SLAC-PUB-250

HIGH ENERGY BEHAVIOR OF LOCAL QUANTUM FIELDS*

Arthur M. Jaffe†

Stanford Linear Accelerator Center and Department of Mathematics
Stanford University, Stanford, California 94305

*Presented to the American Physical Society, December 29, 1966, at Stanford, California.

I. INTRODUCTION

The studies of the past years have yielded many insights into the foundations and the dynamics of quantum field theory. Two main avenues have been followed in discovering these results. On the one hand, people have done many calculations, such as the perturbation theory study of the magnetic moment of the electron, or such as the current commutator derivations of sum rules. On the other hand, some people have followed Wightman to look at the basic structure of quantum field theory.

These people who study the foundations of quantum field theory have always made two simplifying assumptions, the first at low energy, the second at high energy. At low energy, it was always assumed that no zero mass particles enter the theory; photons and neutrinos must be given a small mass. At high energies, it was always assumed that the off-mass-shell amplitudes remain polynomially bounded.

There are many reasons to believe that the assumption about polynomial high energy behavior rules out from the start any consistent theory of weak interactions, many models of strong interactions, or any field theory involving a non-renormalizable coupling.

On the other hand, only on the basis of these two simplifying assumptions did people get through to the construction of an S-matrix, and to the proof of dispersion relations for two particle scattering.

The work which I shall describe fills the second gap in technique. It shows how to deal with quantum field theories that have singular high-energy behavior.

We shall define the notion of a strictly localizable field,[1-5] and show that it is possible to work out the general principles of field theory for any field which is localizable in that sense. In other words we put these fields on an equal footing with the ones usually studied.

Furthermore we derive a useful high-energy bound from strict localizability. In general, the off-mass-shell amplitudes corresponding to a strictly localizable field, need not be polynomially bounded. However it is possible that the on-mass-shell scattering amplitudes have analytic continuations which do have the property of polynomial boundedness.

II. BASIC REQUIREMENTS ON A FIELD

Let us start with a review of five basic, physically motivated requirements on a field theory.[6,7] The first three of these are:

(1) The State Space is a Hilbert Space.

(2) The Fields are Covariant under Lorentz Transformations and under Space-Time Translations. For a scalar field, the transformation law has the simple form

$$U(a,\Lambda)\,A(x)\;U(a,\Lambda)^{-1} = A(\Lambda x + a)$$

where $U(a,\Lambda)$ is a unitary representation of the inhomogeneous Lorentz group.

(3) The Energy Spectrum is Positive

$$U(a,1) = e^{iP^{\mu} a_{\mu}} ,$$

where the spectrum of the energy-momentum operator P^{μ} lies in the forward light cone. There is a unique ground state ψ_o, the vacuum vector, which has zero energy-momentum

$$P^{\mu}\,\psi_o = 0 .$$

III. A(x) IS NOT AN OPERATOR

Let us pause at this point to remark that such a field A(x) can not be a field of operators. That fact, which was first noticed by Wightman,[8] is an easy consequence of the Källén-Lehmann spectral representation for the vacuum expectation value of the product of two fields.[9-11] This spectral representation in turn rests on these three assumptions.

Assuming that $\left(\psi_o, A(x)\,\psi_o\right) = 0$, we have

$$\left(\psi_o, A(x)\, A(y)\, \psi_o\right) = \int_o^\infty dM^2 \int d^4p\, \rho(M^2)\, \delta(p^2-M^2)\, \theta(p_o)\, e^{-ip(x-y)} \quad .$$

Here

$$\rho(M^2) \text{ is positive.}$$

On the other hand, let us compute the length of the vector got by applying A(x) to the vacuum,

$$\| A(x)\, \psi_o \|^2 = \left(\psi_o, A(x)\, A(x)\, \psi_o\right) .$$

By the spectral representation this equals

$$= \int_o^\infty dM^2\, \rho(M^2) \int d\vec{p}\, \frac{1}{2(\vec{p}^{\,2} + M^2)^{1/2}} \quad .$$

Since the volume of each mass hyperboloid is infinite, $A(x)\,\psi_o$ can be a vector only in the trivial case that $\rho(M^2) = 0$.

In other words, a field can not assign an operator to each space-time point. Rather it must be averaged over some space-time region with a smooth function in order to yield an operator.

IV. STRICT LOCALIZABILITY

We now want to talk about the notion of a localizable field. Suppose that a field can be averaged over an arbitrarily small region in the neighborhood of each point x. Then we can recover all the information about the field at the point x. It then makes sense to write A(x) as a local function of x, and, for instance, to write a local equation of motion for A(x).

We will always require that a field can be localized to the neighborhood of any point, and we call such fields strictly localizable fields, or SLF's. Let us now continue to list the basic requirements for fields:

(4) Strict Localizability and Locality.

We assume that A(x) is an SLF, so it can be averaged over an arbitrarily small region around any point. Furthermore it is local, so that

$$[A(x),\ A(y)] = 0, \text{ if } (x-y)^2 < 0 .$$

(5) Particle Interpretation.

The field theory can be connected to physical experiments.

V. HIGH ENERGY BOUNDS

The simplest consequence of strict localizability is that strict localizability imposes a high energy bound on the growth of the momentum space amplitudes of the field.[3] Let us first recall the usual polynomial bound and compare that bound with the bound arising from strict localizability.

The usual assumption is that there is an integer M such that

$$\frac{\left(\psi_o,\ \tilde{A}(p_1)\ \tilde{A}(p_2)\ .\ .\ .\ \tilde{A}(p_n)\psi_o\right)}{(1+\|p\|^2)^{Mn}}$$

is bounded in the momenta. Here $\tilde{A}(p)$ is the Fourier transform of $A(x)$ and

$$\|p\|^2 = \sum_{j=1}^{n} \left\{ (p_{jo})^2 + (\vec{p}_j)^2 \right\} ,$$

is the sum of the squares of all the energy-momentum components. The energy-momentum vectors range over all space, and are not necessarily restricted to the mass shell.

In contrast with this set-up, the requirement of strict localizability allows for a much wider class of high energy behavior. The polynomial growth can be replaced by an entire function of $\|p\|^2$. Now we only ask that

$$\frac{\left(\psi_o,\ \tilde{A}(p_1)\ .\ .\ .\ \tilde{A}(p_n)\ \psi_o\right)}{\left[g(\|p\|^2)\right]^n}$$

is bounded in the momenta, where g is an entire function with positive power series coefficients,

$$g(t^2) = \sum_{r=0}^{\infty} c_{2r}\, t^{2r},\quad c_{2r} \geq 0.$$

The requirement of strict localizability is exactly equivalent to the fact that $g(t^2)$ satisfies the bound

$$\int_1^\infty \frac{\log g(t^2)}{t^2}\, dt < \infty .$$

What does this bound mean? Some examples of acceptable high energy behavior are

$$g(\|p\|^2) = (1+\|p\|^2)^M, \qquad \text{(the usual case)},$$

$$g(\|p\|^2) \sim \exp\left\{\|p\|/(\log \|p\|)^{1+\epsilon}\right\},$$

and

$$g(\|p\|^2) \sim \exp\left\{\|p\|/\log \|p\| (\log\log \|p\|)^{1+\epsilon}\right\}.$$

However a growth as fast as

$$g(\|p\|^2) \sim \exp\left\{\|p\|/\log \|p\|\right\}$$

is not strictly localizable.

The question now arises whether it is possible to carry through the general investigation of strictly localizable fields, and the answer is yes. Mathematically, they may appear difficult to handle, since such fields need not be distributions in the sense of Schwartz. However, they can be given a precise mathematical meaning as generalized functions. Let us now describe some results of this investigation to show that SLF's are on an equal footing with the usual fields.

VI. VACUUM EXPECTATION VALUES

First we mention that instead of characterizing the SLF's by their behavior in momentum space, one can look in configuration space. The configuration space vacuum expectation values are boundary values of analytic functions, and these functions are analytic in the usual domain of the difference variables.[7,5] From this analyticity we conclude that the TCP theorem and the spin and statistics theorem hold for the strictly local field theories. Furthermore it is possible to look at the local singularities which can occur in the configuration space analytic functions as you approach real points. A special feature of strictly localizable fields is that essential singularities can occur in the vacuum expectation values. The possible essential singularities can be classified in terms of the indicator function $g(\|p\|^2)$, which we introduced to characterize the momentum space growth of the vacuum expectation values.[4]

VII. SCATTERING MATRIX

The second point of interest is that we can start from a strictly localizable field and construct the S-matrix. There are two methods, namely the Haag-Ruelle scheme which allows the construction directly from the vacuum expectation values,[12] or the L.S.Z. reduction formulae which works in terms of time-ordered or retarded products. These L.S.Z. reduction formulae can be proved so long as no two asymptotic velocities associated with the scattered particles are equal.[13] The constructions are based on two results which were already known in the case of polynomial high energy behavior, and these results have now been proved for strictly localizable fields. The first property is the space-like cluster property of the configuration space vacuum expectation values. The cluster property makes precise the short range nature of the forces which arise from a theory which involves no zero mass particles. The second feature used in the proofs are some time-like cluster properties which can be proved when the fields are averaged with wave packets associated with non-overlapping regions of velocity space. The final formula for the S-matrix is a familiar one in form:

$$< \vec{p}_1 \ldots \vec{p}_n^{\,out} \left| \vec{q}_1 \ldots \vec{q}_m^{\,in} \right. > = (-i)^{n+m} \Pi \left(p_i^2 - m_i^2\right) \left(q_j^2 - m_j^2\right) < T\left(p_1, \ldots, p_n, -q_1, \ldots, -q_m\right) > \Bigg|_{\substack{p_{io} = \omega_i \\ q_{jo} = \omega_j}}$$

However, $<T\left(p_1, \ldots, p_n\right)>$ is not the Fourier transform of the time-ordered vacuum expectation value of n-fields, and in general it is unknown how to define the time-ordered product starting from the vacuum expectation values. The reason is that the θ functions which do the time ordering are ill-defined at exactly those points where the vacuum expectation values they multiply are singular. It is here that one would expect trouble from an "infinite number of subtractions."

On the other hand it is unnecessary to define sharp time-ordered products. A set of smooth time-ordered products are both well-defined and yield the correct S-matrix on the mass shell. The smooth, time-ordered, vacuum expectation values, the $< T(p_1, \ldots, p_n)>$'s which appear in the reduction formulae, are got by replacing each product of theta functions such as $\theta(t_1 - t_2) \ldots \theta(t_{n-1} - t_n)$, by a regularized product:

$$\int \theta(t_1 - s_1 - t_2 + s_2) \ldots \theta(t_{n-1} - s_n - t_n + s_n)\, f(s_1 - s_2, \ldots, s_{n-1} - s_n)\, d(s_1 - s_2) \ldots d(s_{n-1} - s_n) .$$

Here $f(\xi_1, \ldots, \xi_{n-1})$ is a suitable, smooth, symmetric, positive function of $(n-1)$ variables with total integral equal to one. Such an f does not affect the on-mass-shell S-matrix, but only influences its off-mass-shell extrapolations.

VIII. SHARP TIME ORDERING

However, since these formulae do not arise from sharp time-ordered functions, we can not expect that they will lead to the desired momentum space analyticity properties. This problem can be overcome for the propagator, for the vertex, and for the absorptive part of the two-particle scattering amplitude. The method for the vertex and absorptive part is to prove a Jost-Lehmann-Dyson representation for a strictly local field theory. The explicit representation allows the insertion of a convergence factor, which allows a mathematically precise definition of the necessary retarded functions, and this results in the analyticity. Again this is a method known in the case of tempered fields.

IX. CONVERGENCE FACTOR IN THE PROPAGATOR

Before proceeding to the vertex, let us see how exactly the same procedure can be used to put in a convergence factor into the propagator.[14] Using the Källén-Lehmann representation, we can write the Fourier transform of the vacuum expectation value of the unordered commutator of an SLF in the form

$$f(q) = \int e^{iqx} \left(\psi_o, \left[A(\tfrac{x}{2}), A(-\tfrac{x}{2})\right]\psi_o\right) dx$$

$$= \int_o^\infty dM^2 \; \rho(M^2) \; \delta(q^2-M^2) \; \epsilon(q_o),$$

where

$$\int_o^\infty \frac{\rho(M^2)}{g(M^2)} \; dM^2 < \infty .$$

Here $g(M^2)$ is the square of an acceptable indicator function for the field $A(x)$.

Using the Källén-Lehmann spectral representation, we can give a <u>definition</u> of the expectation value of the retarded commutator, $f^{Ret}(q)$, by replacing $\delta(q^2-M^2)\;\epsilon(q_o)$ with $\left[(q_o + i\epsilon)^2 - \vec{q}^{\,2} - M^2\right]^{-1}$ and by inserting a convergence factor in the resulting integral. This yields

$$f^{Ret}(q) = \lim_{\epsilon \to o} \int_o^\infty dM^2 \; \frac{g(q^2)}{g(M^2)} \; \rho(M^2) \; \frac{1}{(q_o + i\epsilon)^2 - \vec{q}^{\,2} - M^2} \; .$$

This definition is acceptable for the following reasons:[5]

(1) The Fourier transform of $f^{Ret}(q)$ vanishes outside the forward light cone, as every retarded commutator should. In order to ensure that result, it is essential to define $f^{Ret}(q)$ with the aid of a convergence factor $g(q^2)/g(M^2)$ such

that $g(M^2)$ has the following properties:

(a) $g(M^2)$ is an entire function of M^2,

(b) $g(M^2)$ has no zeros for $M^2 \geq 0$, and

(c) $g(M^2)$ grows for large M^2 no faster than an acceptable growth indicator function for an SLF.

However we have originally required the growth-indicator function g to have these properties; it is an entire function with positive power series coefficients, and an appropriate high energy growth. Therefore we have made the convenient, and perfectly acceptable choice of using a power of the growth indicator function itself to make the subtractions in $f^{Ret}(q)$.

(2) Inside the forward light cone, the Fourier transform of $f^{Ret}(q)$ agrees with the Fourier transform of $f(q)$, the unordered commutator.

(3) In fact, the only way in which the Fourier transform of $f^{Ret}(q)$ depends on the indicator function $g(M^2)$ is at the point $x = 0$. We can see that by looking at the difference between two possible choices of the indicator function g. The difference between the two corresponding $f^{Ret}(q)$'s

$$f_1^{Ret}(q) - f_2^{Ret}(q) = \int_0^\infty dM^2 \ \rho(M^2) \left[\frac{g_2(M^2)\, g_1(q^2) - g_1(q^2)\, g_2(M^2)}{g_1(M^2)\, g_2(M^2)\, (q^2 - M^2)} \right] = m(q^2) \ ,$$

is an entire function of q^2, which we call $m(q^2)$. Thus the ambiguity in the configuration space propagator has the form

$$m(-\Box)\ \delta(x),$$

and this generalized function can be shown to be localized at $x = 0$.

(4) The retarded function $f^{Ret}(q)$ is analytic in the cut q^2 plane, as it should be. This property again results from the fact that $g(q^2)$ is entire.

(5) Furthermore, since the ambiguity, $m(q^2)$, is an entire function, every definition of $f^{Ret}(q)$ will have the same residue at the one particle pole.

The last fact is the key to using the Jost-Lehmann-Dyson representation to define form factors and scattering amplitudes. The off-mass-shell vertex or absorptive part will depend on the convergence factor used; however, the physical, on-mass-shell amplitudes are unambiguous.

X. JOST-LEHMANN-DYSON REPRESENTATION

Let us now proceed to the Jost-Lehmann-Dyson representation.[15-18] Consider the matrix elements of the commutator of an SLF between two eigenstates of the energy-momentum.

$$\tilde{f}(P,H,x) = \left< P \left| \left[A(\tfrac{x}{2}),\ A(-\tfrac{x}{2})\right] \right| H \right>.$$

For this generalization of the Källén-Lehmann representation, we must expect a somewhat more complicated integral representation. However it is quite similar in form to the case in which $|P>$ and $|H>$ are the vacuum ψ_o. The Fourier transform of $\tilde{f}(P,H,x)$ can be written as

$$f(P,H,q) = \int d\vec{u} \int dM^2\, \epsilon(q_o)\, \delta\left(q_o^2 - (\vec{q}-\vec{u})^2 - M^2\right) \left\{ \phi_1 + q_o\, \phi_2 \right\},$$

where

$$\phi_j = \phi_j\,(P,H,\vec{u},\ M^2).$$

In this case, the ϕ_j are not necessarily distributions, but they are strictly-localizable generalized functions, in an appropriate sense. Furthermore, they have the property that if $g(\|p\|^2)$ is the square of an indicator function characterizing the growth of the momentum space field, then

$$\frac{\phi_j\,(P,H,\vec{u},\ M^2)}{\left[g\,(M^2)\right]}$$

is a strictly-localizable generalized function which has compact support in $\vec{u}$ and is bounded in M^2. Therefore we can <u>define</u> a matrix element of a retarded commutator exactly as we defined the propagator.

$$f^{Ret}(P,H,q) = \int d\vec{u} \int dM^2 \; \frac{g\left(q_o^2 - (\vec{q}-\vec{u})^2\right)}{g(M^2)} \; \frac{\{\phi_1 + q_o\,\phi_2\}}{(q_o+i\epsilon)^2-(\vec{q}-\vec{u})^2-M^2}$$

This formula not only defines $f^{Ret}(P,H,q)$, but it also gives a high energy bound on its growth for large q. Namely, we see from the explicit representation that

$$\frac{f^{Ret}(P,H,q)}{\left[g(2\,\|q\|^2)\right]} \quad \text{is bounded in } q\,.$$

XI. AMBIGUITIES IN THE PHYSICAL AMPLITUDES

Secondly, it is clear that different choices of the indicator function g give different convergence factors. These different choices for the subtractions will yield different commutators. Nevertheless, this ambiguity will disappear in the form factor or in the S-matrix elements that we define with $f^{\mathrm{Ret.}}$ The argument to show this is similar to the one we just made with the propagator. In the case of the propagator, the residue at the one particle pole was unaffected by the choice of subtractions. Now, two different choices for the indicator function will yield retarded commutators which differ by an entire function in the components of q. However, the reduction formulae which relate the retarded commutators to the physical matrix elements are always accompanied by factors of $(p^2 - m^2)$ for each of the particles reduced. But $(p^2 - m^2)$ times an entire function vanishes on the mass shell, so the physical amplitudes are unambiguous.

XII. APPLICATIONS

Let us consider an example of an application.[19] The form factor $f(q^2)$ is related to the scalar vertex by

$$f(q^2)\,\delta(P+q-H) = \langle P \,|\, \tilde{J}(q) \,|\, H \rangle \;.$$

We can derive a reduction formula which relates the vertex to the matrix element of the retarded commutator of two fields. Using the Jost-Lehmann-Dyson representation, we can show that $f(q^2)$ is analytic in the region of the q^2 plane shown in Fig. 1, namely the cut q^2 plane with some possible complex singularities confined to a finite circle. Furthermore, the representation shows that $f(q^2)$ is bounded in its analyticity domain by $g(\|q^2\|)$, namely

$$\left| f(q^2) \right| \le C g(\| q \|^2) \quad .$$

This analyticity and upper bound on the form factor lead to a lower bound on its decay. This is related to the fact that an analytic function whose envelope decays rapidly must grow rapidly in some other direction. By using precise results of this form, we get two lower bounds on the decay of $f(q^2)$, one for space-like momentum transfer, and the other for time-like momentum transfer.

$$\text{envelope } \left| f(q^2) \right| > c e^{-a\sqrt{-q^2}} \quad , \quad q^2 \longrightarrow -\infty \quad ,$$

and

$$\text{envelope } \left| f(q^2) \right| > \frac{M}{g(q^2)} \quad , \quad q^2 \longrightarrow +\infty \; .$$

This type of bound was discussed by André Martin on the basis of S-matrix dispersion theory.[20-21] Now these bounds are proved on the basis of the fundamental principles of quantum field theory.

Another application of the Jost-Lehmann-Dyson representation is to use a double representation[22] to define the absorptive part of the two particle scattering amplitude as a generalized function.

Once this has been done, it is possible to carry through the proof of dispersion relations for the two particle scattering amplitude. However, instead of polynomial boundedness, it is necessary in general to use an infinite number of subtractions, as characterized by an acceptable indicator function $g(\|p\|^2)$. This can be done, and the dispersion relation can be continued to the mass shell to the manner of Bogoliubov and Hepp. The final step is to capitalize on the unitarity condition and apply the methods of Martin[23] to prove that actually only a finite number of subtractions are necessary on the mass shell.

CONCLUSION

In conclusion, we have seen that it is possible to broaden the framework of a general discussion of quantum field theory to include fields which are strictly localizable, but which have faster than polynomial growth at high energy. Approximate calculations such as perturbation theory, the summation of ladder graphs, or the calculation of Bardakci and Schroer[24] lead one to expect that such behavior in field theories arising, for instance, from non-renormalizable Lagrangians. We have seen that such SLF's can be put on an equal footing with polynomially bounded fields.

There are some interesting unsolved problems associated with strictly localizable fields. In spite of their bad behavior off the mass-shell, it may be possible that all mass shell analytic continuations of the scattering amplitudes are actually polynomially bounded. This question remains open. Also it may be possible to get bounds on other cross sections of interest. For example, it is intriguing to see that the form factor bound looks quite similar to the form of the Orear fit[25] for elastic proton-proton cross sections at high transverse momentum transfer.

REFERENCES AND FOOTNOTES

†N. A. S. -N. R. C. Fellow, supported by the A. F. O. S. R. and in part by the U. S. Atomic Energy Commission.

1. A mathematically precise development of the theory, along with extensive bibliographical references can be found in papers 2-5.
2. A. M. Jaffe, Local Quantum Fields as Operator-Valued Generalized Functions, Proceedings of the M. I. T. Conference on Scattering Theory, March 1966, I. E. Segal, Editor.
3. A. M. Jaffe, High Energy Behavior in Quantum Field Theory I, Strictly Localizable Fields, Stanford preprint.
4. A. M. Jaffe, High Energy Behavior of Strictly Localizable Fields II, Mathematical Tools, Stanford preprint.
5. A. M. Jaffe, Properties of Strictly Localizable Fields, in preparation.
6. A. S. Wightman, Phys. Rev. 101, 860 (1956).
7. R. F. Streater and A. S. Wightman, TCP, Spin and Statistics and All That, W. A. Benjamin Company, New York (1964).
8. A. S. Wightman, Ann. de l'Institut Henri Poincaré I, 403 (1964).
9. H. Umezawa and S. Kamefuchi, Prog. Theor. Phys. 6, 543 (1951).
10. G. Källén, Helv. Phys. Acta 25, 417 (1952).
11. H. Lehmann, Nuovo Cimento 11, 342 (1954).
12. D. Ruelle, Helv. Phys. Acta 35, 34 (1962).
13. K. Hepp, Comm. Math. Phys. 1, 95 (1965).
14. O. Steinmann, J. Math. Phys. 4, 583 (1963).
15. R. Jost and H. Lehmann, Nuovo Cimento 5, 1598 (1957).
16. F. Dyson, Phys. Rev. 110, 1460 (1958).

17. A.S. Wightman, "Analytic Functions of Several Complex Variables" in Dispersion Relations and Elementary Particles, C. de Witt and R. Omnès, Editors (Hermann, Paris 1960).

18. V.S. Vladimirov, Methods of the Theory of Functions of Many Complex Variables (M.I.T. Press 1966).

19. A.M. Jaffe, Phys. Rev. Letters 17, 661 (1966).

20. A. Martin, Nuovo Cimento 37, 671 (1965).

21. Proton and Deuteron Form Factors are plotted in S. Drell, A. Finn, and M. Goldhaber, Electromagnetic Form Factors for Composite Particles at High Momentum Transfer, SLAC-PUB-237.

22. K. Hepp, Helv. Phys. Acta, 37, 639 (1964).

23. A. Martin, Nuovo Cimento 42, 901 (1966).

24. K. Bardakci and B. Schroer, J. Math. Phys. 7, 10 (1966).

25. J. Orear, Phys. Rev. Letters 12, 112 (1964).

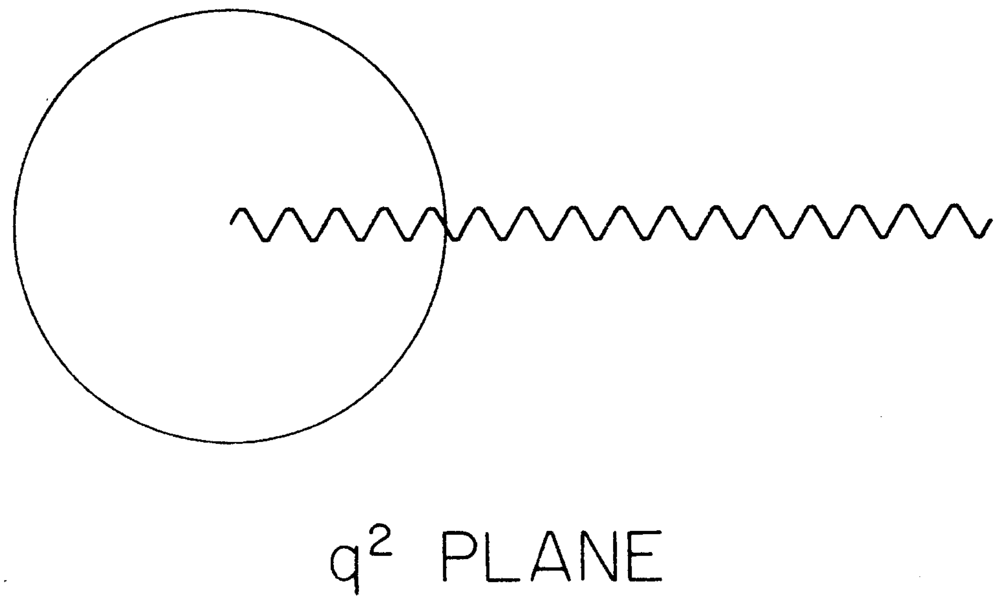

Fig. 1--Domain of analyticity of the form factor $f(q^2)$.