COMMUTATION RELATIONS AND FIELD DEPENDENCE OF VECTOR AND TENSOR CURRENTS*

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ABSTRACT

It is shown, directly from proper Lorentz-invariance and a positive Hilbert-space metric, that the vacuum expectation value $<0\left|\left[j_o(\vec{x}),\ j_k^\dagger(\vec{y})\right]\right|0>$ cannot vanish unless $j_\mu(x)|0> = j_\mu^\dagger(x)|0> \equiv 0$. Neither locality nor Källén-Lehmann type representations are needed. The same is demonstrated for $<0\left|\left[S_{k\ell}(\vec{x}),\ S_{om}^\dagger(\vec{y})\right]\right|0>$, for any antisymmetric tensor $S_{\mu\nu}$. The explicit dependence of j_μ and $S_{\mu\nu}$ on the fields with which they interact is an immediate consequence in our approach. Similarly, it is immediate to show that $\chi(x)$, $\chi(x) = \partial_\mu j^\mu(x)$, does not commute with $j_o^\dagger(y)$ for $y_o = x_o$, unless $\chi(x)|0> = \chi^\dagger(x)|0> \equiv 0$.

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I. INTRODUCTION

Schwinger[1] demonstrated that the equal time commutator $\left[j_o(\vec{x}), j_k(\vec{y})\right]$ of the time and space components of a conserved current $j_\mu(x)$ cannot vanish.[2] It was afterwards realized that the above commutator does not vanish also for the case of a non-conserved j_μ.[3-7] In those derivations, the assumptions of a local theory and the existence of Källén[8] - Lehmann[9] type representations were made by the authors. We show that the non-vanishing of $<0\left|\left[j_o(\vec{x}), j_k^\dagger(\vec{y})\right]\right|0>$ (we consider polar or axial vector currents, not necessarily hermitian) is independent of the latter assumptions,[10] provided we do not have $j_\mu\left|0>\equiv 0\right.$ and $j_\mu^\dagger\left|0>\equiv 0\right.$ simultaneously. Only proper Lorentz invariance, no massless particles and a positive Hilbert-space metric are assumed. In the same approach, the non-vanishing of the vacuum expectation value $<0\left|\left[S_{k\ell}(\vec{x}), S_{om}^\dagger(\vec{y})\right]\right|0>$ for any antisymmetric tensor $S_{\mu\nu}$, is demonstrated. This fact was pointed out via spectral representations in a local theory, by Boulware and Deser.[11] The explicit dependence of j_μ and $S_{\mu\nu}$ on the fields with which they interact[4,11,12] is an immediate consequence in our approach. This is demonstrated for a scalar field gradient coupled to a vector current and for a massive vector field with vector and tensor sources. In the same approach, the non-vanishing of $<0\left|\left[j_o^\dagger(\vec{x}), \chi(\vec{y})\right]\right|0>$,[11] where $\chi(x)=\partial^\mu j_\mu(x)$, is also immediate, provided not both χ and $\chi^\dagger$ annihilate the vacuum $\left|0\right>$.

II. THE VECTOR CASE

Let us decompose the vector $j_\mu(x)$ into

$$j_\mu(x) = J_\mu(x) + \partial_\mu \phi(x) \tag{1}$$

where $\phi(x)$ is defined by[13]

$$\phi(x) = \Box^{-1} \partial^\mu j_\mu(x) \tag{2}$$

consequently

$$\partial^\mu J_\mu(x) = 0 \tag{3}$$

It follows that

$$< 0 \left| \left[J_\mu(x),\ j_\nu^\dagger(y) \right] \right| 0 > = \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box} \right) F(x - y) \tag{4}$$

where $F(x - y)$ is invariant under proper Lorentz transformations. Thus

$$< 0 \left| \left[J_\mu(x),\ \phi^\dagger(y) \right] \right| 0 > = 0 \tag{5}$$

and hence

$$< 0 \left| \left[j_\mu(x),\ j_\nu^\dagger(y) \right] \right| 0 > = < 0 \left| \left[J_\mu(x),\ J_\nu^\dagger(y) \right] \right| 0 > + < 0 \left| \left[\partial_\mu \phi(x),\ \partial_\nu \phi^\dagger(y) \right] \right| 0 > \tag{6}$$

Suppose that

$$< 0 \left| \left[j_o(\vec{x}),\ j_k^\dagger(\vec{y}) \right] \right| 0 > = 0 \tag{7}$$

then, from (3) and (7), taking the three-divergence of the latter,

$$< 0 \left| \left[J_o(\vec{x}),\ \partial^o J_o^\dagger(\vec{y}) \right] \right| 0 > + < 0 \left| \left[\phi(\vec{x}),\ \partial^o \partial^k \partial_k \phi^\dagger(\vec{y}) \right] \right| 0 > = 0 \tag{8}$$

Using also

$$\partial_\mu A(x) = i \left[P_\mu, A(x) \right] \tag{9}$$

where P_μ are the generators of space-time translations, we obtain

$$< 0 \left| J_o(\vec{x}) \; H J_o^\dagger \; (\vec{y}) \right| 0 > + < 0 \left| J_o^\dagger \; (\vec{y}) \; H J_o(\vec{x}) \right| 0 > +$$

$$+ < 0 \left| \phi \; (\vec{x}) \; H\vec{P}^2 \; \phi^\dagger(\vec{y}) \right| 0 > + < 0 \left| \phi^\dagger(\vec{y}) \; H\vec{P}^2 \; \phi(\vec{x}) \right| 0 > = 0 \qquad (10)$$

from which, using positive definiteness,

$$J_o(x) \left| 0 > \doteq J_o^\dagger \; (x) \right| 0 > = 0 \qquad (11a)$$

$$\phi \; (x) \left| 0 > = \phi^\dagger (x) \right| 0 > = 0 \qquad (11b)$$

these entail

$$j_\mu(x) \left| 0 > = j_\mu^\dagger(x) \right| 0 > = 0 \qquad (11c)$$

Thus we have shown that the assumption

$$< 0 \left| \left[j_o(\vec{x}), \; j_k^\dagger \; (\vec{y}) \right] \right| 0 > = 0$$

leads to (11c). In a local theory, (11c) implies $j_\mu(x) \equiv 0$ for a local j_μ.[14]

Consider now a scalar field $\phi(x)$ gradient coupled to a vector current $j_\mu(x)$. The field canonically conjugate to $\phi^\dagger(x)$ is

$$\phi_{(o)}(x) = \partial_o \phi(x) - j_o(x) \qquad (12)$$

Suppose that

$$< 0 \left| \left[j_k^\dagger \; (\vec{x}), \; \phi_{(o)}(\vec{y}) \right] \right| 0 > = 0 \qquad (13)$$

It then follows[15]

$$< 0 \left| \left[j_o^\dagger(\vec{x}), \; \partial_k \; \phi(\vec{y}) - j_k \; (\vec{y}) \right] \right| 0 > = 0 \qquad (14)$$

and assuming that $< 0 \left| \left[j_o^\dagger \; (\vec{x}), \; \phi(\vec{y}) \right] \right| 0 >$ vanishes leads to a contradiction. Thus it is impossible to assume that $< 0 \left| \left[j_k^\dagger(\vec{x}), \; \phi_{(o)}(\vec{y}) \right] \right| 0 >$ and

$< 0 \left| \left[j_o^\dagger(\vec{x}),\ \phi(\vec{y}) \right] \right| 0 >$ vanish simultaneously. This was obtained by Boulware and Deser[4,11] from detailed spectral representation arguments.

Consider now a vector-meson field $A_\mu(x)$, coupled to vector j_μ and tensor $S_{\mu\nu}$ sources. The field equations are

$$\partial^\mu F_{\mu\nu} + m_o^2 A_\nu = j_\nu \tag{15a}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - S_{\mu\nu} \tag{15b}$$

where m_o is the bare mass of the vector meson. Assuming

$$< 0 \left| \left[j_k^\dagger(\vec{x}),\ F_{\ell o}(\vec{y}) \right] \right| 0 > = 0 \tag{16a}$$

and

$$< 0 \left| \left[j_o^\dagger(\vec{x}),\ A_k(\vec{y}) \right] \right| 0 > = 0 \tag{16b}$$

we immediately obtain that $j_\mu \left| 0 > \equiv j_\mu^\dagger \right| 0 > \equiv 0$. This is so because Eq. (16a) implies

$$\begin{aligned} 0 = &< 0 \left| \left[j_k^\dagger(\vec{x}),\ \partial^\ell F_{\ell o}(\vec{y}) \right] \right| 0 > = < 0 \left| \left[j_k^\dagger(\vec{x}),\ j_o(\vec{y}) \right] \right| 0 > \\ &- m_o^2 < 0 \left| \left[j_k^\dagger(\vec{x}),\ A_o(\vec{y}) \right] \right| 0 > \end{aligned} \tag{17}$$

and (16b) implies that $< 0 \left| \left[j_k^\dagger(\vec{x}),\ A_o(\vec{y}) \right] \right| 0 > = 0$ (by arguments similar to those in Footnote 15).

In case that only (16b) holds we get

$$< 0 \left| \left[j_k^\dagger(\vec{x}),\ \partial^\ell F_{\ell o}(\vec{y}) \right] \right| 0 > = < 0 \left| \left[j_k^\dagger(\vec{x}),\ j_o(\vec{y}) \right] \right| 0 > \neq 0 \tag{18}$$

which may serve to determine the form of the dependence of $j_k(\vec{x})$ on $A_\ell(\vec{y})$.[11]

Another application is to theories where a relation of the type $\partial^\mu j_\mu(x) = \chi(x)$ holds.[11] Using Eq. (6) we get

$$<0\left|\left[\chi(x),\ j_o^\dagger(y)\right]\right|0> \;=\; <0\left|\left[\Box\,\phi(x),\ \partial_o\,\phi^\dagger(y)\right]\right|0>$$
$$= -\,i\left\{<0\left|\phi(x)P^2\,H\,\phi^\dagger(y)\right|0> + <0\left|\phi^\dagger(y)P^2H\,\phi(x)\right|0>\right\} \tag{19}$$

Thus

$$<0\left|\left[\chi(\vec{x}),\ j_o^\dagger(\vec{y})\right]\right|0> = 0 \tag{20}$$

implies

$$\chi(x)\,\big|0> \;=\; \chi^\dagger(x)\,\big|0> \;=\; 0 \tag{21}$$

In a local theory this also implies $\chi(x) \equiv 0$.[14]

III. THE TENSOR CASE

Consider the antisymmetric tensor $S_{\mu\nu} = -S_{\nu\mu}$. Let us define

$$V_\nu(x) = \Box^{-1}\,\partial^\mu S_{\mu\nu}(x) \tag{22}$$

Then

$$S_{\mu\nu}(x) = \widetilde{S}_{\mu\nu}(x) + \left(\partial_\mu V_\nu(x) - \partial_\nu V_\mu(x)\right) \tag{23}$$

where

$$\widetilde{S}_{\mu\nu} = -\,\widetilde{S}_{\nu\mu} \tag{24a}$$

and

$$\partial^\mu V_\mu(x) = 0 \qquad\qquad \partial^\mu \widetilde{S}_{\mu\nu}(x) = 0 \tag{24b}$$

Now,

$$\sum_m \langle 0 \left| \left[S_{km}(x),\ S^\dagger_{om}(y) \right] \right| 0 \rangle \quad =$$

$$= \sum_m \langle 0 \left| \left[\tilde{S}_{km}(x),\ \tilde{S}^\dagger_{om}(y) \right] \right| 0 \rangle - \langle 0 \left| \left[\partial_k V_\nu (x) - \partial_\nu V_k (x),\ \partial_o V^{\nu\dagger}(y) - \partial^\nu V^\dagger_o(y) \right] \right| 0 \rangle$$

$$- \langle 0 \left| \left[\tilde{S}_{k\nu} (x),\ \partial_o V^{\nu\dagger}(y) - \partial^\nu V^\dagger_o (y) \right] \right| 0 \rangle - \langle 0 \left| \left[\partial_k V_\nu (x) - \partial_\nu V_k(x), \tilde{S}^{\dagger\nu}_o (y) \right] \right| 0 \rangle$$

$$= \sum_m \langle 0 \left| \left[\tilde{S}_{km}(x),\ \tilde{S}^\dagger_{om} (y) \right] \right| 0 \rangle$$

$$- \langle 0 \left| \left[\partial_k V_\nu (x),\ \partial_o V^{\nu\dagger}(y) \right] \right| 0 \rangle - \langle 0 \left| \left[\partial_\nu V_k(x),\ \partial^\nu V^\dagger_o(y) \right] \right| 0 \rangle \tag{25}$$

where we have used $\langle 0 \left| \left[\tilde{S}_{\mu\nu} (x),\ V^{\nu\dagger}(y) \right] \right| 0 \rangle = 0$.[16] From

$$\langle 0 \left| \left[V_\mu (x),\ V^\dagger_\nu (y) \right] \right| 0 \rangle = \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\Box} \right) G(x-y) \tag{26}$$

we obtain

$$\langle 0 \left| \left[\partial_k V_\nu (x), \partial_o V^{\nu\dagger} (y) \right] \right| 0 \rangle = -3 \langle 0 \left| \left[\partial_\nu V_k(x),\ \partial^\nu V^\dagger_o (y) \right] \right| 0 \rangle \tag{27}$$

and thus, combined with (25),

$$\sum_m < 0 \left| \left[\partial^k S_{km}(x), S^\dagger_{om}(y) \right] \right| 0 > =$$

$$= \sum_m < 0 \left| \left[\tilde{S}_{om}(x), \partial^o \tilde{S}^\dagger_{om}(y) \right] \right| 0 > - 2 < 0 \left| \left[\Box V_o(x), \partial^o V^\dagger_o(y) \right] \right| 0 >$$

$$= i \left\{ \sum_m \left[< 0 \left| \tilde{S}_{om}(x) H \tilde{S}^\dagger_{om}(y) \right| 0 > + < 0 \left| \tilde{S}^\dagger_{om}(y) H \tilde{S}_{om}(x) \right| 0 > \right] + \right.$$

$$\left. + 2 \left[< 0 \left| V_o(x) P^2 H V^\dagger_o(y) \right| 0 > + < 0 \left| V^\dagger_o(y) P^2 H V_o(x) \right| 0 > \right] \right\} \tag{28}$$

Thus

$$< 0 \left| \left[S_{km}(\vec{x}), S^\dagger_{o\ell}(\vec{y}) \right] \right| 0 > = 0 \tag{29}$$

implies

$$S_{\mu\nu} \left| 0 > \equiv S^\dagger_{\mu\nu} \right| 0 > \equiv 0 \tag{30}$$

by arguments similar to those following Eq. (10). Again, locality implies also $S_{\mu\nu} \equiv 0$.[14]

Returning now to the vector meson field A_μ of the former section, we can immediately show that it is impossible to have

$$< 0 \left| \left[S^\dagger_{o\ell}(\vec{x}), A_m(\vec{y}) \right] \right| 0 > = 0 \tag{31a}$$

and

$$< 0 \left| \left[S^\dagger_{m\ell}(\vec{x}), F_{o\ell}(\vec{y}) \right] \right| 0 > = 0 \tag{31b}$$

simultaneously. For if it were so, then we would get

$$< 0 \left| \left[S^{\dagger}_{m\ell} \; (\vec{x}), \; S_{o\ell}(\vec{y}) \right] \right| 0 > \; = \; < 0 \left| \left[S^{\dagger}_{m\ell} \; (\vec{x}), \; \partial_o A_\ell(\vec{y}) \; - \; \partial_\ell A_o(\vec{y}) \right] \right| 0 > \qquad (32)$$

However, since

$$< 0 \left| \left[S^{\dagger}_{\mu\nu} \, (x), \; A_\lambda \; (y) \right] \right| 0 > \; = (g_{\mu\lambda} \, \partial_\nu \; - g_{\nu\lambda} \, \partial_\mu) \; R_1 \; (x-y) \, + \, \epsilon_{\mu\nu\lambda\sigma} \partial^\sigma R_2(x-y) \qquad (33)$$

we get from (31a),

$$\left[\partial_o \; R_1 \; (x) \right]_{x_o = o} = 0 \qquad\qquad \partial_r \; R_2 \; (\vec{x}) = 0 \qquad (34)$$

Eqs. (33) and (34) together imply that the right hand side of (32) is zero, thus obtaining a contradiction, unless (30) holds.

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REFERENCES AND FOOTNOTES

1. J. Schwinger, Phys. Rev. Letters 3, 296 (1959).

2. We are not concerned here with the operator properties of this equal-time commutator. In particular, $\left[j_0(x),\ j_k(y)\right]$ may need a smearing in the relative time $x_0 - y_0$, as well as in the space coordinates, in order to yield an operator.

3. S. L. Brown, private communication (in 1962) to D. Boulware and S. Deser (see Reference 4).

4. D. Boulware and S. Deser, Physics Letters 22, 99 (1966).

5. F. Csikor and G. Pocsik, Nuovo Cimento 42, A413 (1966).

6. J. D. Bjorken, Phys. Rev. 148, 1467 (1966). See footnote 7 there.

7. S. Okubo, Nuovo Cimento 44, A1015 (1966).

8. G. Källén, Helv. Phys. Acta. 25, 417 (1952).

9. H. Lehmann, Nuovo Cimento 11, 342 (1954).

10. Okubo (in Nuovo Cimento 19, 574 (1961)) pointed out that Källén-Lehmann spectral representations for two-point functions may not exist even in certain local field theories. However, A. M. Jaffe informed me that such representations do exist also in the cases considered by Okubo in the framework of "Strictly Localizable Fields." See A. M. Jaffe, "High Energy Behaviour in Quantum Field Theory," SLAC preprint.

11. D. G. Boulware and S. Deser, Phys. Rev. 151, 1278 (1966).

12. K. Johnson, Nucl. Phys. 25, 431 (1961).

13. Since we assume that there are no massless particles in our theory and since $\langle 0 \left| \partial^\mu j_\mu(x) \right| 0 \rangle = 0$, $\phi(x) \left| 0 \right\rangle$ is well defined, and this is all we need.

14. P. Federbush and K. Johnson, Phys. Rev. 120, 1926 (1960).

15. This is easily derived from (13) and

$$<0 \left| \left[j^{\dagger}_{\mu}(x), \phi_{(\nu)}(y) \right] \right| 0> = g_{\mu\nu} F_1(x-y) - \partial_{\mu} \partial_{\nu} F_2(x-y)$$

where F_1 and F_2 are invariant under proper Lorentz transformations and $\phi_{(\nu)}(y) = \partial_{\nu}\phi(y) - j_{\nu}(y)$.

16. This follows from

$$<0 \left| \left[\widetilde{S}_{\mu\nu}(x), V^{\dagger}_{\lambda}(y) \right] \right| 0> = \epsilon_{\mu\nu\lambda\alpha} \partial^{\alpha} H(x-y)$$

where $H(x)$ is invariant under proper Lorentz transformations. We have used Eqs. (24a)-(24b) to derive this form.