\sim}{p}-\underset{\sim}{p}')^2+\mu^2\right)^{1/2}}\right) \qquad (2.21)$$

Calling

$$\left(q_1\right)_f = E/2 + q \;==\; (\underset{\sim}{p}\,,\; E/2 + ip_o) = \left(\underset{\sim}{p}\,,\; (\underset{\sim}{p}^2 + m_1^2)^{1/2}\right)$$

$$\left(q_2\right)_f = E/2 - q' = (-\underset{\sim}{p}'\,,\; E/2 - ip_o') = \left(-\underset{\sim}{p}'\,,\; \left((-\underset{\sim}{p}')^2 + m_2^2\right)^{1/2}\right)$$

$$\left(q_3\right)_f = \left(\underset{\sim}{p}' - \underset{\sim}{p}\,,\; \left((\underset{\sim}{p}' - \underset{\sim}{p})^2 + \mu^2\right)^{1/2}\right)$$

we note that the terms in Eq. (2.21) correspond to terms occurring in two sums of Feynman diagrams for inelastic production of one quantum of mass μ (see Fig. 6). In fact, these are the very production graphs which were mentioned in Section I; they represent a sum over simple ladder diagrams in which the last quantum absorbed by particle 1 (Fig. 7a) or by particle 2 (Fig. 7b) "escapes." (The sum over all ladder exchanges preceding radiation is contained in the function T.) Thus some elements of the restricted inelastic cross section discussed in Section I do enter our expression for $\frac{4\pi}{k}$ Im f(0) when $2\mu > KE > \mu$. However, although the interference between Figs. 7a and 7b appears in Eq. (2.21), the square of each figure alone is missing. Thus unitarity is not satisfied.[5]

One now can consider how the expression obtained above for $\frac{4\pi}{k}$ Im f(0) will change if dressed propagators are substituted for bare propagators in the ladder diagram equation. First consider propagators which only include iterated bubble graphs:

$$\Delta_{(i)}(q_i^2) \longrightarrow \frac{1}{q_i^2 - m_o^2 - \Sigma_{(i)}(q_i^2)}$$

where

$$\Sigma_{(i)}(q_i^2) = \frac{ie_i^2}{(2\pi)^4} \int \left(\frac{1}{\left[q_i'^2 - m_{o(i)}^2\right]} \; \frac{d^4 q_i'}{\left[(q_i - q_i')^2 - \mu^2\right]} \right) \tag{2.22}$$

(The subscript $i = 1, 2$ refers to one of the scattering particles. Subscripts in parenthesis are only necessary if $e_1^2 \neq e_2^2$ and will often be deleted below.) Deforming the q_{io}' contour to the imaginary axis (see Fig. 8), one can impose a cut-off Λ on $|q_i'|$ in Eq. (2.22) in order to avoid the ultraviolet divergence

in Σ. Then $\Delta(q_i^2)$ will have a pole associated with the "dressed mass" at $q_i^2 = m^2$ where $m^2 = m_o^2 + \Sigma(m^2)$, (We will use $m = m_1 = m_2$ to denote this dressed mass and $m_{o(i)}$ to denote the bare masses which need not be the same if $e_i^2 \neq e_2^2$.) The self-energy $\Sigma(m^2)$ is negative for our spinless case, but, at least if the couplings e_i^2 are weak, m^2 will be positive, as we shall assume. Further analysis of the structure of $\Sigma(q_i^2)$ shows it to be an analytic function of q_{io} except for two branch cuts starting at $\pm q_i^2 + (m_o + \mu)^2\ ^{1/2} \mp i\epsilon$ and extending along the real q_{io} axis to $\pm\infty \mp i\epsilon$. On the real axis, between these cuts, Σ is <u>real.</u> However, along its cuts Σ has a (noninfinitesimal) imaginary part.

We now propose a simple mechanism for getting new contributions to Eq.(2.17). Since $\Sigma(q_i^2)$ has an imaginary part along its cuts, there will be new contributions to $1/D - 1/D^\dagger$ whenever points along these cuts enter the transformed equation. But so long as these cuts are not encountered along $C_{\underset{\sim}{p}2}$ for any $\underset{\sim}{p}$, our expression for $\frac{4\pi}{k}$ Im f(0) will remain essentially the same as for the bare case. The largest detour in $C_{\underset{\sim}{p}2}$ (see Fig. 9) has extension $E/2 - m$, while the lowest excursion of the upper cut in $\Sigma(q_i^2)$ is $m_o + \mu - E/2$. Thus, in order to "see" this cut in the transformed equation we must have $E > m + m_o + \mu$. However, we want to see the cuts for $E > 2m + \mu$ where the preceding analysis indicates the need for a new contribution to Eq. (2.17). One would require then that m_o be equal to m. But this equality is impossible since the self-energy, $\Sigma(m^2)$, will not vanish. As explained in Section I, iterated bubble graphs alone are not expected to produce unitarity. Already the fundamental failing of these graphs is clear. We could take the view that e_i^2 is small so that m_o is nearly equal to m. A more precise way of proceeding will be to adopt the approach of first order renormalization, treat m as given, and set $m_o = m$ in calculating $\Sigma(q_i^2)$, but, in forming $\Delta = 1/\left(q_1^2 - m_o^2 - \Sigma(q_i^2)\right)$, readjust m_o^2 so that the poles in Δ do lie at $q_i^2 = m^2$.

It should be noted, however, that the propagator so formed is not strictly equal to the sum of iterated bubble graphs. Any finite sum of iterated bubble graphs has a pole at $q_i^2 = m_o^2$ with residue 1.

Consider how the previous analysis is affected by this change in propagators. At its poles $\Delta(q_i^2)$ no longer has a residue of 1, but rather looks like

$$\frac{z_i}{q_i^2 - m^2}$$

where (2.23)

$$\frac{1}{z_i} = 1 - \left[\frac{d}{dq_i^2}\left(\Sigma(q_i^2)\right)\right]_{q_i^2 = m^2} .$$

Then the elastic scattering amplitude given by the asymptotic form of $\psi(\vec{x}, 0)$[6] is

$$f(\Omega_{out}) = z_1 z_2 \left(\frac{2\pi^3}{E}\right) T(\underset{\sim}{p}_{out}, p)$$

and the imaginary part of $\frac{1}{D}$ is given by (2.24)

$$\frac{1}{D} - \frac{1}{D^\dagger} = z_1 z_2 \left(\frac{2\pi^2 i}{E}\right) \delta(\underset{\sim}{p}^2 - k^2)\, \delta(p_o) + \text{terms not seen for } E < 2m + \mu .$$

If one uses these expressions to recalculate $\frac{4\pi}{k}$ Im f(0) for $E < 2m + \mu$, one finds that the new factors z_i precisely cancel so that elastic unitarity is again verified.

Turning to the range of one quantum production, $2m + \mu < E \quad 2m + 2\mu$, we first reinvestigate the contribution of $\psi^\dagger [V - V^\dagger] \psi$ to Eq. (2.17). Since in the calculation of this contribution one collects opposite residues in $\psi^\dagger(p')$ and $\psi(p)$, the final answer, Eq. (2.21), is now multiplied by the factor $z_1 \cdot z_2$. Also the off mass shell propagators in Eq. (2.21) associated with residues in $\psi^\dagger$, i.e., terms appearing with T^*, are replaced by their complex conjugates, so that the interpretation of Eq. (2.21) as the interference of Figs. 7a and 7b remains valid.

Now consider the new contribution to

$$T^\dagger\left(\frac{1}{D} - \frac{1}{D^\dagger}\right) T = \iint T^\dagger(p) \left[\frac{1}{D(p)} - \frac{1}{D^\dagger(p)}\right] T(p) .$$

For $E < 2m + 2\mu$ the cuts in $1/D$ only touch the detour loops of C_{p2} but do not cross the real p_o axis(see Fig. 9 with $m_o = m$ in Σ). Then for $E < 2m + 2\mu$ the only singularities of $1/D - 1/D^\dagger$ inside these loops lie at $\pm\ E/2 - (p^2 + m^2)^{1/2}$. Also for $E < 2m + 2\mu$ no singularity in T or $T^\dagger$ will enter the loops. Thus all we need consider for $E < 2m + 2\mu$ are the residues at the poles in $1/D$. Near the poles of interest we now have

$$\frac{1}{D(p\ ,\ p_o)} - \frac{1}{D.(p,\ -p_o^*)} = \frac{-i}{2(p^2 + m^2)^{1/2}} \frac{z_1}{\left(p_o - i\left[E/2 - (p^2 + m^2)^{1/2}\right]\right)}$$

$$\cdot \left[\frac{M_2^{2*} - M_2^2}{(E/2 + i p_o)^2 - (-p)^2 - M_2^2}\right]_{p_o = i\left[E/2 - (p^2 + m^2)^{1/2}\right]}$$

$$+ \frac{i}{2\left((-p)^2 + m^2\right)^{1/2}} \frac{z_2}{\left(p_o + i\left[E/2 - \left((-p)^2 + m^2\right)^{1/2}\right]\right)}$$

$$\cdot \left[\frac{M_1^{2*} - M_1^2}{(E/2 - i p_o)^2 - p^2 - M_1^2}\right]_{p_o = -i\left[E/2 - \left((-p)^2 + m^2\right)^{1/2}\right]} \tag{2.25}$$

where

$$M_i^2 = m^2 + \Sigma(q_i^2) - \Sigma(m^2) \tag{2.26}$$

Setting $m_o = m$ in Σ, one finds for real q_{io}

$$\Sigma(q_i,\ q_{io}) - \Sigma^*(q_i,\ q_{io}) = \frac{-\pi^2 e_i^2 i}{(2\pi)^4} \int \frac{d^3 q_i'}{(q_i'^2 + m^2)^{1/2}}$$

$$\frac{\delta\left((q_i'^2 + m^2)^{1/2} + \left((q_i' - q_i)^2 + \mu\right)^{1/2} - |q_{io}|\right)}{\left((q_i' - q_i)^2 + m^2\right)^{1/2}} \tag{2.27}$$

Now using Eqs. (2.25) and (2.27) to collect residues we get the following inelastic contribution to $T^\dagger(1/D - 1/D^\dagger)T$, after relabeling $\underset{\sim}{q}_i$, $\underset{\sim}{q}_i'$, etc.:

$$\pi^3 i \int \frac{d^3\underset{\sim}{p}}{(\underset{\sim}{p}^2 + m^2)^{1/2}} \int \frac{d^3\underset{\sim}{p}}{\left((-\underset{\sim}{p}')^2 + m^2\right)^{1/2}}$$

$$\frac{\delta\left((\underset{\sim}{p}^2 + m^2)^{1/2} + \left((-\underset{\sim}{p}')^2 + m^2\right)^{1/2} + \left((\underset{\sim}{p}' - \underset{\sim}{p})^2 + \mu^2\right)^{1/2} - E\right)}{\left[(\underset{\sim}{p} - \underset{\sim}{p}')^2 + \mu^2\right]^{1/2}}$$

$$\left\{ z_1 \left| \frac{e_2\, T\left(\underset{\sim}{p},\, i\left[E/2 - (\underset{\sim}{p}^2 + m^2)^{1/2}\right]\right)}{\left(E - (\underset{\sim}{p}^2 + m^2)^{1/2}\right)^2 - (-\underset{\sim}{p})^2 - M_2^2\left(q_2 = \left(-\underset{\sim}{p}, E - (\underset{\sim}{p}^2 + m^2)^{1/2}\right)\right)} \right|^2 \right.$$

$$\left. + z_2 \left| \frac{e_1\, T\left(\underset{\sim}{p}',\, -i\left[E/2 - \left((-\underset{\sim}{p}')^2 + m^2\right)^{1/2}\right]\right)}{\left(E - \left((-\underset{\sim}{p}')^2 + m^2\right)^{1/2}\right)^2 - p'^2 - M_1^2\left(q_1 = \left(\underset{\sim}{p}', E - \left((-\underset{\sim}{p}')^2 + m^2\right)^{1/2}\right)\right)} \right|^2 \right\} \tag{2.28}$$

We have here very nearly the term needed to complete the square of the sum of Figs. 7a and 7b. However, a fault remains: the factors z_1 and z_2 appear in (2.28) separately, whereas $\psi^\dagger\left[V - V^\dagger\right]\psi$ from Eq. (2.21) now contains the factor $z_1 \cdot z_2$. But this difficulty disappears if the propagator used in calculating $\Sigma(q_i^2)$ is not the bare propagator, $1/\left(q_i'^2 - m_o^2\right)$, but rather the dressed propagator $\Delta(q_i'^2)$ itself. To achieve this effect we propose use of a propagator $\Delta(q_i^2)$ which satisfies the non-linear equation

$$\Delta\left(q_i^2\right) = \frac{1}{q_i^2 - m_o^2 - \Sigma_\Delta\left(q_i^2\right)}$$

where (2.29)

$$\Sigma_\Delta(q_i^2) = \frac{i e_i^2}{(2\pi)^4} \int \frac{d^4 q_i'\, \Delta(q_i'^2)}{\left[(q_i - q_i')^2 - \mu^2\right]}$$

Assuming this propagator has the same poles and cuts found for first order renormalization, the above analysis is easily adapted to treat it. The mass entering $\Sigma_\Delta(q_i^2) - \Sigma_\Delta^*(q_i^2)$ is now automatically m, not m_o. Furthermore, in calculating $\Sigma_\Delta(q_i^2) - \Sigma_\Delta^*(q_i^2)$ one picks up a residue from the dressed propagator itself, thus yielding a new factor of z_i so that the product $z_1 \cdot z_2$ now appears in (2.28). It is now easy to combine (2.21) and (2.28) to give the following contribution to (2.17) for $E < 2m + 2\mu$:

$$\left(\frac{E}{\pi^3}\right) z_1^{-1} z_2^{-1} \, \mathrm{I\,m} f(0) = \left(\frac{E}{\pi^3}\right) z_1^{-1} z_2^{-1} \frac{k}{4\pi} \sigma_{el}$$

$$+ \frac{\pi^3}{(2\pi)^4} z_1 \cdot z_2 \int \frac{d^3\underset{\sim}{p}}{(\underset{\sim}{p}^2 + m^2)^{1/2}} \int \frac{d^3\underset{\sim}{p}}{\left((-\underset{\sim}{p}')^2 + m^2\right)^{1/2}}$$

$$\cdot \frac{\delta\left((\underset{\sim}{p}^2 + m^2)^{1/2} + \left((-\underset{\sim}{p}')^2 + m^2\right)^{1/2} + \left((\underset{\sim}{p} - \underset{\sim}{p}')^2 + \mu^2\right)^{1/2} - E\right)}{\left[(\underset{\sim}{p} - \underset{\sim}{p}')^2 + \mu^2\right]^{1/2}}$$

$$\cdot \left| \frac{e_2 T\left(\underset{\sim}{p}, i\left[E/2 - (\underset{\sim}{p}^2 + m^2)^{1/2}\right]\right)}{\left[\left(E - (\underset{\sim}{p}^2 + m^2)^{1/2}\right)^2 - (-\underset{\sim}{p})^2 - M_2^2\left(-\underset{\sim}{p}, E - (\underset{\sim}{p}^2 + m^2)^{1/2}\right)\right]} + \right.$$

$$\left. + \frac{e_1 T\left(-\underset{\sim}{p}', -i\left[E/2 - \left((-\underset{\sim}{p}')^2 + m^2\right)^{1/2}\right]\right)}{\left[\left(E - \left((-\underset{\sim}{p}')^2 + m^2\right)^{1/2}\right)^2 - \underset{\sim}{p}'^2 - M_1^2\left(\underset{\sim}{p}', E - \left((-p')^2 + m^2\right)^{1/2}\right)\right]} \right|^2 \tag{2.30}$$

Multiplying Eq. (2.30) on both sides by $(\pi^3/E)\, z_1 z_2\, (4\pi/k)$ we get $(4\pi/k)\mathrm{Im}\, f(0) = \sigma_{elastic} + \sigma_{inelastic}$, where $\sigma_{inelastic}$ can be shown to be precisely the result given by a (dressed propagator) Feynman diagram calculation of the sum of Figs. 7a and 7b. We have now realized the restricted form of unitarity described in Section I. Furthermore, the nonlinear propagator Eq. (2.29) just includes the "nested bubble" self-energy graphs discussed in Section I. To see

this, one can rewrite Eq. (2.29) as

$$\Delta(q_1) = \frac{1}{q_i^2 - m_o^2} + \frac{1}{q_i^2 - m_o^2} \Sigma_\Delta(q_i)\, \Delta(q_i) \tag{2.31}$$

and expand in a formal Born series. In this sense the propagator formed here, unlike the renormalized iterated bubble propagator formed above, does maintain equality to a sum of simple Feynman graphs. Note that Eq. (2.31) is just a simplified form of Dyson's equation for the fully dressed propagator.

Equation (2.29) has the special advantage that it can be renormalized before being solved. In the manner of renormalization theory we can treat m_o^2 as a nonessential parameter, requiring only that $m_o^2 = m^2 - \Sigma(m^2)$ even if $\Sigma(m^2)$ is infinite in the limit of infinite cut-off. We may then write the renormalized equation

$$\Delta(q_i^2) = \frac{1}{q_i^2 - m^2 - \int K(q_i, q_i')\, \Delta(q'^2)\, d^4 q_i'} \tag{2.32}$$

where

$$K(q_i, q_i') = \left[\frac{1}{(q_i - q_i')^2 - \mu^2} - \frac{1}{(\tilde{m} - q_i)^2 - \mu^2} \right] \tag{2.33}$$

$\tilde{m}$ being any q_i such that $q_i^2 = m^2$, e.g., $m = (\vec{0}, m)$.

III. SOLUTION OF PROPAGATOR EQUATION

Approximate solution

The first order renormalization discussed above suggests a method for solving Eq. (2.29). One may view the propagator formed there as being an approximate solution to Eq. (2.29). In effect, one "guessed" $\Delta \cong \Delta_o = 1/\left(q_i^2 - m^2\right)$, inserted this function on the right of Eq. (2.29) and used the equation to calculate a "better" solution. This new solution was almost good enough to produce a unitary ladder diagram Bethe-Salpeter equation, failing only because of the factors z_i.

Now suppose we make a better guess:[7] $\Delta \cong \tilde{\Delta} = \dfrac{\tilde{z}}{q_i^2 - m^2}$, and use Eq. (2.29) to calculate Δ. We then have a free parameter, $\tilde{z}$, which we can adjust so that $\tilde{\Delta}$ and Δ both have the same residue at $q_i^2 = m^2$. Specifically, we require

$$\frac{1}{1 - \dfrac{1}{\tilde{z}^{-1}} A} = \frac{1}{\tilde{z}^{-1}} \tag{3.1}$$

where $A = \left[\dfrac{d}{dq_i^2}\left(\dfrac{ie_i^2}{(2\pi)^4}\displaystyle\int \frac{\Delta_o(q_i')}{(q_i - q_i')^2 - \mu^2}\right)\right]_{q_i^2 = m^2}$

is finite without cut-off. (A closer inspection shows $A < 0$.) From Eq. (3.1) one has the quadratic equation

$$\left(\tilde{z}^{-1}\right)^2 - \left(\tilde{z}^{-1}\right) + A = 0 \tag{3.2}$$

In the limit $e_i^2 \longrightarrow 0$, we expect $z^{-1} \longrightarrow 1$, so we take

$$\tilde{z}^{-1} = \frac{1 + \sqrt{1 - 4A}}{2} \tag{3.3}$$

But now since Δ has not only the same poles as $\tilde{\Delta}$, but also the same residue at these poles, we will get the required factors z_i in $\Sigma_i - \Sigma_i^*$, and we may use this Δ to form $1/D\,(\underset{\sim}{p}, p_o)$, thereby obtaining a unitary ladder exchange Bethe-Salpeter equation in the range KE $< 2\mu$.[8]

Since $A < 0$, Eq. (3.3) always yields $\tilde{z}$ positive and $\tilde{z} < 1$, a result predicted by field theory for the residue of a physical spin zero propagator.

Precise solution for Δ_i

One may conjecture that if the process of using an approximate Δ on the right of Eq. (2.29) to calculate a "better" Δ is iterated many times, each time using the preceding result to calculate a yet better Δ, the propagators so calculated will converge the proper solution of Eq. (2.29). As a first step toward a formal analysis of this idea, one can use the fact that the propagator is only dependent on q_i^2 to eliminate all variables except $-q_i^2 = s$ from Eq. (2.29). The easiest way to go from the integral over q_i' in Eq. (2.29) to an integral over s' is to rotate the q_{io}' contour to the imaginary axis and take

$$-i d^4 q_i' \longrightarrow \frac{1}{2}\, d\Omega_4'\, s'\, ds' \,.$$

However, as shown in Fig. 8, the ability to make a complete rotation in

$$\int \frac{\Delta(q_i')}{\left(q_i - q_i'\right)^2 - \mu^2}\, dq_{io}'\, d^3 \underset{\sim}{q}_i'$$

depends on what q_i is. Taking $q_i = \left(\vec{0}, \sqrt{-s}\right)$, non-obstructed rotation is possible only if $s > -\mu^2$. But the only reason for considering negative values of s is to renormalize Δ at $s = -m^2$. Thus if $m^2 < \mu^2$, one need never consider values of s for which complete rotation of q_{io}' is impossible. It will be very convenient, therefore, to assume $m < \mu$. (Although the case $m > \mu$ is more complicated, it submits to the same type of analysis.)

With $m < \mu$ we have the simple equation

$$\underset{\sim}{\Delta}(s) = \frac{1}{s + m^2 + \int_0^\infty K(s, s')\underset{\sim}{\Delta}(s')\,ds'} \tag{3.4}$$

for all $s > 0$, where $\underset{\sim}{\Delta}(s) \equiv -\Delta(q_i)$, and

$$K(s, s') = H(s, s') - H(-m^2, s') \tag{3.5}$$

with[9]

$$H(s,s') = \frac{-e_i^2 s'}{2(2\pi)^4}\int \frac{d\Omega_4'}{-(q_i - q_i')^2 + \mu^2} = \frac{-2\pi^2 e_i^2}{(2\pi)^4}\,\frac{s'}{s + s' + \mu^2 + \left(\left(s - s' + \mu^2\right)^2 + 4s'\mu^2\right)^{1/2}}$$

so that $H(s, s') < 0$. (The "self-energy" is negative.)

However,

$$\frac{d}{ds}H(s,s') = \frac{2\pi^2 e_i^2}{(2\pi)^4}\,\frac{1 + \dfrac{s - s' + \mu^2}{\left(\left(s - s' + \mu^2\right)^2 + 4s'\mu^2\right)^{1/2}}}{\left(s + s' + \mu^2 + \left(\left(s - s' + \mu^2\right)^2 + 4s'\mu^2\right)^{1/2}\right)^2}\,s' > 0 \tag{3.6}$$

for $s' > 0$ and any s.

Thus $K(s, s') = H(s, s') - H(-m^2, s') > 0$ for $s > 0$, $s' > 0$, which is the range of s, s' being considered. (Note $\frac{d}{ds}H(s, s') > 0$ is essentially equivalent to the assertion that in Eq. (3.1) A is negative.)

To simplify the discussion further, one can introduce a finite mesh of points s_ν on the interval $0 < s < \Lambda < \infty$ and consider the restriction of Eq. (3.4) to the set s_ν, replacing ds by d_ν and $K(s, s')$ by $K_{\nu\omega} = K(s_\nu, s_\omega)d_\omega$. Defining $U(s) = s + m^2$, $U_\nu = U(s_\nu) > 0$, $\underset{\sim}{\Delta}_\nu = \underset{\sim}{\Delta}(s_\nu)$, Eq. (3.4) becomes

$$\underset{\sim}{\Delta}_\nu = \frac{1}{U_\nu + \sum_\omega K_{\nu\omega}\,\underset{\sim}{\Delta}_\omega} \tag{3.7}$$

For physical reasons (consider the Lehmann representation for $\underset{\sim}{\Delta}$), one is only interested in solutions to Eq. (3.4), (3.7) which are positive, $\underset{\sim}{\Delta}_\nu > 0$ for all ν.

We now show that starting with any choice of $\underset{\sim}{\Delta}^{(1)} > 0$, iteration of Eq. (3.7) converges to a positive solution of Eq. (3.7) and that the resulting solution is the only positive solution of Eq. (3.7).

To show uniqueness, suppose $\underset{\sim}{\Delta}_\nu$ is a positive solution of Eq. (3.7). Since U and K are also positive, Eq. (3.8) indicates $\underset{\sim}{\Delta}_\nu < U_\nu^{-1}$. But then $K_{\nu\omega} \underset{\sim}{\Delta}_\omega < K_{\nu\omega} U_\omega^{-1}$, so that

$$U_\nu^{-1} > \underset{\sim}{\Delta}_\nu > \left(U_\nu + \sum_\omega K_{\nu\omega} U_\omega^{-1}\right)^{-1} > 0 \tag{3.8}$$

It is thus established that $\underset{\sim}{\Delta}$ can be neither too large nor too small. Now suppose we have a second positive solution $\underset{\sim}{\Delta}'$, i.e., both

$$\underset{\sim}{\Delta}_\nu = 1/\left(U_\nu + K_{\nu\omega} \underset{\sim}{\Delta}_\omega\right) \tag{3.9a}$$

and

$$\underset{\sim}{\Delta}'_\nu = 1/\left(U_\nu + K_{\nu\omega} \underset{\sim}{\Delta}'_\omega\right) \tag{3.9b}$$

Subtracting, combining denominators, and using Eq. (3.9a), one finds

$$\underset{\sim}{\Delta}_\nu - \underset{\sim}{\Delta}'_\nu = \underset{\sim}{\Delta}_\nu \left(\frac{\sum_\omega K_{\nu\omega}\left(\underset{\sim}{\Delta}'_\omega - \underset{\sim}{\Delta}_\omega\right)}{U_\nu + \sum_\omega K_{\nu\omega} \underset{\sim}{\Delta}'_\omega}\right) \tag{3.10}$$

Let $C > 0$ be the minimum number such that $|\underset{\sim}{\Delta}_\nu - \underset{\sim}{\Delta}'_\nu| \leq C \underset{\sim}{\Delta}'_\nu$ for all ν, $C = \underset{\nu}{\mathrm{Max}}\,(|\underset{\sim}{\Delta}_\nu - \underset{\sim}{\Delta}'_\nu| / \underset{\sim}{\Delta}'_\nu)$. Since $\underset{\sim}{\Delta}'_\nu$ obeys Eq. (3.8), no $\underset{\sim}{\Delta}'_\nu$ can be zero; therefore, C exists. Furthermore,

$$\begin{aligned} |\underset{\sim}{\Delta}'_\nu - \underset{\sim}{\Delta}_\nu| / \underset{\sim}{\Delta}'_\nu &< 1 + \underset{\sim}{\Delta}_\nu / \underset{\sim}{\Delta}'_\nu \\ &< 1 + U_\nu^{-1} / \left[\left(U_\nu + \sum_\omega K_{\nu\omega} U_\omega^{-1}\right)^{-1}\right] \\ &= 2 + U_\nu^{-1} \sum_\omega K_{\nu\omega} U_\omega^{-1} \end{aligned}$$

and hence one has a bound on C involving only U and K:

$$C < 2 + M$$

where

$$M \equiv \underset{\nu}{\mathrm{Max}} \left(U_\nu^{-1} \sum_\omega K_{\nu\omega} U_\omega^{-1} \right) \tag{3.11}$$

Now using $|\Delta'_\omega - \Delta_\omega| \leq C \Delta'_\omega$ on the right in Eq. (3.10), one finds

$$|\Delta_\nu - \Delta'_\nu| \leq \Delta_\nu C \left(\frac{\sum_\omega K_{\nu\omega} \Delta'_\omega}{U_\nu + \sum_\omega K_{\nu\omega} \Delta'_\omega} \right) \tag{3.12}$$

where

$$\frac{\sum_\omega K_{\nu\omega} \Delta'_\omega}{U_\nu + \sum_\omega K_{\nu\omega} \Delta'_\omega} = \frac{U_\nu^{-1} \sum_\omega K_{\nu\omega} \Delta'_\omega}{1 + U_\nu^{-1} \sum_\omega K_{\nu\omega} \Delta'_\omega} \leq \frac{U_\nu^{-1} \sum_\omega K_{\nu\omega} U_\omega^{-1}}{1 + U_\nu^{-1} \sum_\omega K_{\nu\omega} U_\omega^{-1}} \leq \frac{M}{1+M} \equiv \gamma < 1. \tag{3.13}$$

M is given in Eq. (3.11). Thus $|\Delta_\nu - \Delta'_\nu| < \gamma C \Delta_\nu$. But now switching $\Delta \leftrightarrow \Delta'$ and repeating the argument, one finds $|\Delta_\nu - \Delta'_\nu| < \gamma^2 C \Delta'_\nu < C \Delta'_\nu$, which contradicts the assumption that C is a minimum greater than zero. Therefore, Δ and Δ' cannot be distinct. (Note nothing has been said to exclude the possible existence of a distinct non-positive solution.)

Now to prove convergence. Starting with any $\Delta_\nu^{(1)} > 0$, inductively define

$$\Delta_\nu^{(n+1)} = \frac{1}{U_\nu + \sum_\omega K_{\nu\omega} \Delta_\omega^{(n)}} \tag{3.14}$$

Note that due to the positive nature of $\Delta^{(1)}$, K and U, $\Delta_\nu^n < U_\nu^{-1}$ for all $n > 1$, and hence

$$U_\nu^{-1} > \Delta_\nu^{(n)} > \left(U_\nu + \sum_\omega K_{\nu\omega} U_\omega^{-1} \right)^{-1} \tag{3.15}$$

for all $n > 2$. Thus it causes no loss of generality to assume $\Delta_\nu^{(1)}$, and hence all $\Delta^{(n)}$, satisfy Eq. (3.14), as we shall now do.

Borrowing from the approach outlined above and using Eq. (3.14), one has

$$\underset{\sim}{\Delta}_{\nu}^{(n+1)} - \underset{\sim}{\Delta}_{\nu}^{(n)} = \underset{\sim}{\Delta}_{\nu}^{(n+1)} \left(\frac{\sum_{\omega} K_{\nu\omega} \left(\underset{\sim}{\Delta}_{\omega}^{(n-1)} - \underset{\sim}{\Delta}_{\omega}^{(n)} \right)}{U_{\nu} + \sum_{\omega} K_{\nu\omega} \underset{\sim}{\Delta}_{\omega}^{(n-1)}} \right) \tag{3.16a}$$

and also

$$\underset{\sim}{\Delta}_{\nu}^{(n+1)} - \underset{\sim}{\Delta}_{\nu}^{(n)} = \underset{\sim}{\Delta}_{\nu}^{(n)} \left(\frac{\sum_{\omega} K_{\nu\omega} \left(\underset{\sim}{\Delta}_{\omega}^{(n-1)} - \underset{\sim}{\Delta}_{\omega}^{(n)} \right)}{U_{\nu} + \sum_{\omega} K_{\nu\omega} \underset{\sim}{\Delta}_{\omega}^{(n)}} \right) \tag{3.16b}$$

For any n, define $C_n^{\pm}$ to be the minimum number such that $C_n^{\pm} \underset{\sim}{\Delta}_{\nu}^{(n)} \geq \left| \underset{\sim}{\Delta}_{\nu}^{(n\pm1)} - \underset{\sim}{\Delta}_{\nu}^{(n)} \right|$ for all ν.

Using Eq. (3.16a), (3.14) and the previous line of analysis, one finds

$$C_{n+1}^{-} < \gamma C_{n-1}^{+} \tag{3.17a}$$

and from (3.16b) with (3.14),

$$C_n^{+} < \gamma C_n^{-} \tag{3.17b}$$

where $\gamma = M/1 + M$, as defined above. Thus

$$C_n^{+} < \gamma C_n^{-} < \gamma^2 C_{n-2}^{+} < \gamma^4 C_{n-4}^{+} \quad \cdots$$

and, therefore, for $n \geq 0$,

$$C_{2n+2}^{+} < \gamma^{2n} C_1^{+} \tag{3.18a}$$

i.e.,

$$\left| \underset{\sim}{\Delta}_{\nu}^{(2n+2)} - \underset{\sim}{\Delta}_{\nu}^{(2n+1)} \right| < \gamma^{2n} C_1^{+} \underset{\sim}{\Delta}_{\nu}^{(2n+1)} < \gamma^{2n} C_1^{+} U_{\nu}^{-1} \tag{3.18b}$$

Also,

$$C_{2n+1}^{-} < \gamma C_{2n-1}^{+} < \gamma^{2n-1} C_1^{+} \tag{3.19}$$

for $n \geq 1$, so that

$$\left|\Delta_\nu^{(2n+1)} - \Delta_\nu^{(2n)}\right| < \gamma^{2n-1} C_1^+ U_\nu^{-1} \tag{3.20}$$

and hence for any $j \geq 1$

$$\left|\Delta_\nu^{(j+1)} - \Delta_\nu^{(j)}\right| < \gamma^{j-1} C_1^+ U_\nu^{-1} \tag{3.21}$$

(Further, because Eq. (3.15) holds for $\Delta_\nu^{(1)}$, there is a bound on C_1^+ just like the bound on C found above: $C_1^+ < 2 + M$).

Because γ is distinctly less than 1, successive differences in $\Delta_\nu^{(n)}$ decrease geometrically as $n \to \infty$. Therefore, the iteration procedure does converge. Also, from the properties of geometric convergence, setting $\Delta_\nu \equiv \lim_{n\to\infty} \Delta_\nu^{(n)}$, Eq. (3.21) gives a bound on the discrepancy between $\Delta_\nu^{(n)}$ and Δ_ν :

$$\left|\Delta_\nu - \Delta_\nu^{(n)}\right| < \frac{\gamma^{n-1}}{1-\gamma} C_1^+ U_\nu^{-1} \quad , \tag{3.22}$$

and so

$$\frac{\left|\Delta_\nu - \Delta_\nu^n\right|}{\Delta_\nu} < \frac{\gamma^{n-1}}{1-\gamma} C_1^+ \frac{U_\nu^{-1}}{\Delta_\nu} < \frac{\gamma^{n-1}}{1-\gamma} C_1^+ (1+M). \tag{3.23}$$

(Note Δ_ν , being the limit of $\Delta_\nu^{(n)}$, must obey Eq. (3.9) since all $\Delta_\nu^{(n)}$ obey (3.15)). Thus in the manner established above,

$$\left|\frac{1}{\left(U_\nu + \sum_\omega K_{\nu\omega} \Delta_\omega\right)} - \frac{1}{\left(U_\nu + \sum_\omega K_{\nu\omega} \Delta_\omega^{(n)}\right)}\right| < \frac{\gamma^{n-1}}{1-\gamma} C_1^+ (1+M) \Delta_\nu^{n+1}$$

$< \frac{\gamma^{n-1}}{1-\gamma} C_1^+ (1+M) U_\nu^{-1} \longrightarrow 0$ in the limit $n \to \infty$, so that we have

$$\frac{1}{\left(U_\nu + \sum_\omega K_{\nu\omega} \Delta_\omega\right)} = \lim_{n\to\infty}\left(\frac{1}{U_\nu + \sum_\omega K_{\nu\omega} \Delta_\omega^{(n)}}\right) = \lim_{n\to\infty} \Delta_\mu^{(n+1)} = \Delta_\nu \tag{3.24}$$

Therefore, Δ_ν is a unique positive solution of the finite point equation (3.7).

The above analysis can be applied immediately to the full integral equation (3.4) with little elaboration. The only questions which arise concern infinities which could develop from terms which are obviously finite for a finite set, but not obviously finite for an infinite set. For example, $\int K(s,s')\Delta^{(n)}(s')$ might diverge; one is clearly constrained to start the iteration with a $\Delta^{(1)}(s)$ such that $\int K(s,s')\,\Delta^{(1)}(s')ds'$ converges for all s, e.g., $\Delta^{(1)} \equiv 0$. Once this constraint is observed, however, all $\Delta^{(n)}$ for $n > 1$ will satisfy $\Delta^{(n)}(s) \leq U^{-1}(s)$, and since $\int K(s,s')U^{-1}(s')ds'$ converges for all s, so, too, will $\int K(s,s')\Delta^{(n)}(s')$ for all $n > 1$.

The only remaining point to discuss concerns

$$M = \underset{0 < s < \infty}{\mathrm{Max}}\, U^{-1}(s) \int K(s,s')\, U^{-1}(s')\, ds'.$$

Such a maximum may not exist; one needs to know that $U^{-1}(s)\int K(s,s')U^{-1}(s')ds'$ is a bounded function of s for $s \geq 0$.* From Eq. (3.5) we can find a bound on K:

$$K(s,s') \leq \frac{2\left(2\pi^2 e_i^2\right)}{(2\pi)^4}\, s' \, \frac{s + m^2}{\left(s + s' + \mu^2\right)\left(s' + \left(\mu^2 - m^2\right)\right)} \tag{3.25}$$

(We already know $K(s, s') > 0$). Thus $U^{-1}(s)\int K(s,s')U^{-1}(s')ds'$ is bounded by

$$\frac{2\left(2\pi^2 e_i^2\right)}{(2\pi)^4} \int_0^\infty \frac{s'ds'}{\left(s + s' + \mu^2\right)\left(s' + \left(\mu^2 - m^2\right)\right)\left(s' + m^2\right)} \tag{3.26}$$

which is a convergent integral in s' and, by inspection, a bounded function of $s \geq 0$. Therefore, M is finite and the convergence factor, γ, is distinctly less than 1. Further, since the bounds C and C_1^+ used above are themselves bounded by $2 + M$, their existence now is established by the finiteness of M. Thus the proofs of uniqueness and convergence are valid for the full integral equation

---

* The simplicity of this condition makes it a natural one to use here although $M/(1 + M)$ is only an upper bound on the convergence factor. Convergence can occur in cases where M is not finite.

(3.4) as well as Eq. (3.7). Also the above proof that the limit of convergence is a solution of the equation has been specially constructed to maintain validity so long as M is finite. Therefore, $\underset{\sim}{\Delta}(s) = \lim_{n \to \infty} \underset{\sim}{\Delta}^{(n)}(s)$ is a unique positive solution of Eq. (3.4).

Since Eq. (3.26) gives a bound on M which is proportional to e_i^2, convergence being governed by $\gamma = M/1 + M$, one should expect that convergence will be slow for large values of e_i^2. To illustrate the application of this method a finite mesh was used to compute the values of $\underset{\sim}{\Delta}^{-1}(s)$ shown in Fig. 10.

It is interesting that the propagators formed here cannot develop "ghost" poles--i.e., poles other than the pole at $s = -m^2$. For real $s < -\mu^2$ such poles are excluded by Eq. (3.6) which shows $\frac{d}{ds}\left(K(s,s')\right)$ is positive. Hence, $\underset{\sim}{\Delta}^{-1}(s)$ is an increasing function of real $s > -\mu^2$, and can have only one zero there. The same observation can be made for all real $s > -(m + \mu)^2$ by considering the extra part of K which enters if $s < -\mu^2$ (see Appendix A). Again one finds $\frac{d}{ds} K > 0$ so that $\underset{\sim}{\Delta}^{-1}$ is increasing. More generally, as outlined in Appendix B, one can show that, starting iteration of Eq. (3.4) with $\underset{\sim}{\Delta}^{(1)}(s) = 1/(s + m^2)$, all $\underset{\sim}{\Delta}^{(n)}(s)$ and $\underset{\sim}{\Delta}(s)$ have a Lehmann spectral representation and that no ghost poles appear for real or complex s.

IV. CONCLUSION

In treating inelastic unitarity and self-energy in the Bethe-Salpeter equation we have developed a procedure that maintains a precise formal correspondence to a set of Feynman graphs, but deals directly with the amplitudes that appear in the equation. As a result, it has been possible to include a large class of self-energy graphs and also maintain the physical properties of the Lehmann spectral representation. However, there are many, many more complex graphs which have not been included here, and a better understanding of the significance of such graphs must be obtained before the Bethe-Salpeter equation for strong interactions can be fully understood or effectively used.

APPENDIX A

KERNAL FOR $s < -\mu^2$

For $s < -\mu^2$ we get an addition to $\int K\underset{\sim}{\Delta}$ in Eq. (3.4) of the form $\int K_2(s, s')\,\underset{\sim}{\Delta}(s')\,ds'$ (we will now refer to the K of Eq. (3.5) as K_1) where

$$K_2(s,s') = H_2(s,s') - H_2(-m^2, s') = H_2(s,s') \qquad \text{if} \quad m < \mu ,$$

$$H_2(s,s') = \frac{4\pi^2 e^2}{(2\pi)^4}\,\theta(-s')\,\theta\left(s' + (\sqrt{-s} - \mu)^2\right) \cdot \frac{\left[(s + s' + \mu^2)^2 - 4ss'\right]^{1/2}}{s} < 0 \qquad \text{(A1)}$$

for $s < 0$, $\theta(x) = 1$, $x > 0$; $=0$, $x < 0$;

while

$$\frac{d}{ds} H_2(s, s') = \frac{4\pi e^2}{(2\pi)^4}\,\theta(-s')\,\theta\left(s' + (\sqrt{-s} - \mu)^2\right) \cdot \frac{(-2s'\mu^2 - 2s\mu^2 - s'^2 + ss')}{s^2\left[(s + s' + \mu^2)^2 - 4ss'\right]^{1/2}} > 0 \qquad \text{(A2)}$$

for $s < s' < 0$.

Eqs. (A2) and (3.6) insure that $\underset{\sim}{\Delta}^{-1}(s)$ calculated from Eq. (3.4) will be an increasing function of real $s > -(m + \mu)^2$. If $m > \mu$, the K in Eq. (3.4) becomes $K_1(s,s') + K_2(s, s')$ and the integral equation involves all $s, s' > -(m - \mu)^2$. However, Eqs. (A2) and (3.6) insure that K will still be positive. The proof that iteration of Eq. (3.4) converges can then be generalized to include $m > \mu$.

The case $\mu = 0$ is special in that $U(s)^{-1} \int K(s,s')\,U(s')$ is then unbounded as $s \longrightarrow -(m - \mu)^2 = -m^2$. However, convergence can still be verified by separating $s' > -m^2 + \epsilon$ from $s' < -m^2 + \epsilon$, and showing that the error in $\underset{\sim}{\Delta}(s)$ for $s > -m^2 + \epsilon$ vanishes as $\epsilon \longrightarrow 0$.

APPENDIX B

SPECTRAL REPRESENTATION FOR Δ

Starting iteration of Eq. (3.4) with $\underset{\sim}{\Delta}^{(1)}(s') = 1/(s' + m^2)$ one can perform the q'_{io} integration exactly, obtaining

$$\underset{\sim}{\Delta}^{(2)}(s) = \frac{1}{s + m^2 - \int_{(m+\mu)^2}^{\infty} f^{(2)}(\sigma)\left(\frac{1}{s+\sigma} - \frac{1}{-m^2+\sigma}\right) d\sigma} \tag{B1a}$$

$$= \frac{1}{(s + m^2)\left(1 + \int_{(m+\mu)^2}^{\infty} \frac{g^{(2)}(\sigma)}{s+\sigma}\, d\sigma\right)} \tag{B1b}$$

with

$$f^{(2)}(\sigma) = (\sigma - m^2)\, g^{(2)}(\sigma) = a\, F(\sigma, m)$$

where

$$a = \frac{e_i^2 \pi^2}{(2\pi)^4}, \qquad F(\sigma, m) = \frac{[\sigma - (m+\mu)^2]^{1/2}[\sigma - (m-\mu)^2]^{1/2}}{\sigma}$$

Note: $1 > F(\sigma, m) > 0, \quad \frac{a}{\sigma - m^2} > g^{(2)}(\sigma) > 0.$

From $g^{(2)}(\sigma) > 0$, it follows for $\mathrm{Re}(s) > -(m+\mu)^2$ that $\left(\underset{\sim}{\Delta}^{(2)}(s)\right)^{-1}$ is only zero at $s = -m^2$, since the integral in Eq. (B1b) then has positive real part. Also from Eq. (B1a) one has

$$\left|\underset{\sim}{\Delta}^{(2)}(s)^{-1}\right| \geq \left|\mathrm{Im}\left(\underset{\sim}{\Delta}^{(2)}(s)^{-1}\right)\right| = |\mathrm{Im}\, s| + \int_{(m+\mu)^2}^{\infty} f^{(2)}(\sigma) \times \frac{|\mathrm{Im}\, s|}{\left(\mathrm{Re}(s) + \sigma\right)^2 + |\mathrm{Im}\, s|^2}\, d\sigma \tag{B2}$$

Since $f^{(2)}(\sigma) > 0$, ghost poles are excluded for $|\mathrm{Im}(s)| > 0$ and they are also excluded for $\mathrm{Im}(s) = \pm i\epsilon, \mathrm{Re}(s) < -(m+\mu)^2$ where Eq. (B2) gives $\left|\mathrm{Im}\left(\underset{\sim}{\Delta}^{(2)}(s)^{-1}\right)\right| = \pi f^{(2)}\left(\mathrm{Re}(s)\right) > 0$. Thus the only singularities in $\underset{\sim}{\Delta}^{(2)}(s)$ are a pole at $s = -m^2$ and a cut from $-(m+\mu)^2$ to $-\infty$. To write a Lehmann spectral representation for $\underset{\sim}{\Delta}^{(2)}(s)$ we need to know that $\oint \underset{\sim}{\Delta}^{(2)}(s')/(s'-s)\, ds'$ vanishes over a large circle (with detours around the pole and cut in $\underset{\sim}{\Delta}^{(2)}(s')$) of radius $R \to \infty$. Along any ray away from the cut, Eq. (B1b) with $g^{(2)}(\sigma) \leq a/(\sigma - m^2)$ gives $\underset{\sim}{\Delta}^{(2)}(s') \longrightarrow 1/(s' + m^2)$ so that the integral over this part of the circle is easily seen to vanish as $R \longrightarrow \infty$. Near the cut we may use Eq. (B2) and the fact that $f^{(2)}(\sigma)$ is greater than a step function of the form $\mathcal{F}(\sigma) = b\theta(\sigma - c)$ with $b > 0$, $c > (m+\mu)^2$, e.g., $f^{(2)}(\sigma) \geq \mathcal{F}(\sigma) \equiv a\, F\left((m+2\mu)^2, m\right) \theta\left(\sigma - (m+2\mu)^2\right)$, to find a lower bound on $\left|\underset{\sim}{\Delta}^{(2)}(s)^{-1}\right|$ for $\mathrm{Re}(s) < -c$:

$$\left|\underset{\sim}{\Delta}^{(2)}(s)^{-1}\right| \geq \left|\mathrm{Im}\left(\underset{\sim}{\Delta}^{(2)}(s)^{-1}\right)\right| \geq \left|\mathrm{Im}(s)\right| + \frac{b\pi}{2} \tag{B3}$$

so that

$$\left|\underset{\sim}{\Delta}^{(2)}(s)\right| \leq \frac{1}{\left|\mathrm{Im}(s)\right| + \frac{b\pi}{2}} \quad .$$

One can then show that the integral of $\Delta(s')/(s' - s)$ over an arc of the circle touching the cut vanishes as $R \to \infty$, thus obtaining a spectral representation for $\underset{\sim}{\Delta}^{(2)}(s)$:

$$\underset{\sim}{\Delta}^{(2)}(s) = \frac{z^{(2)}}{s + m^2} + \int_{(m+\mu)^2}^{\infty} \frac{\rho^{(2)}(\sigma)}{s + \sigma}\, d\sigma \tag{B4}$$

with

$$z^{(2)} = \left(\left[\frac{d}{ds}\left(\underset{\sim}{\Delta}^{(2)}(s)^{-1}\right)\right]_{s=-m^2}\right)^{-1} > 0 \tag{B5}$$

and

$$\rho^{(2)}(\sigma) = \left|\underset{\sim}{\Delta}(\sigma \pm i\epsilon)\right|^2 f^{(2)}(\sigma) > 0 \quad . \tag{B6}$$

Since $\underset{\sim}{\Delta}^{(2)}(s) \to 1/(s+m^2)$ as $s \to +\infty$, we can multiply Eq. (B4) on both sides by $s+m^2$ and taking $s \to +\infty$ get the familiar relation

$$1 = z^{(2)} + \int_{(m+\mu)^2}^{\infty} \rho^{(2)}(\sigma)\, d\sigma \tag{B7}$$

Now we can go to the next iteration, writing $\underset{\sim}{\Delta}^{(3)}(s)$ in the form of Eqs. (B1a,b) with

$$f^{(3)}(\sigma) = (\sigma - m^2)\, g^{(3)}(\sigma) = a\Bigg(z^{(2)} F(\sigma, m) + \theta\Big(\sigma - (m+2\mu)^2\Big) \times \int_{(m+\mu)^2}^{(\sqrt{\sigma}-\mu)^2} \rho^{(2)}(\sigma')\, F(\sigma, \sqrt{\sigma'})\, d\sigma'\Bigg) \tag{B8}$$

From Eqs. (B7), (B8), with $F(\sigma, m)$, $F(\sigma, \sqrt{\sigma'}) < 1$, we get

$$0 < z^{(2)} a\, F(\sigma, m) \leq f^{(3)}(\sigma) < a \tag{B9}$$

so again

$$0 < g^{(3)}(\sigma) < \frac{a}{\sigma - m^2} \tag{B10}$$

and

$$f^{(3)}(\sigma) \geq z^{(2)}\, \mathcal{F}(\sigma) \tag{B11}$$

Thus we can eliminate ghost poles from $\underset{\sim}{\Delta}^{(3)}$, write a spectral representation for $\underset{\sim}{\Delta}^{(3)}$, ... etc. Equations (B1) – (B11) then hold inductively with $2 \to n$, $3 \to n+1$ for an n greater than 2.

Now from Section III and Appendix A we know the $\underset{\sim}{\Delta}^{(n)}(s)$ formed here converge for real $s > s_{min} = -\Big(\mathrm{Min}(0, \mu - m)\Big)^2$. But then, using the contour in Fig.8, we can show $\underset{\sim}{\Delta}^{(n+1)}(s)^{-1}$ converges for real $s > -(\sqrt{-s_{min}} + \mu)^2$. In this way we can move down the negative real s axis, on either side for

$s < -(m+\mu)^2$, showing $\underset{\sim}{\Delta}^{(n)}(s)^{-1}$ converges. (We can also reach any complex s in this way by moving parallel to the real q_{io} axis in Fig. 8). But then $z^{(n)}$, the inverse of the slope of $\underset{\sim}{\Delta}^{(n)}(s)^{-1}$ at $s = -m^2$ converges and $g^{(n)}(\sigma)$, the cut discontinuity in $\left[\left(\underset{\sim}{\Delta}^{(n)}(s)^{-1}\right)/(s+m^2)\right] - 1$, converges as $n \to \infty$ for any bounded range of σ. Since $g^{(n)}(\sigma) < a/(\sigma - m^2)$ for every n we can conclude

$$\underset{\sim}{\Delta}(s) = \frac{1}{(s+m^2)\left(1 + \int \frac{g(\sigma)}{s+\sigma}\right)} \tag{B12}$$

where

$$\underset{\sim}{\Delta}(s) = \lim_{n\to\infty} \underset{\sim}{\Delta}^{(n)}(s) \tag{B13}$$

$$g(\sigma) = \lim_{n\to\infty} g^{(n)}(\sigma) \tag{B14}$$

Then we can write Eq. (B1a) for $\underset{\sim}{\Delta}(s)$ with $f(\sigma) = (\sigma - m^2)\, g(\sigma) = \lim_{n\to\infty} f^{(n)}(\sigma) \geq z\, \mathcal{F}(\sigma)$, $z = \lim_{n\to\infty} z^{(n)} > 0$. Thus $\underset{\sim}{\Delta}(s)$ has no ghost poles and we can write

$$\underset{\sim}{\Delta}(s) = \frac{z}{s+m^2} + \int_{(m+\mu)^2}^{\infty} \frac{\rho(\sigma)}{s+\sigma}\, d\sigma \tag{B15}$$

where $\rho(\sigma) = \left|\underset{\sim}{\Delta}(\sigma \pm i\epsilon)\right|^2 f(\sigma)$ satisfies

$$1 = z + \int_{(m+\mu)^2}^{\infty} \rho(\sigma)\, d\sigma \tag{B16}$$

In conclusion note that the approximate solution to Eq. (3.4) given first in Section III also has a spectral representation without ghost poles.

ACKNOWLEDGMENTS

The author would like to thank Professor S. D. Drell, Dr. Y. Frishman, Dr. J. D. Sullivan, Dr. S. J. Brodsky and Professor J. D. Bjorken for their informative and stimulating conversations.

LIST OF FOOTNOTES AND REFERENCES

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4. Robert Saenger, "The Bethe-Salpeter Equation in Momentum Space," SLAC-PUB-231, November 1966.
5. This result confirms the analysis of Ref. 3(b).
6. We are taking $\psi(\vec{x}, o) = \int e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} \tilde{\psi}(p)\, d^4p$ as in Ref. 4.
7. The author wishes to thank J. D. Sullivan for suggesting the possible usefulness of this approximation.
8. The propagator given in Ref. 3(b) also results in a unitary equation, but the device of slope readjustment used there differs from our prescription for adjusting $\tilde{z}$. For large couplings the resulting propagators are totally different.
9. R. Haag and Th. A. Maris, Phys. Rev. 132, 2325 (1963); Y. Frishman and A. Katz, Phys. Letters 11, 172(1964).

LIST OF FIGURES

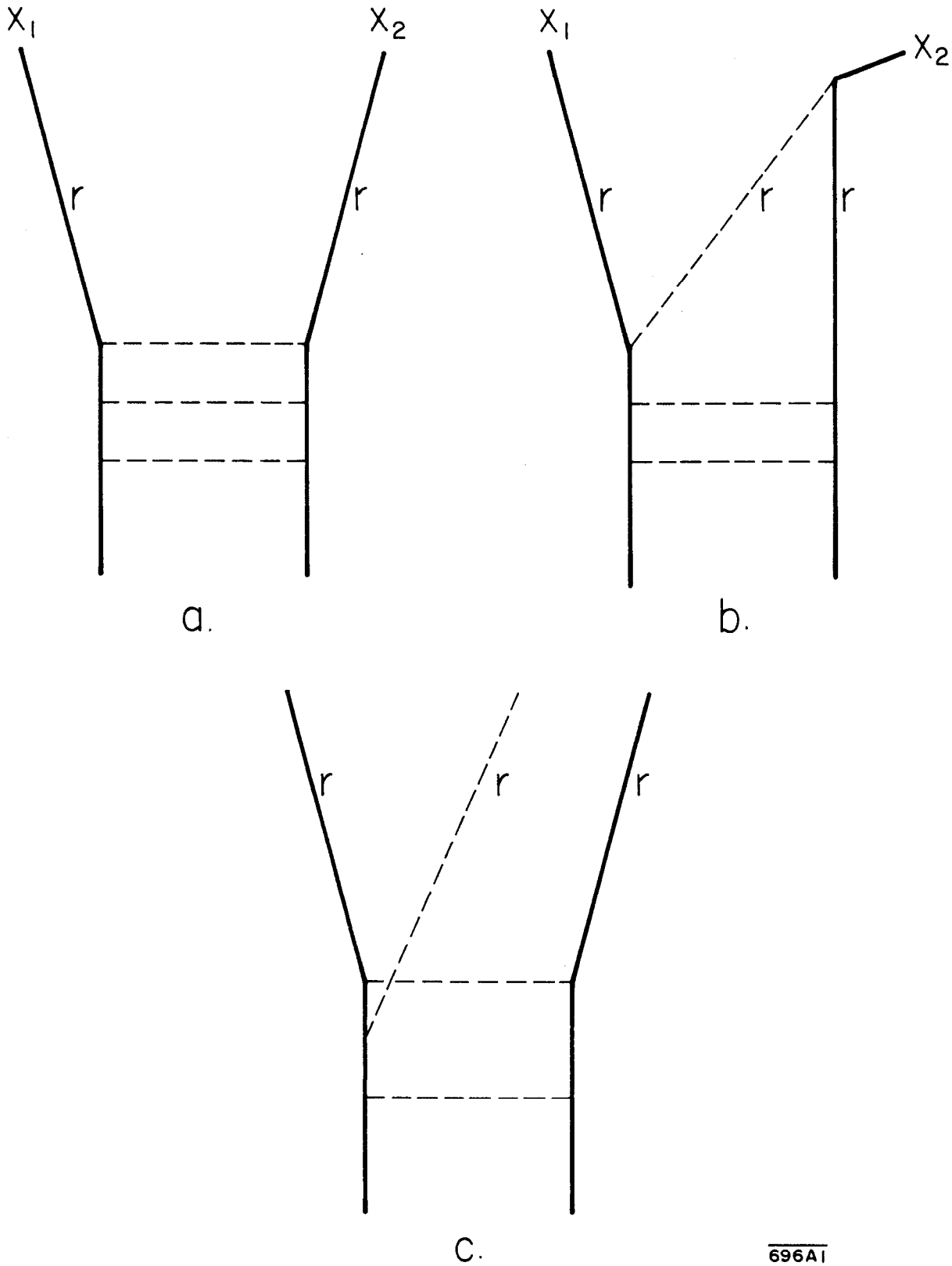

FIG. 1

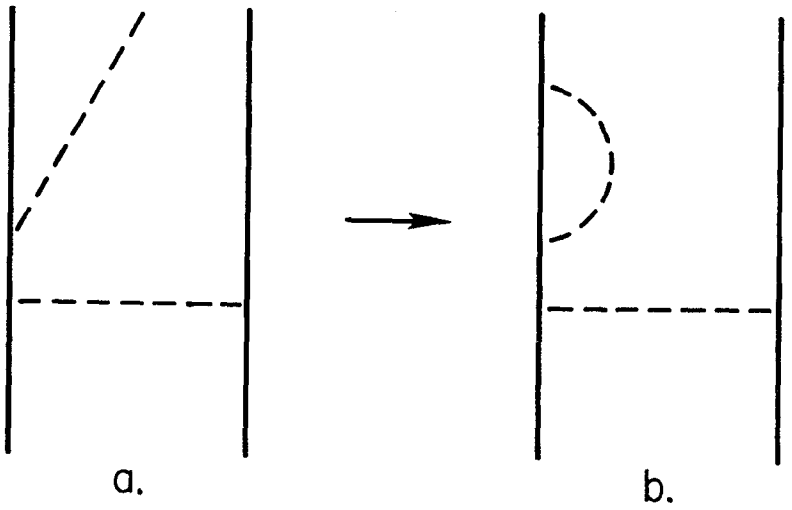

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Fig. 2

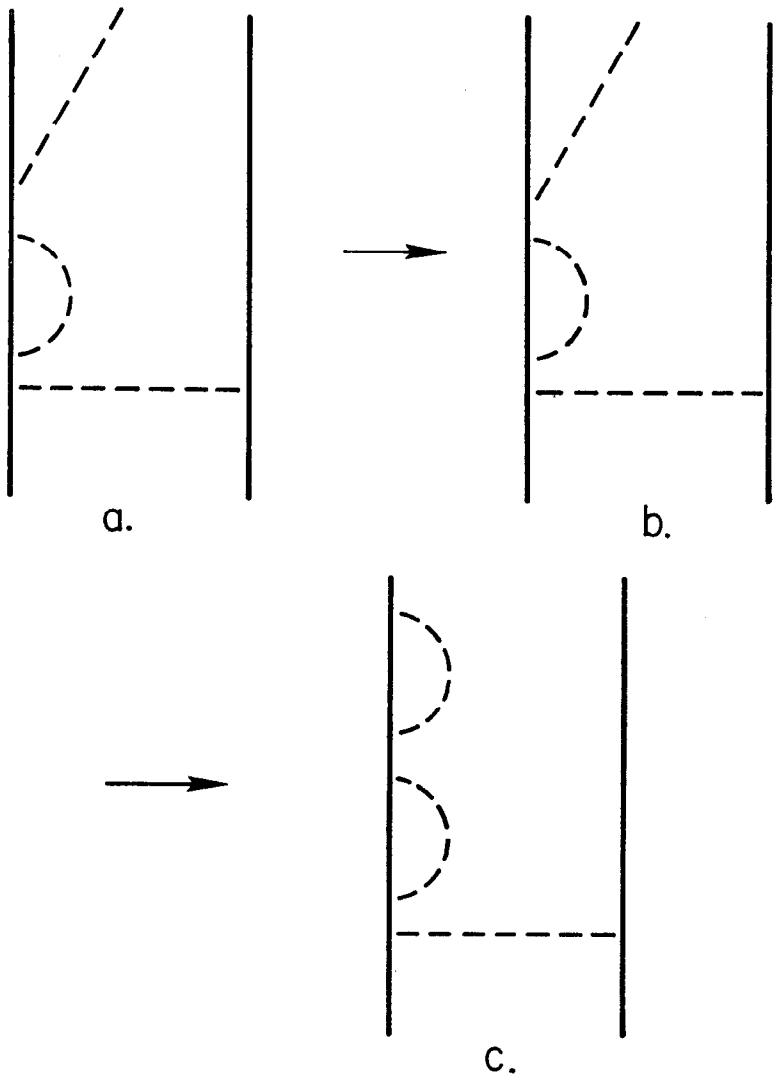

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Fig. 3

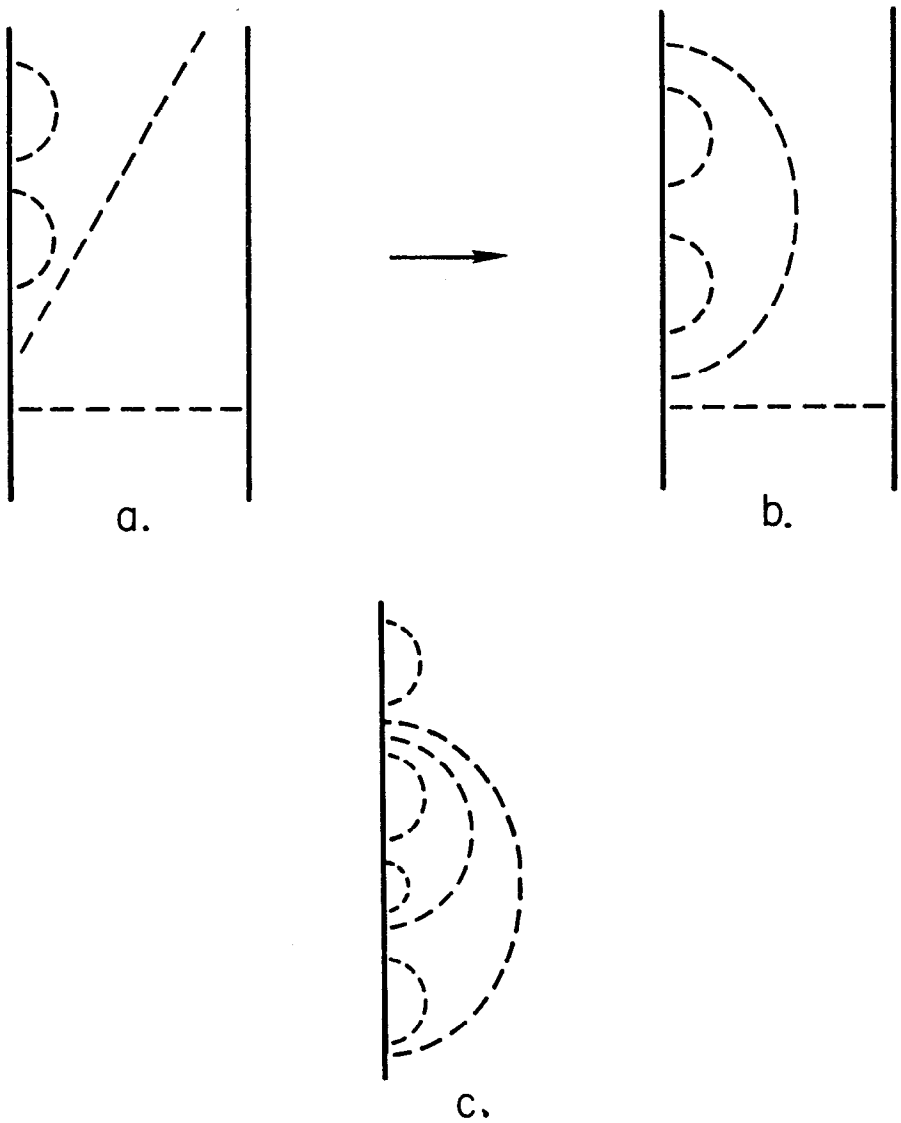


Fig. 4

696A4

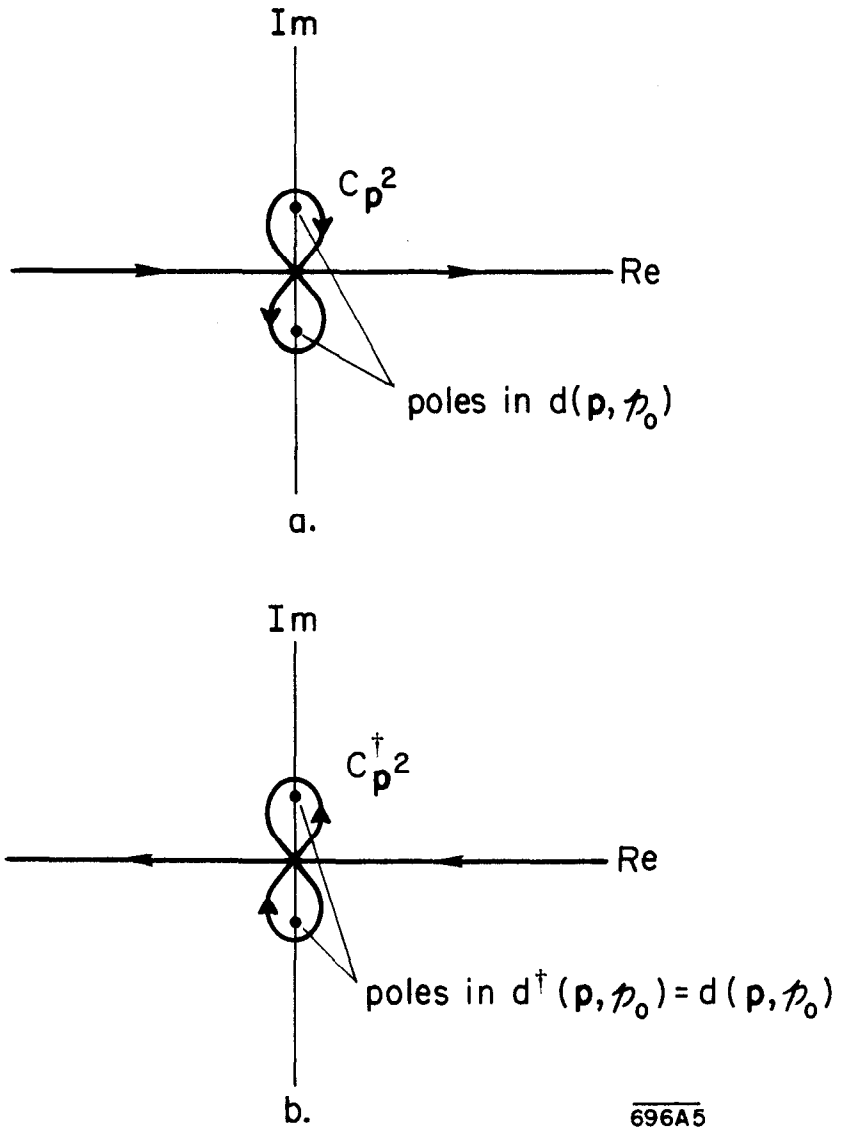

Im
C_{p^2}
Re
poles in $d(\mathbf{p}, \mathcal{P}_0)$
a.
Im
$C^{\dagger}_{p^2}$
Re
poles in $d^{\dagger}(\mathbf{p}, \mathcal{P}_0) = d(\mathbf{p}, \mathcal{P}_0)$
b.
696A5

Fig. 5

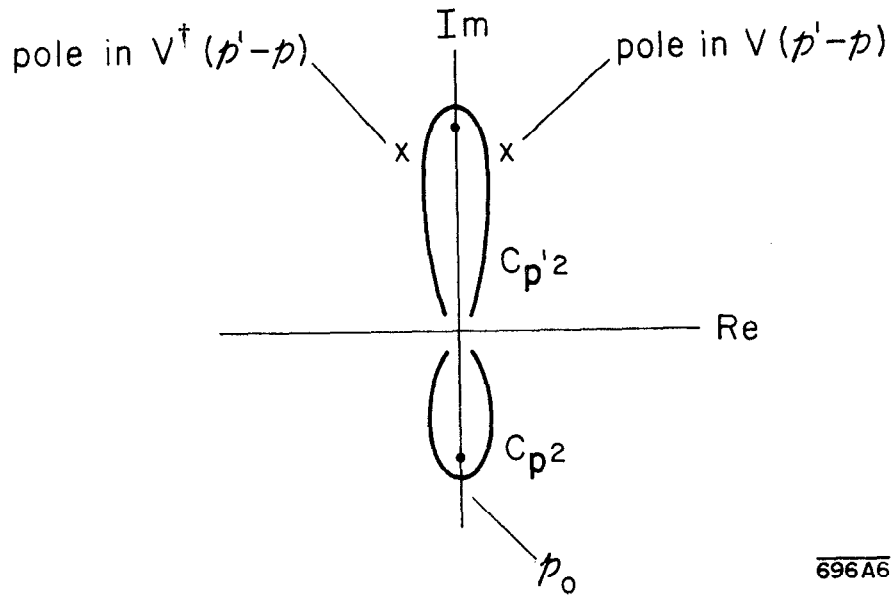


FIG. 6

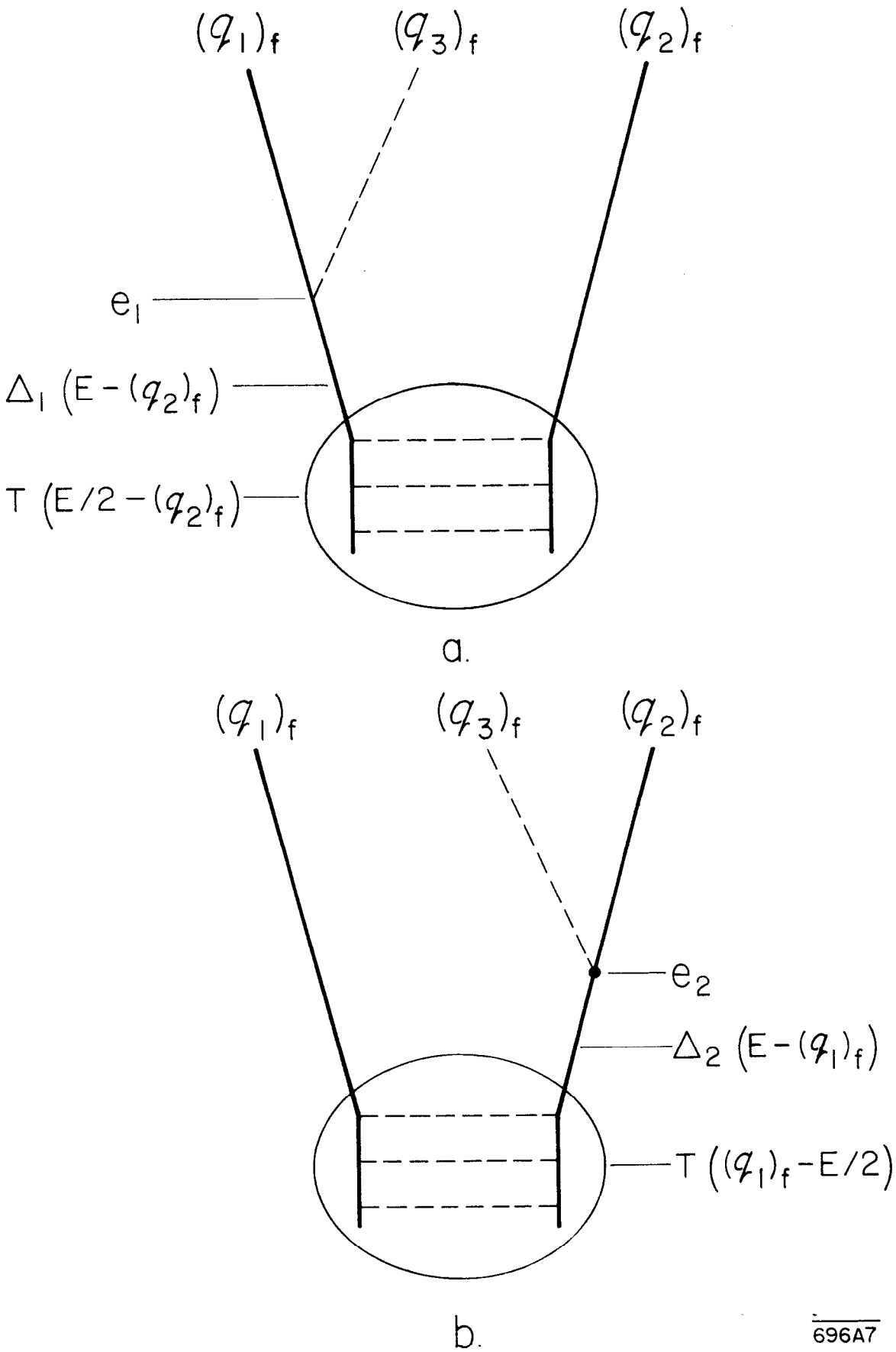

$(q_1)_f$
$(q_3)_f$
$(q_2)_f$
e_1
$\Delta_1 (E-(q_2)_f)$
$T (E/2-(q_2)_f)$
a.
$(q_1)_f$
$(q_3)_f$
$(q_2)_f$
e_2
$\Delta_2 (E-(q_1)_f)$
$T ((q_1)_f-E/2)$
b.
696A7

FIG. 7

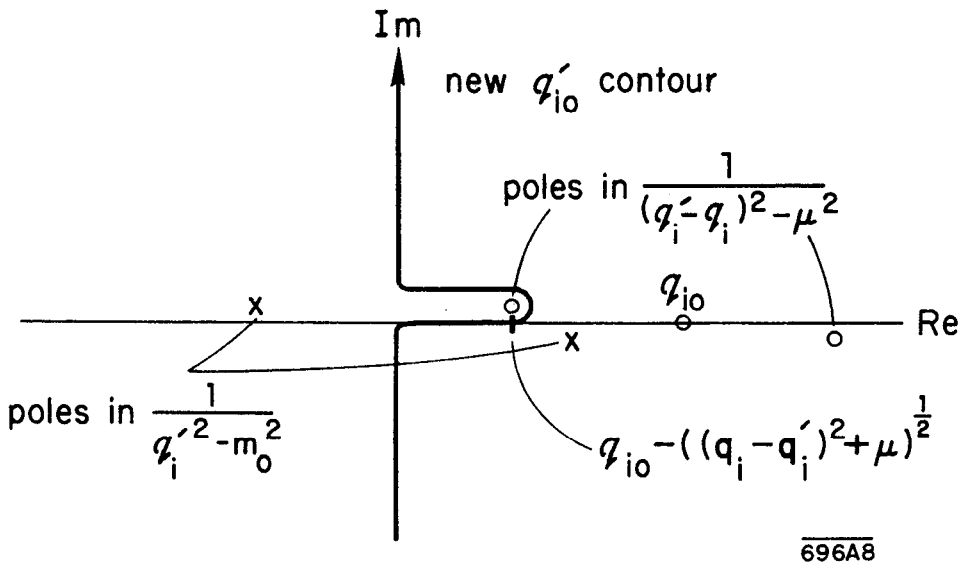
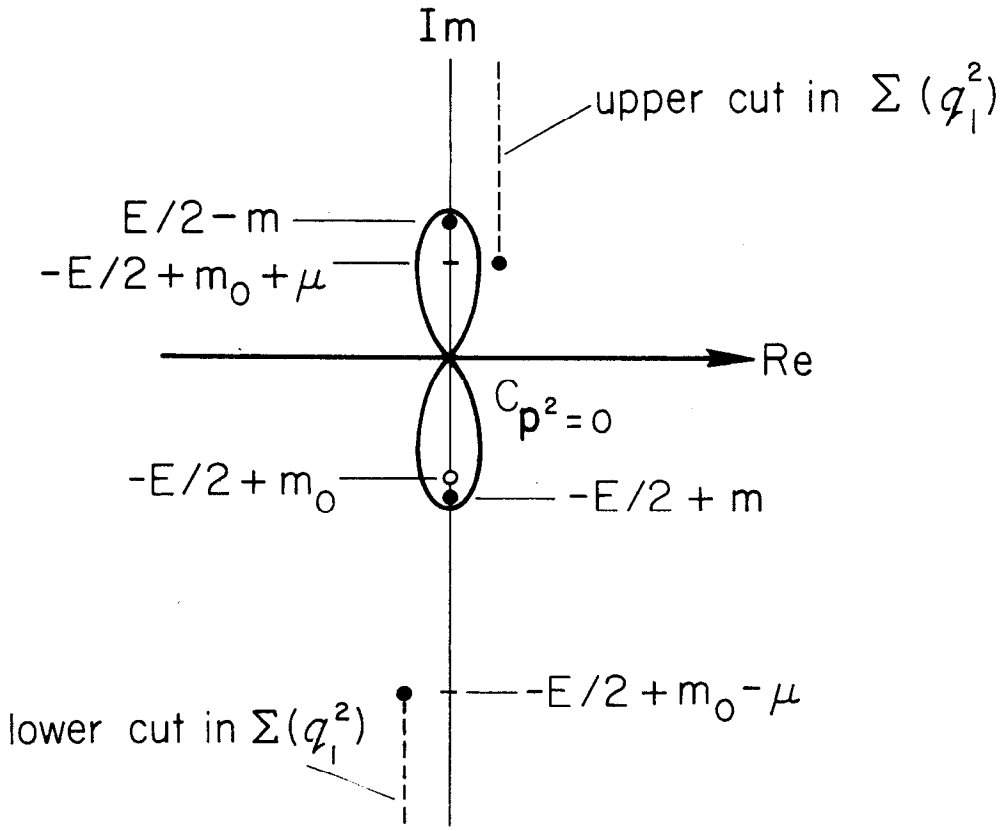

Im
new q'_{io} contour
poles in $\frac{1}{(q'_i - q_i)^2 - \mu^2}$
q_{io}
Re
poles in $\frac{1}{q_i'^2 - m_o^2}$
$q_{io} - ((q_i - q'_i)^2 + \mu)^{\frac{1}{2}}$
696A8

Fig. 8


Im
upper cut in $\Sigma(q_1^2)$
$E/2-m$
$-E/2+m_0+\mu$
Re
$C_{p^2=0}$
$-E/2+m_0$
$-E/2+m$
$-E/2+m_0-\mu$
lower cut in $\Sigma(q_1^2)$

696A9

Fig. 9

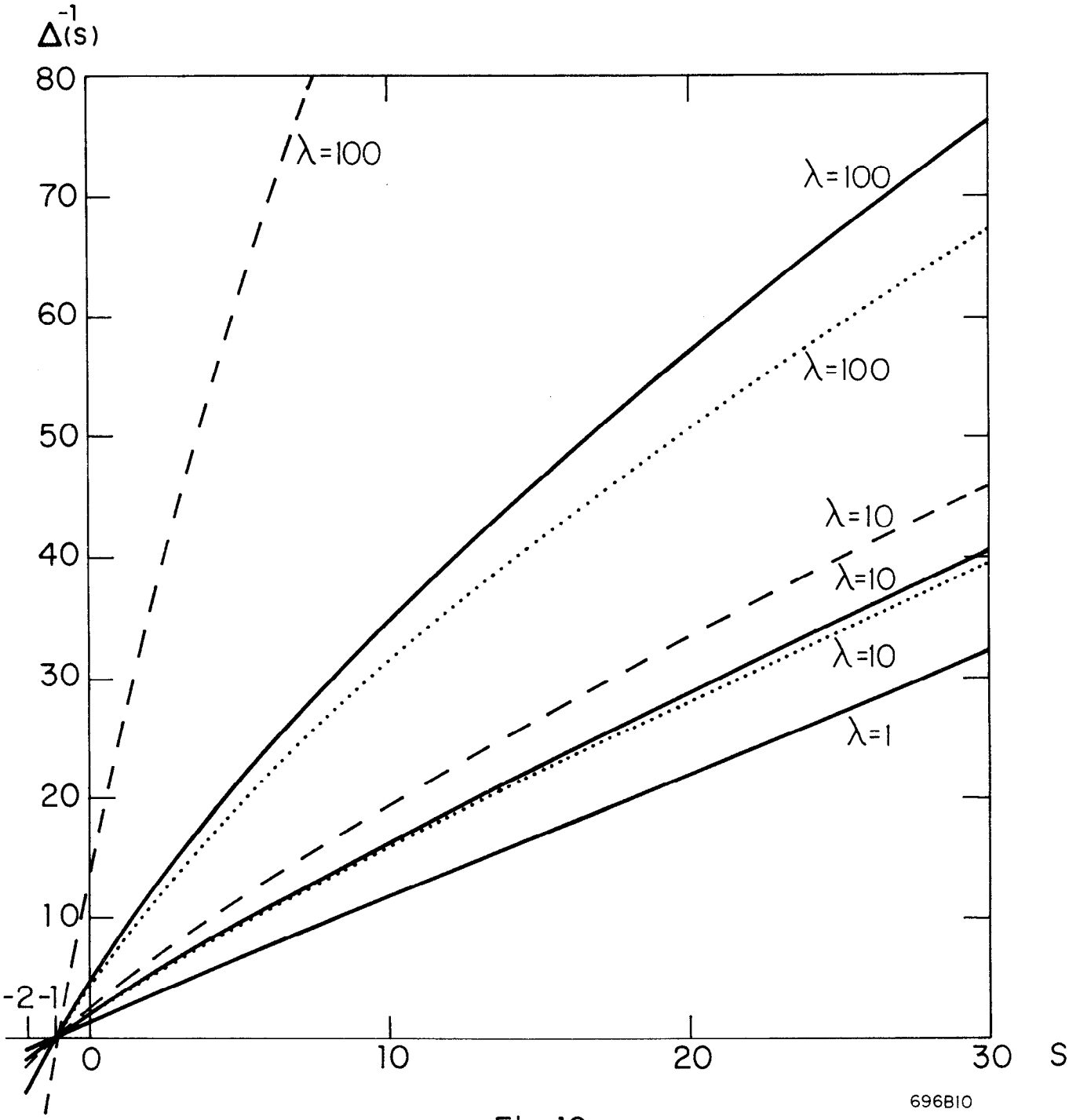

$\Delta^{-1}(s)$
80
70
60
50
40
30
20
10
-2
-1
0
10
20
30
S
λ=100
λ=100
λ=100
λ=10
λ=10
λ=10
λ=1
696B10

Fig. 10