SLAC-PUB-246
December 1966

THE DESIGN OF LOW-BETA INSERTIONS FOR STORAGE RINGS†

P. L. Morton and J. R. Rees
Stanford Linear Accelerator Center
Stanford, California

The space-charge limit on the beam current which can be usefully stored in a colliding-beam storage ring is determined by the current density in the interaction region and, in the case of a thin, flat ribbon beam, by the value of β_y, the vertical β-function there. The quantity which is limited is the dimensionless parameter,

$$\frac{J\beta_y r_e}{\gamma} \lesssim \text{a constant} \qquad (1)$$

where J is the current density in the interaction region, r_e the classical electron radius, and γ the energy in rest-mass units. In the static "Amman-Ritson" limit, β_y appears because of its presence in frequency-shift formulae.[1] By the same token, β_y appears in coherent-instability limits when the only damping mechanism present is Landau damping by the non-linearity of the force between beams.[2] The appearance of the form (1) in the incoherent "Courant" limit is on a more speculative basis,[3] but it is probably warranted.

The luminosity is proportional to the product of the current in one beam and the current density in the other. Evidently, if the stored current is limited, as for example, by the radiofrequency power available, the current density should be increased to the space-charge limit by adjusting the beam dimensions in the interaction region. If, on the other hand, there is no limitation on beam current, the limitation on luminosity will be set by the aperture that can be filled at the interaction region. In the former case, there is a clear gain to be got from decreasing β_y in the interaction region, and, in the latter case, there may be a gain, depending on other details of the storage ring. All high-energy storage rings now contemplated will be current-limited at high energies because of the rf power required for synchrotron radiation, so it is advantageous to design the magnet lattice to produce a small value of β_y at the interaction region.

In the design of cyclic particle accelerators in the past, structures yielding abnormal β-functions, i.e., β-functions differing greatly from R/ν, have been studiously avoided; primarily because the large values of β, which tend to accompany the small values, aggravate the tolerances on the magnetic guide field, and, in an accelerator, no significant advantage accrued from abnormal values of β.

That this was not the case in storage rings, where one beam is the target for the other beam, was realized by the French storage-ring group which provided in the design of ACO for operating in a strong-focussing regime in which β_y at the interaction region is compressed to about 1/50 of its maximum value and about 1/10 of its average value. A more extreme compression of the vertical β-function is proposed to convert the Cambridge Electron Accelerator into a storage ring.[4] The CEA scientists hope to achieve a reduction factor of about a hundred in a single "bypass" or beam siding where their detector will be located.

It is possible to incorporate special sections into the guide field of any accelerator in such a way that the β-functions outside the special sections are undisturbed. Such sections are "matched" to the unperturbed structure. Collins first noted that an extra-long straight section could be inserted into a normal AGS structure and matched at a single momentum using only two quadrupoles,[5] and other workers have elaborated the idea of long straight sections.

A truly matched insertion is one which, when interposed in the base structure, does not disturb either the β-functions or the off-momentum equilibrium orbits. A straight section--one without bends of any kind--may have this property only if its radial transport matrix is the unit matrix, i.e., if it has a radial phase shift of 2π. On the other hand, a general insertion, which may include bends, has no restriction on the betatron phase shifts. It is to the design of such structures that we address ourselves here.

Suppose we have a storage ring or accelerator in which we denote the equilibrium-orbit path-length coordinate by s, the radial excursion by x and the vertical excursion by y. Let us denote two particular positions on the circumference by g and h, i.e., s = g and s = h. (See Fig. 1.)

Let the machine matrices at g, the transport matrices once around the whole machine, from g back to g, be

$$\left.\begin{aligned}
M_x &= \begin{pmatrix} \cos\mu_x+\alpha_x(g)\sin\mu_x & \beta_x(g)\sin\mu_x & M_{13}(g) \\ -\gamma_x(g)\sin\mu_x & \cos\mu_x-\alpha_x(g)\sin\mu_x & M_{23}(g) \\ 0 & 0 & 1 \end{pmatrix} \\
M_y &= \begin{pmatrix} \cos\mu_y+\alpha_y(g)\sin\mu_y & \beta_y(g)\sin\mu_y \\ -\gamma_y(g)\sin\mu_y & \cos\mu_y-\alpha_y(g)\sin\mu_y \end{pmatrix} \\
\text{where} \quad \gamma &= \frac{1+\alpha^2(g)}{\beta(g)} \\
M_{13}(g) &= \eta(g)(1-\cos\mu_x-\alpha_x(g)\sin\mu_x) - \eta'(g)\beta_x(g)\sin\mu_x \\
M_{23}(g) &= \eta(g)\gamma_x(g)\sin\mu_x + \eta'(g)(1-\cos\mu_x+\alpha_x(g)\sin\mu_x)
\end{aligned}\right\} \quad (1)$$

$\eta(s)$ is the function describing the displaced equilibrium orbit for an off-momentum particle.*

The transport matrices from any point g to any other point h are given by the matrices $R^x(g,h)$, a 3×3 matrix, and $R^y(g,h)$, a 2×2 matrix.

† Work supported by the U. S. Atomic Energy Commission.

* The vector operated on by M_x is $(x, x', \Delta p/p)$; that operated on by M_y is (y, y'). The notation is largely that of Courant and Snyder, Annals of Phys. 3, 1 (1958).

(1967 U.S. National Particle Accelerator Conference, Washington, D.C., March 1-3, 1967)

$$
\left.\begin{aligned}
R_{11}^{x,y} &= \sqrt{\frac{\beta_{x,y}(h)}{\beta_{x,y}(g)}}\left[\cos\psi_{x,y} + \alpha_{x,y}(g)\sin\psi_{x,y}\right] \\
R_{12}^{x,y} &= \sqrt{\beta_{x,y}(g)\,\beta_{x,y}(h)}\;\sin\psi_{x,y} \\
R_{21}^{x,y} &= \frac{-1}{\sqrt{\beta_{x,y}(g)\,\beta_{x,y}(h)}}\left\{\left[1+\alpha_{x,y}(g)\,\alpha_{x,y}(h)\right]\sin\psi_{x,y} \right. \\
&\qquad\qquad \left. + \left[\alpha_{x,y}(g) - \alpha_{x,y}(h)\right]\cos\psi_{x,y}\right\} \\
R_{22}^{x,y} &= \sqrt{\frac{\beta_{x,y}(g)}{\beta_{x,y}(h)}}\left[\cos\psi_{x,y} - \alpha_{x,y}(h)\sin\psi_{x,y}\right] \\
R_{13}^{x} &= \eta(h) - R_{11}^{x}\,\eta(g) - R_{12}\,\eta'(g) \\
R_{23}^{x} &= \eta'(h) - R_{21}^{x}\,\eta(g) - R_{22}^{x}\,\eta'(g) \\
R_{33}^{x} &= 1
\end{aligned}\right\} \quad (2)
$$

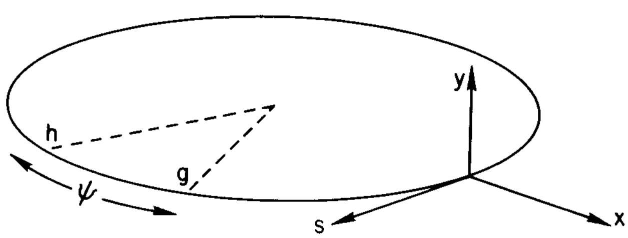

Fig. 1--Coordinate System

The angles ψ_x and ψ_y are the betatron phase shifts in the radial and vertical coordinates respectively through the section of the orbit between g and h.

These expressions, Eq. (2), may be used as recipes for designing sections of guide field giving desired orbit properties. For example, if we start with a uniform periodic structure such as that of a conventional alternating gradient synchrotron, we may design "insertions" which do not perturb the orbit properties outside themselves, or in other words, matched insertions. We can "pull apart" the base structure at some point and interpose a special section as shown in Fig. 2. In this case, Eq. (2) takes on an especially simple form. If the momentum terms are ignored, the recipe for a Collins straight section results. Since phase shifts ψ_x and ψ_y are added, the betatron frequencies are shifted by $\psi_x/2\pi$ and $\psi_y/2\pi$.

For a high-luminosity storage ring, an insertion is needed with very small values of one or both of the beta functions at the place where the beams collide. Referring to Fig. 2, Eq. (2) applies between g and h and again between h and g'; thus, the recipe involves the properties of the base structure (at g and g') and those desired at the collision region, h. It is worth noting that the phase shifts ψ_x and ψ_y are inherently free parameters in achieving desired values of the β's and α's at the collision region. Their values affect the total betatron frequencies, of course, and, consequently, are not wholly unrestricted; but, outside of narrow excluded regions, they may be chosen arbitrarily.

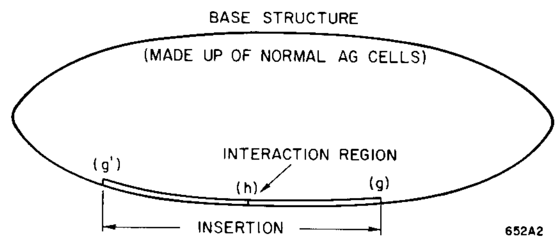

Fig. 2

In the SLAC 3-GeV electron-positron storage ring, the base structure was chosen symmetric about the insertion, so the insertion is also symmetric about the collision region. Thus, the problem of design is simplified to that of synthesizing a transport system from g to h giving the desired properties at h.

Two approaches were made to the problem of synthesis: First, we used pencils and paper and thin-lens formulae, and, second, using the guidance thus obtained, we used the SLAC-developed beam-transport design computer code TRANSPORT to obtain final fits and exact properties.

Pencil-And-Paper Approach

To simplify the problem, we arbitrarily separated the problems of matching the momentum function, η, and that of obtaining the desired properties in terms of the transverse coordinates.

Figure 3 shows the two sections of the insertion. In the momentum matching section between g and i, we ignored the transverse-coordinate transformation properties, concentrating on reducing both η and η' to zero at i so that they remain zero all the way to h. This done, any combination of lenses (or any elements with no magnetic field on their axes) may be used between i and h without affecting η anywhere.

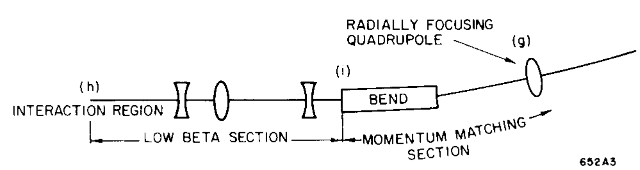

Fig. 3

Momentum Matching Section

Consider a particle at g with no betatron oscillations. Its state of motion is specified by the vector

$$x_g = \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} \frac{\Delta p}{p} \tag{3}$$

We require

$$\begin{pmatrix} R^x_{11} & R^x_{12} & R^x_{13} \\ R^x_{21} & R^x_{22} & R^x_{23} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \eta \\ \eta' \\ 1 \end{pmatrix} = \begin{pmatrix} R^x_{11}\eta + R^x_{12}\eta' + R^x_{13} \\ R^x_{21}\eta + R^x_{22}\eta' + R^x_{23} \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4}$$

where the first matrix is the transport matrix from g to i. If both R_{13} and R_{23} were zero, Eq. (4) could not be satisfied, because

$$\begin{matrix} R^x_{11}\eta + R^x_{12}\eta' = 0 \\ R^x_{21}\eta + R^x_{22}\eta' = 0 \end{matrix} \quad \text{and} \quad \begin{vmatrix} R^x_{11} & R^x_{12} \\ R^x_{21} & R^x_{22} \end{vmatrix} = 1$$

have only the trivial solution $\eta = \eta' = 0$. The same argument applies to any interval of the structure. Thus η must go to zero in a bending magnet (i.e., a magnet in which M_{13} and M_{23} are not zero), and, moreover, η and η' must reach zero at the exit from a bending magnet if they are to remain zero in the following drift.

For the case of an $n = 0$ magnet with bending angle θ, a drift, and a lens, (see Fig. 4) it is easy to see that the drift distance, s, and the focal length, f can be adjusted to satisfy

$$\theta = -\eta' + \frac{\eta}{f}$$
$$s = \frac{\eta}{\theta} - \frac{\ell}{2} \tag{5}$$

with the desired result. (See Fig. 4.)

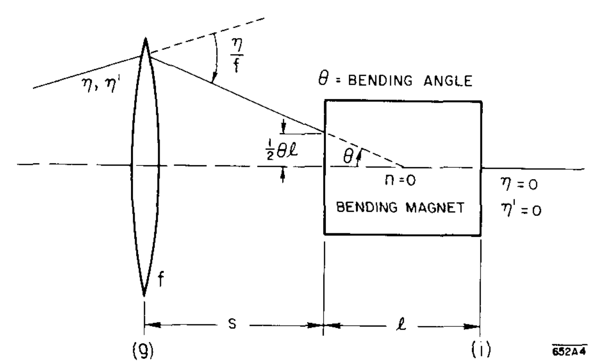

Fig. 4--The Momentum Matching Section

Low-Beta Section

In the low-beta section, there are no bends, so we may drop the last column and row of the radial matrices.

In reducing η to zero, we paid no attention to the betatron functions. We now have the task of tailoring a section to transform the given functions at i, $\beta_x(i)$, $\beta_y(i)$, $\alpha_x(i)$, $\alpha_y(i)$ to those specified at the interaction region $\beta_y(h)$ (small), $\alpha_x(h) = \alpha_y(h) = 0$. We place no initial restriction on $\beta_x(h)$. The R-matrix Eq. (2) has 10 parameters of which we have fixed 7, so three parameters are variables. It is to be equated to a transport matrix, T, involving only the parameters of the transport system. For example, we might seek a solution with the doublet of Fig. 5 which has 5 parameters.

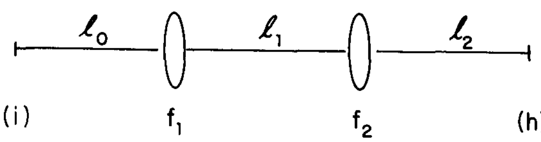

Fig. 5

$$R\left(\beta_x(h), \psi_x, \psi_y\right) = T\left(\ell_o, f_1, \ell_1, f_2, \ell_2\right) \tag{6}$$

This unimodular matrix relation represents 6 equations in 8 unknowns, so we may fix any two. In practice, we would fix ℓ_2 to make room for the detector and fix the sum $\ell_o + \ell_1 + \ell_2$ to preserve the path length for rf reasons. Then the equation may admit one real solution in the region of interest. Of course, since Eq. (6) is non-linear and transcendental, a real solution is not guaranteed. In our case, real solutions were found to exist with values of $\beta_y(h)$ as small as 5 cm.

In practice, it may be desirable to be able to vary $\beta_y(h)$ during operation with ℓ_o, ℓ_1 and ℓ_2 fixed. This clearly requires a triplet so that the three focal lengths fill out the required 6 variables. This is the solution we have settled upon.

Having settled the question of what system to use, we terminated our pencil-and-paper work and turned to a computer to get exact design figures.

Some general conclusions can be reached from Eq. (2) independent of the low-beta system chosen. If we require $\alpha_x(h) = \alpha_y(h) = 0$, we can obtain from a trigonometric identity

$$\beta^2_{x,y}(h) - \frac{\beta_{x,y}(i)}{\left(R^{x,y}_{22}\right)^2}\,\beta_{x,y}(h) + \left(\frac{R^{x,y}_{12}}{R^{x,y}_{22}}\right)^2 = 0 \tag{7}$$

with solutions

$$\beta_{x,y}(h) = \frac{\beta_{x,y}(i)}{2\left(R^{x,y}_{22}\right)^2} \pm \left\{\left(\frac{\beta_{x,y}(i)}{2\left(R^{x,y}_{22}\right)^2}\right)^2 - \left(\frac{R^{x,y}_{12}}{R^{x,y}_{22}}\right)^2\right\}^{1/2} \tag{8}$$

Complex solutions represent unstable motion, so we must ensure that the discriminant is positive.

$$\beta_{x,y}(i) \geq \left| 2\, R_{12}^{x,y}\ R_{22}^{x,y} \right| \tag{9}$$

The smaller of the two solutions is that with negative sign, and that solution satisfies the inequality

$$\frac{\left(R_{12}^{x,y}\right)^2}{\beta_{x,y}(i)} \leq \beta_{x,y}(h) \leq 2\, \frac{\left(R_{12}^{x,y}\right)^2}{\beta_{x,y}(i)} \tag{10}$$

Thus, in general, we need large $\beta(i)$ and small R_{12}.

We found for the triplet system that, in trying to produce smaller R_{12}^{y} (to get smaller $\beta_y(h)$), we always struck the limit on x-stability, Eq. (9).

The Computer-Designed System

We have available two large computer programs to aid us.

TRANSPORT is a code intended for the design of beam transport systems.[6] The formulation is in terms of transport matrices and a matrix describing the behavior of an ellipsoid in 6-dimensional phase space. In addition to manipulating the matrices, the program is capable of varying any chosen transport-system parameters (e.g., gradients, fields, positions) seeking a property of the beam matrix or the transport matrix specified by the user. This code, currently in SUBALGOL, a Stanford language, is being implemented in FORTRAN IV.

SYNCH is a code, now rather widely used, for the design of synchrotrons.[7] We used TRANSPORT to adjust parameters and to achieve the desired properties and SYNCH to tabulate the resulting orbit functions.

The values of the characteristic functions at g were

$$\text{at g} \begin{cases} \beta_x = 13.22 \text{ m} & \alpha_x = -2.06 \quad \eta = 2.28 \text{ m} \\ \beta_y = 4.52 \text{ m} & \alpha_y = 1.44 \quad \eta' = 0.380 \end{cases}$$

The configuration of the momentum matching section is shown in Fig. 6

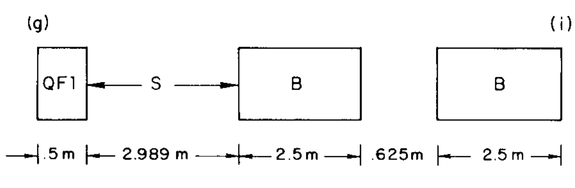

Fig. 6--Computer Designed Momentum Matching Section

QF1 is a quadrupole and B is a bending magnet. They are standard ring components, the same as those used in the basic structure. The separation between the bending magnets was set as small as physically possible, and TRANSPORT was instructed to vary the strength of QF1 and the drift distance S to reduce η and η' to zero at i. Then the characteristic functions at i were

$$\text{at i} \begin{cases} \beta_x = 9.63 \text{ m} & \alpha_x = -1.96 \quad \eta = 0 \\ \beta_y = 22.62 \text{ m} & \alpha_y = -1.72 \quad \eta' = 0 \end{cases}$$

The field in the bending magnets was 7.867 kilogauss (fixed by the requirement of orbit closure), and the gradient in QF1 was 66.99 kilogauss/m.

The total length of the low-beta section was fixed for rf reasons, and the distance from the last quadrupole to the interaction region was chosen to be 2.5m to make room for the detector. The low-beta section is shown in Fig. 7.

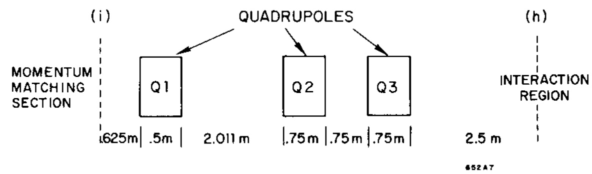

Fig. 7--Computer Designed Low-Beta Section

It was found that, in order to achieve the lowest β_y at the interaction region, Q1 should be close to the momentum matching section and Q2 and Q3 should be close together.

The values of the betatron function at i, the entrance of the low-beta section, along with the constraint that $\alpha_x = \alpha_y = 0$ at the interaction region, were used in the TRANSPORT program to obtain the values for the quadrupole field gradients for various values of β_y at the interaction region. Table 1 gives both the quadrupole field gradients and the value of β_x at the interaction region for various values of β_y at the interaction region.

Table 1

at h

β_y meters	β_x meters	K_{Q1} kG/m	K_{Q2} kG/m	K_{Q2} kG/m
.1	1.84	-21.53	83.31	-98.55
.4	.93	-50.44	88.34	-87.47
1.11	.66	-60.80	86.35	-70.08
2.5	.49	-63.45	80.82	-47.48

These values do not agree well with values obtained using thin lens approximations, because the magnets are not well represented in that approximation.

Conclusion

This design technique separates the problem of producing a small beam size at the interaction region from that of not disturbing the equilibrium orbits of off-momentum particles outside the insertion. It may be used to produce desired values of β_x and β_y in the middle of the insertion where $\eta = 0$. The same technique is applicable to the design of synchrotron sections with specially chosen beam properties.

References

1. F. Amman and D. Ritson, 1961 International Conference on High Energy Accelerators, pp. 262 and 471 (Brookhaven National Lab., Upton, N.Y., 1961).
2. D. Ritson and J. Rees, "Limitations on Storage Ring Reaction Rates," SLAC-Internal Report, Stanford Linear Accelerator Center (November 1965).
3. E. D. Courant, private communication.
4. K. W. Robinson and G. A. Voss, "Possible Use of the CEA Directly as a Colliding Beam Facility," Cambridge Electron Accelerator Internal Report CEAL-TM-149, October 22, 1965.
5. T. L. Collins, "Long Straight Sections for AG Synchrotrons," Cambridge Accelerator Report CEA-86, July 10, 1961.
6. C. H. Moore, S. K. Howry and H. S. Butler, "TRANSPORT, A Computer Program for Designing Beam Transport Systems," SLAC Internal Report (unpublished).
7. A. A. Garren and J. W. Eusebio, "SYNCH, A Computer System for Synchrotron Design and Orbit Analysis," Lawrence Radiation Laboratory Internal Report UCID-10153, April 1965 (unpublished).