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THE DISCRETE SYMMETRIES OF ELEMENTARY PARTICLE PHYSICS, I*

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ABSTRACT

The discrete symmetries P, C and T are discussed in terms of Lie algebra extensions of the Poincaré Lie algebra. This formulation leads to certain problems of Lie algebra theory which will be presented in this and succeeding papers.

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I. INTRODUCTION

Recent work on the rate of the discrete symmetries P, C and T in elementary particle physics has, at the minimum, pointed out the need for a careful discussion of their true nature. (Note particularly the comments of T. D. Lee and G. C. Wick [4] on the ambiguity of the definition of C.) This paper will suggest a purely (Lie) algebraic model where one can readily formulate some of the problems in a clear-cut mathematical way, and is the first in a series in which the relevant mathematical problems will be examined. Since the basic idea is to use the various extensions of the Poincaré Lie algebra, the first problem (which will be the main problem in this paper) will be to survey the methods for classifying extensions of the Poincaré Lie algebra.

First, however, let us present the main idea. Start off with the Lie algebra $\underset{\sim}{G}$ of the Poincaré group G as a semidirect sum $\underset{\sim}{L} + \underset{\sim}{T}$ of the homogeneous Lorentz algebra L and the abelian ideal T of translations. Explicitly, we have:

$$[\underset{\sim}{L}, \underset{\sim}{L}] \subset \underset{\sim}{L}, \quad [\underset{\sim}{T}, \underset{\sim}{T}] = 0; \; [\underset{\sim}{L}, \underset{\sim}{T}] \subset \underset{\sim}{T} \; .$$

On this "geometric" level, P, and T (parity and time inversion) are isomorphisms of $\underset{\sim}{G}$. Thus, if (X_u), $u = 0, 1, 2, 3$, is the basis for T,

$$P(X_i) = -X_i, \quad i = 1, 2, 3$$
$$P(X_o) = X_o$$
$$T(X_o) = -X_o; \; T(X_i) = X_i, \; i = 1, 2, 3 \; .$$

The action of these automorphisms on $\underset{\sim}{L}$ is determined by the condition that each be an automorphism of $\underset{\sim}{G}$.

Now, as L. Michel has emphasized [5] to define an elementary particle system, one must also be given an extension of $\underset{\sim}{G}$, i.e., a Lie algebra $\underset{\sim}{G}'$, together with homomorphism of $\phi : \underset{\sim}{G}'$ onto $\underset{\sim}{G}$. (The physical states can then be defined by representations of G' by operators on a Hilbert space.) If we call $\underset{\sim}{K}$ the kernel of ϕ (which is an ideal of $\underset{\sim}{G}'$, i.e., $[\underset{\sim}{G}', \underset{\sim}{K}] \subset \underset{\sim}{K}$) then $\underset{\sim}{G}$ is the quotient algebra $\underset{\sim}{G}'/\underset{\sim}{K}$. From this point of view, it is natural to define the "physical" discrete symmetries, which we shall call P', T', as automorphism of $\underset{\sim}{G}'$ such that:

$$\phi P' = P\phi : \ \phi T' = T\phi$$

We shall present below the method for finding all such extensions, based on the exposition of the classification of abelian extensions given in [2, Part III]. In principle then, it is possible to find all such operators P', T' by a definite algebraic procedure. If we further want the "physical" transformations P', T' to generate the same group as does P and T (i.e., ${P'}^2 = {T'}^2$) then in many cases they are quite determined.

The transformation of charge-conjugation, C, is not so obviously defined in the general case, since it is not tied so clearly to "geometric" discrete symmetries. However, by examining the usual derivation for the Dirac equation we shall be able to pinpoint at least one way of defining C that has general validity and that leads to a well-posed mathematical problem.

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II. ABELIAN EXTENSIONS OF THE POINCARÉ LIE ALGEBRA

Let G be an arbitrary Lie algebra. As we have said, an extension of G is a pair (G', ϕ) consisting of a Lie algebra G' and a homomorphism ϕ of G', onto G. Let K be the kernel of ϕ. It is an ideal of G', and G isomorphic to the quotient algebra G'/K. In [2, Part III] we have given a short exposition of the standard results describing how the second cohomology groups classify to abelian extensions (i.e., the case where K is abelian). In this section we will apply this to the case where G is the Poincaré algebra, an exercise that does not seem to have been done in full detail before.*

Recall how Lie algebra cohomology is related to abelian extensions. Suppose K and G are Lie algebras with K abelian. Let ϕ be a representation of G by linear transformations on K, and let ω: $(X, Y) \to \omega(X, Y)$ be a 2-cocycle of G with coefficients in K, i.e.,

$$\omega(X,Y) = -\omega(Y,X)$$

$$\omega(X,[Y,Z]) - \omega([X,Y]Z) - \omega(Y,[X,Z])$$

$$= \phi(Y)\big(\omega(X,Z)\big) - \phi(Z)\big(\omega(X,Y)\big) - \phi(X)\big(\omega(Y,Z)\big)$$

for $X, Y, Z \in$ G.

Construct Lie algebra G' in the following way:

G as a vector space is just K $\oplus$ G.

The bracket $[\,,\,]_1$ in G' is constructed as follows:

$$[X,Y]_1 = 0 \text{ for } X, Y \in K.$$

$$[X,Y]_1 = \phi(X)(Y) \text{ for } X \in G,\ Y \in K. \qquad (2.1)$$

$$[X,Y]_1 = [X,Y] + W(X,Y)$$

for $X \in$ K, $Y \in$ G .

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*We note however a preprint by A. Galindo, "An extension of the Poincairé group" giving an example of a non-trivial abelian extension. Such an example was also discovered earlier by A. Glashow, and mentioned briefly in E. Stein's talk at the 1965 Trieste conference. It will be presented in more detail here.

Let α denote the map: $\mathbf{G}' \rightarrow \mathbf{G}$ which sends $X + Y$ into $\alpha(X + Y) = Y$ for $X \in \mathbf{K}$, $Y \in \mathbf{G}$. Then, 2.1 tells us that α is a homomorphism with kernel $\mathbf{K}$, i.e., $\mathbf{G}'$ is an extension of $\mathbf{G}$ by $\mathbf{K}$. Standard results assert that every extension of $\mathbf{G}$ by $\mathbf{K}$ arises in this way, and that this is a semidirect sum, i.e., there exists a homomorphism $\beta : \mathbf{G} \rightarrow \mathbf{G}'$ such that $\alpha\beta$ = identity, is and only if the cocycle ω is cohomologous to zero.

Note one sidepoint that is of interest for the theory of deformation of Lie algebras: If formulas 2.1 are used, with $\omega(X, Y)$ replaced by $\lambda\omega(X, Y)$, where λ is a real parameter, we obtain a one-parameter family of Lie algebras, each of which is an extension of $\mathbf{G}$ by $\mathbf{K}$, which for $\lambda = 1$ is $\mathbf{G}'$, and for $\lambda = 0$ is the semidirect product of $\mathbf{G}$ and $\mathbf{K}$.

Let us now turn to the case where $\mathbf{G}$ is the Poincaré algebra $\mathbf{L} + \mathbf{T}$, where $\mathbf{L}$ is the homogeneous Lorentz algebra and $\mathbf{T}$ is the abelian ideal formed by translations. Suppose $0 \rightarrow \mathbf{K} \rightarrow \mathbf{G}' \rightarrow \mathbf{G} \rightarrow 0$ is an abelian extension. As we have seen in [2], we can choose ω in its cohomology class (which only changes $\mathbf{G}'$ up to an isomorphism) so that

$$X(\omega) = 0 \quad \text{for} \quad X \in L \tag{2.2}$$

$$X \lrcorner\, \omega = 0 \quad \text{for} \quad X \in L \tag{2.3}$$

Equation 2.3 then says that ω is determined uniquely by its reduction to $\mathbf{T} \times \mathbf{T}$, i.e., we are given a "tensor" mapping skew-symmetrically $\mathbf{T} \times \mathbf{T} \rightarrow \mathbf{K}$. In addition, 2.2 says that this tensor is invariant under the action of $\mathbf{L}$ on these spaces. Now $\mathbf{L}$ is the Lie algebra of $SL(2, C)$. All invariant tensors of this group can readily be found by the usual Clebsch-Gordan analysis. Thus, in principle, we could solve the problem of writing down all possible abelian extensions of the Poincaré algebra.

Of course, not every such invariant tensor will satisfy the cocycle condition. Let us examine this in more detail. Suppose then that $\omega : \underset{\sim}{G} \times \underset{\sim}{G} \to \underset{\sim}{K}$ satisfies

$$X \lrcorner \omega = 0 = X(\omega) \text{ for all } X \in \underset{\sim}{L} . \tag{2.4}$$

Let us look for the condition $d\omega = 0$. Now,

$$X(\omega) = X \lrcorner d\omega + d(X \lrcorner \omega) ;$$

hence from 2.4 above follows the condition

$$X \lrcorner d\omega = 0 \text{ for } X \in \underset{\sim}{L} .$$

Suppose $X \in \underset{\sim}{T}$. The condition that $d\omega = 0$ is now:

$$X \lrcorner d\omega (Y, Z) = 0 \text{ for } X \in \underset{\sim}{T},\ Y,\ Z \in \underset{\sim}{G},$$

or

$$X(\omega)(Y, Z) - d(X \lrcorner \omega)(Y, Z) = 0$$

or

$$\phi(X)\Big(\omega(Y,Z)\Big) - \omega\Big([X,Y],Z\Big) - \omega\Big(Y,[X,Z]\Big) - \phi(Y)\Big(\omega(X,Z)\Big) + \phi(Z)\Big(\omega(X,Y)\Big) + \omega\Big(X,[Y,Z]\Big) = 0 . \tag{2.5}$$

Let us work through conditions 2.5

<u>Case 1</u>: $Y, Z \in \underset{\sim}{T}$. The condition is then:

$$\phi(X)\Big(\omega(Y,Z)\Big) - \phi(Y)\Big(\omega(X,Z)\Big) + \phi(Z)\Big(\omega(X,Y)\Big) = 0. \tag{2.6}$$

<u>Case 2</u>: $Y, Z \in \underset{\sim}{L}$: Then 2.5 is automatically satisfied.

<u>Case 3</u>: $Y \in \underset{\sim}{L}$, $Z \in \underset{\sim}{T}$. Then onlv the following terms survive from 2.5:

$$-\omega\Big([X,Y],Z\Big) - \phi(Y)\Big(\omega(X,Z)\Big) + \omega\Big(X,[Y,Z]\Big) = 0,$$

or

$$-\omega\Big([X,Y],Z\Big) - \omega\Big([Y,X],Z\Big) - \omega\Big(X,[Y,Z]\Big) + \omega\Big(X,[Y,Z]\Big) = 0,$$

which again satisfied identically. Then 2.6 is the only nontrivial condition. Note that it too is automatically satisfied if, for example,

$$\phi(\underset{\sim}{T})(\underset{\sim}{K}) = 0 \tag{2.7}$$

Thus we have proved the following:

Theorem 2.1 Suppose 2.7 is satisfied. Then every bilinear, skew-symmetric mapping $\omega: \underset{\sim}{T} \times \underset{\sim}{T} \to \underset{\sim}{K}$ which is invariant under $\underset{\sim}{L}$ defines an extension of $\underset{\sim}{G}$ by $\underset{\sim}{K}$.

Next, we should inquire if the abelian extension by $\underset{\sim}{K}$ constructed in this way are semidirect products. Suppose otherwise, i.e., the corresponding 2-cocycle ω satisfies

$$0 = X(\omega) = X \lrcorner \omega \quad \text{for } X \in \underset{\sim}{L} \, ,$$

and in addition:

$$\omega = d\theta \, , \text{ for some 1-cochain } \theta \, .$$

Then, $d\Big(X(\theta)\Big) = 0$ for $X \in \underset{\sim}{L}$, i.e., $X(\theta)$ is a 1-cocycle. If, for example, $\underset{\sim}{K}$ contains no subspace that transforms under $\phi(\underset{\sim}{L})$ like the representation of Ad $\underset{\sim}{L}$ in T, we know from [2] that $H^1(\phi) = 0$, i.e., there exists a $W_X \in \underset{\sim}{K}$ such that

$$X(\theta) = dW_X \text{ for } X \in \underset{\sim}{L} \, .$$

If also, $\phi(\underset{\sim}{L})$ acting on $\underset{\sim}{K}$ has no invariant vectors, one sees that the assignment $X \to W_X$ is linear, and is invariant under the action of $\underset{\sim}{L}$ also. Then, if $\underset{\sim}{K}$ contains no subspaces that transform under $\phi(\underset{\sim}{L})$ like the adjoint representation of $\underset{\sim}{L}$ in itself, we see that $W_X = 0$ for $X \in \underset{\sim}{L}$, i.e., θ is an L-invariant linear mapping of $\underset{\sim}{T} \to \underset{\sim}{K}$, forcing $\omega = 0$ if no such mappings exist. Summing up, we have proved:

Theorem 2.2 If $\phi(\underset{\sim}{L})$ acting in $\underset{\sim}{K}$ has no invariant vectors, and no subspaces transforming like Ad $\underset{\sim}{L}$ in $\underset{\sim}{L}$ or $\underset{\sim}{T}$, then every nonzero

2-cocycle determined by an $\mathbf{L}$ invariant map: $\mathbf{T} \times \mathbf{T} \to \mathbf{K}$ is not a coboundry (as a corollary, the extension of $\mathbf{G}$ by $\mathbf{K}$ determined by the cocycle is not isomorphic to a semi-direct product).

The simplest example of an ω satisfying the conditions of theorems 2.1 and 2.2 can be described as follows: Let $\mathbf{K}$ be the vector space consisting of the skew-symmetric bilinear forms on $\mathbf{T}$, which we can symbolize by $\mathbf{T} \wedge \mathbf{T}$. Let ω be the skew-symmetric tensor product $\mathbf{T} \times \mathbf{T} \to \mathbf{T} \wedge \mathbf{T}$. This leads to an abelian extension of $\mathbf{G}$, the Poincaré algebra, by a six-dimensional abelian kernel. This algebra was first constructed by S. Glashow (and was the example that started this investigation). We will call it the *Glashow Algebra*.

In summary, we might say that a good technique exists for studying and classifying the abelian extensions of the Poincaré algebra. How to reduce (finite dimensional) extensions by arbitrary Lie algebras to this case is more or less known, although difficult to find when needed; hence we will now present a short exposition.

III. EXTENSIONS WITH NONABELIAN KERNELS

In this section we will briefly review the standard material describing how, in many favorable cases, extension of Lie algebra by nonabelian kernels can be reduced to extensions by abelian ones.

Suppose G' is a Lie algebra, and K, K' are two ideals of G, with $K' < K$, and $G = G'/K$. There is a linear map:

$$G'/K' \to G'/K = G ,$$

with kernel K/K'. It is readily verified that this map is a Lie algebra homomorphism. Hence, if K/K' is abelian, we have "resolved" the extension $0 \to K \to G' \to G \to 0$ into a sequence of two extensions:

$$0 \to K/K' \to G'/K' \to G \to 0 \qquad (3.1)$$

and

$$0 \to K' \to G' \to G'/K' \to 0 \qquad (3.2)$$

Suppose now that K is a <u>solvable</u> Lie ideal of G'. Choose K' as $[K, K]$. Then, K' is an ideal in K and, by the Jacobi identity, even an ideal in G'. Hence this remark applies, and we see that we may consider the problem of classifying all extensions by <u>solvable</u> Lie algebras as "solved" if it is "solved" for abelian ones.

Let us examine the situation in case G' is the Poincaré group, and K is a "two-step" solvable algebra, i.e.,

$$K' = [K, K] ,$$
$$[K', K'] = 0 .$$

Now, 3.1 determines a representation ϕ of G by linear transformations in K/K', and an element $\omega \in Z^2(\phi)$. Recall that G'/K' is described as follows:

$$G'/K' = G \oplus K/K' \quad \text{(as a vector space)}.$$
$$[X, Y]' = [X, Y] + \omega(X, Y) \quad \text{for } X, Y \in G$$
$$[X, Y]' = 0 \quad \text{for } X, Y \in K/K' \tag{3.3}$$
$$[X, Y]' = \phi(X)(Y) \quad \text{for } X \in G, \; Y \in K/K'.$$

($[\,,]$ is the bracket in G'/K').

Let $G = L + T$, with L the Lorentz subgroup, T the translations. We knew that ω can be chosen so that

$$\omega(X, Y) = 0 \quad \text{for } X \in L, \; Y \in G \tag{3.4}$$

Thus, L is a subalgebra of G', not merely identified with a subspace.

Now, 3.2 determines a homomorphism ϕ' if G' by linear transformations on K', and a 2-cocycle $\omega' \in Z^2(\phi')$. Notice from 3.3 that $T + K/K'$ is an ideal in G'/K'. (In fact, $[T + K/K', T + K/K'] \subset K/K'$, so that this algebra is itself solvable. It is nilpotent if and only if $\phi(T) = 0$.) Thus the rules given in [2] for computing the second cohomology group for semidirect products applies again, and we see, qualitatively, that everything can be reduced to computing tensors of $SL(2,C)$. We will not go further with the details here.

Extensions by semisimple algebras are readily described. If

$$0 \to K \to G \to G \to 0 , \tag{3.5}$$

with K semisimple; then, by the Levi-Malcev decomposition [3],

$$G' = R + S ,$$

where R is a maximal solvable ideal, the "radical", and S is a maximal semisimple subalgebra, which we can suppose contains K. Since K is an ideal of G', $[R, K] = 0$. By the theory of semisimple Lie algebras, S can be written as the direct sum $K + H$ of two ideals. Hence $G = G'/K$ is just $R + H$, and G' is the direct sum (as a Lie algebra) of $R + H$ and K.

This "triviality" of extensions by semisimple algebra enables us to reduce the general extension problem to that for the solvable case, i.e., ultimately, to the case of abelian extension. Suppose that we are given an extension 3.5, with K an arbitrary of Lie algebra.* Applying the Levi-Malcev theorem again,

$$K = R + S ,$$

with R a solvable ideal, S a simisimple subalgebra. Then, we have an exact sequence:

$$0 \rightarrow K/R \rightarrow G'/R \rightarrow G'/K \rightarrow 0 ,$$

or

$$0 \rightarrow S \rightarrow G'/R \rightarrow G \rightarrow 0 .$$

By our preceding remarks on extensions by semisimple algebra,

$$G'/R \text{ is just } G \oplus S .$$

Thus, the problem is reduced to finding extensions of $G + S$ by a solvable Lie algebra.

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*Of course, to use the Levi-Malcev theorem, we can only consider finite dimensional Lie algebras.

IV. DISCUSSION OF THE GEOMETRIC DISCRETE SYMMETRIES P AND T

$\underset{\sim}{G}$ will continue as the Poincare Lie algebra $\underset{\sim}{L} + \underset{\sim}{T}$. Suppose ϕ is a representation of $\underset{\sim}{G}$ by linear operators on a vector space V, with a given field of scalars (say, the real or complex numbers). Suppose $\underset{\sim}{K}$ is a Lie algebra (under commutator) of linear transformations on V, such that

$$\phi(\underset{\sim}{G}) , \underset{\sim}{K} \subset \underset{\sim}{K} .$$

Then, of course, $\underset{\sim}{G}'$ can be constructed as the semidirect product algebra $\underset{\sim}{G} + \underset{\sim}{K}$, with

$$\underset{\sim}{G}, \underset{\sim}{K} = \phi(\underset{\sim}{G}), \underset{\sim}{K} .$$

The "physical" P and T, denoted by P' and T', will be invertible linear transformations: $V \rightarrow V$ such that:

$$\phi(P(X) = P' \phi(X) P'^{-1}$$

$$\phi (T(X) = T' \phi(X) T'^{-1} \quad \text{for} \quad X \epsilon \underset{\sim}{G}$$

If such a P' and T' can in addition be chosen so that

$$P' \underset{\sim}{K} P'^{-1} = \underset{\sim}{K} ,$$

$$T' \underset{\sim}{K} T'^{-1} = \underset{\sim}{K} ,$$

we obviously have succeeded in defining the "physical" discrete symmetries as automorphisms of $\underset{\sim}{G}'$.

The commutation relations that are satisfied by P' and T' are also easy to discuss via Shur's lemma (if $\rho(\underset{\sim}{G})$ acts irreducibly on V) . For then

$$P'^2 \rho(X) = \rho(X) P'^2$$

$$T'^2 \rho(X) = \rho(X) T'^2 \quad \text{for} \quad X \epsilon \underset{\sim}{G} .$$

Then, if V is a complex vector space, if $\rho(G)$ act complex-linearly and irreducible, if P'^2 and T'^2 are complex linear (recall that this will be so even if T' is complex antilinear), i.e.,

$$T'(\lambda v) = \lambda^* T'(v)$$ for a complex number $\lambda, v \in V$, with λ^* the complex conjugate

then they are multiples of the identity.

Since this analysis is, in effect, done in every back on relativistic quantum mechanics, and is very straightforward when done from this point of view, we shall leave it at this point.

V. DISCUSSION OF CHARGE CONJUGATION, C

As we have just seen, there is a straightforward algebraic motivation for the definition of the "physical" P' and T' that one finds in quantum mechanics books. Let us turn to charge conjugation. In effect, we will give in general the "explanation" for C that one finds in quantum mechanics books in various special cases.

Suppose again that $\underset{\sim}{G}$ is the Poincaré algebra, with ρ a representation of $\underset{\sim}{G}$ by linear transformations on a complex vector space V. Suppose also that V has a "complex conjugation" transformation $v \to v^*$ which has the following properties:

It is linear over the real numbers.

$(\lambda v)^* = \lambda^* v^*$ for each complex number λ .

$v^{**} = v$ for $v \in V$.

Let ρ^* denote the following representation of $\underset{\sim}{G}^*$ by linear transformations on V:

$$\rho^*(X)(v) = \left(\rho(X)(v^*)\right)^* \quad \text{for } v \in V,\ X \in \underset{\sim}{G} .$$

Notice that $\rho^*(X)$ is also a complex linear transformation of V, and $X \to \rho^*(X)$ is also a homomorphism of G by linear transformations on V. It may be equivalent to the original representation, i.e., there may be a complex-linear transformation C: $V \to V$ such that:

$$\rho(X) = C^{-1}\rho^*(X)C \quad \text{for } X \in \underset{\sim}{G} ,$$

$$\rho(X)(v) = C^{-1}\left(\rho(X)(v^*)\right)^* C \quad \text{for } v \in V .$$

Now let C' be the following linear transformation of V:

$$C'(v) = Cv^* \quad \text{for } v \in V .$$

Suppose K is a Lie algebra of (complex linear) transformations on $\underset{\sim}{K}$, with

$$C'^{-1} K C' \subset \underset{\sim}{K} \quad ,$$

$$\rho(\underset{\sim}{G}) , \underset{\sim}{K} \subset \underset{\sim}{K} \quad .$$

We can construct the semidirect product $\underset{\sim}{G}'$ of $\underset{\sim}{G}$ with $\underset{\sim}{K}$, as before. Note that C' is an anticomplex linear transformation of V, i.e., satisfies:

$$C'(\lambda v) = \lambda^* C'(v) \quad \text{for } \lambda \text{ a complex number}, \quad v \in V,$$

and C' commutes with $\rho(\underset{\sim}{G})$. Thus, C' induces an automorphism of $\underset{\sim}{G}'$. Notice also that C' takes a "positive energy state" of $\underset{\sim}{G}$ into a "negative energy state." Let X_o be the generator of time translations in $\underset{\sim}{G}$. A positive energy state is an eigenvector of $i\rho(X_o)$ corresponding to a positive eigenvalue:

$$i\rho(X_o)(v) = Ev , \quad \text{or}$$

$$\rho(X_o)(v) = -iEv , \quad \text{with } E > O.$$

Then,

$$\begin{aligned} \rho(X_o)(C'v) &= C' C'^{-1} \rho(X_o) C'(v) \\ &= C' \rho(X_o)(v) \\ &= C'(-iEv) \\ &= iE(C'v) \quad , \end{aligned}$$

i.e., $C'v$ is a "negative energy state."

Then, we see that C' has the algebraic properties to be expected or the charge conjugation operator for the one-particle states, e.g., solutions of the Dirac equation, before second quantization.

The operator C can also be readily interpreted as an automorphism of an extension of $\underset{\sim}{G}$. Suppose that:

$$C\underset{\sim}{K}C^{-1} \subset \underset{\sim}{K} \quad .$$

Let G' as a vector space equal

$$G \oplus K_1 \oplus K_2 \quad ,$$

where K_1 and K_2 are two "copies" of K

$$[G, K_1 \quad = [\rho(G), K]$$

$$[G, K_2 \quad = [\rho^*(G), K_2] \quad .$$

Define $C: G' \rightarrow G'$ as the identity on G, so that

$$C(K_1) = K_2 \text{ , } C(K_2) = K_1 \quad ,$$

and

C intertwines the action of G on K_1 and K_2

It should be clear that this is the algebraic version of the reinterpretation of the Dirac one-particle theory by constructing "anti-particles," with charge conjugation sending particles into anti-particles.

VI. AUTOMORPHISMS OF EXTENSIONS OF THE POINCARÉ ALGEBRA, BY SEMISIMPLE ALGEBRAS

Having briefly reviewed some of the algebra involved in the usual definition of P, T and C, let us inquire if the same qualitative results can be derived from a simpler set of assumptions. Suppose $\underset{\sim}{G}'$ is a Lie algebra with a semisimple (finite dimensional) ideal $\underset{\sim}{K}$, such that the quotient $\underset{\sim}{G}'/\underset{\sim}{K}$ is the Poincaré algebra. As we have seen, $\underset{\sim}{G}'$ is a semidirect sum of a subalgebra isomorphic to $\underset{\sim}{G}$ (which we also denote by $\underset{\sim}{G}$) and $\underset{\sim}{K}$, i.e.,

$$\underset{\sim}{G}' = \underset{\sim}{G} \oplus \underset{\sim}{K}$$

$$[\underset{\sim}{G}, \underset{\sim}{K}] \subset \underset{\sim}{K} \quad .$$

Let A be an automorphism of $\underset{\sim}{G}'$.

Theorem 6.1: A maps $\underset{\sim}{K}$ into itself.

Proof : $A(\underset{\sim}{K})$ must be a semisimple ideal of $\underset{\sim}{G}'$. Its projection into $\underset{\sim}{G}$ must be a semisimple ideal of $\underset{\sim}{G}$, the Poincaré algebra. There are none, hence its projection is zero, i.e., $A(\underset{\sim}{K}) \subset \underset{\sim}{K}$.

Now, let π be the projection of $\underset{\sim}{G}'$ onto $\underset{\sim}{G}$. From theorem 6.1, we see that πA, considered as a mapping $\underset{\sim}{G} \to \underset{\sim}{G}$, is an automorphism. Let us denote by A' this automorphism of $\underset{\sim}{G}$.

The bracket $[G, K] \subset K$ determines a linear representation ϕ of $\underset{\sim}{G}$ by derivations of $\underset{\sim}{K}$.

Suppose that :

For every automorphism A': $\underset{\sim}{G} \to \underset{\sim}{G}$, there is an automorphism

$$\alpha : \underset{\sim}{K} \to \underset{\sim}{K} ,$$

such that

$$\phi\left(A'(X)\right) = \alpha \phi(X)\, \alpha^{-1} \qquad \text{for } X \in \underset{\sim}{G} \tag{6.1}$$

(For example, this is so if ϕ is the representation of $\underset{\sim}{L}$ by Dirac matrices, with $\phi(\underset{\sim}{T}) = 0$. The physicists' version of this statement is that automorphisms of the Dirac matrices can be found corresponding to parity and time reversal, which are the only outer automorphisms of the Poincaré algebra. Note that the existence of α is automatic if A is an inner automorphism of $\underset{\sim}{G}$.)

6.1 can be reinterpreted as follows: Define $A'' : \underset{\sim}{G}' \rightarrow \underset{\sim}{G}'$ as follows:

$$A''(X) = A'(X) \quad \text{for} \quad X \in \underset{\sim}{G}$$

$$A''(X) = \alpha(X) \quad \text{for} \quad X \in \underset{\sim}{K} \quad .$$

Then A'' is an automorphism of $\underset{\sim}{G}'$.

Let us compare A and A'' , i.e., put

$$B = AA''^{-1} \ .$$

Note that πB is the identity on $\underset{\sim}{G}$, and $B\underset{\sim}{K} \subset \underset{\sim}{K}$. However, B does not necessarily map $\underset{\sim}{G}$ into itself.

Since $\underset{\sim}{K}$ is semisimple, every derivation of $\underset{\sim}{K}$ is an inner derivation. In particular, we see that there is a homomorphism.

$$\phi : \underset{\sim}{G} \longrightarrow \underset{\sim}{K}$$

such that

$$[X, Y] = [\phi(X), Y] \qquad \text{for } X \in \underset{\sim}{G},\ Y \in \underset{\sim}{K} \ .$$

For $X \in \underset{\sim}{G}$, put

$$\omega(X) \leftrightharpoons BX - X \ ,$$

i.e., $\omega(X)$ is the projection of BX in $\underset{\sim}{K}$.

Then, for $X \in \underset{\sim}{G}$, $Y \in \underset{\sim}{K}$,

$$\begin{aligned} B[X, Y] &= [BX, BY] \\ &= B[\phi(X), Y] \\ &= [B\phi(X), BY] \\ &= [X, BY] + [\omega(X), BY] \\ &= [\phi(X), BY] + [\omega(X), BY] \ . \end{aligned}$$

Since K has zero center, we have

$$B\alpha(X) = \phi(X) + \quad (X),$$

or

$$BX = X + \phi(X) - B\phi(X) \tag{6.2}$$

for $X \epsilon G$.

Now we have proved:

Theorem 6.2. Suppose condition 6.1 is satisfied. Then every automorphism of G' is the product of one satisfying 6.1 and one satisfying 6.2.

The automorphisms of type 6.1 are essentially determined by the automorphisms of G, i.e., are like parity and time reversal, while those satisfying 6.2 are essentially determined by the automorphisms of "internal symmetry group" K, i.e., are like charge conjugation.

Let us ask whether conversely any automorphism of K will serve to define such an automorphism of G'. Suppose then that $B: K \rightarrow K$ is an automorphism, and we use 6.2 to extend B to G, hence to G'. Reversing the steps leading to 6.2 shows that $B[X, Y] = [BX, BY]$ for $X \epsilon K$, $Y \epsilon G$. We must investigate the case where $X, Y \epsilon G$.

$$\begin{aligned}
B[X, Y] &= [X, Y] + \phi[X, Y] - B\phi[X, Y] \\
&= [X, Y] + [\phi(X), \phi(Y)] - [B\phi(X), B\phi(Y)]
\end{aligned}$$

$$\begin{aligned}
[BX, BY] &= [X + \phi(X) - B\phi(X), Y + \phi(Y) - B\phi(Y)] \\
&= [X, Y] + [X, \phi(Y)] - [X, B\phi(Y)] \\
&\quad + [\phi(X), Y] + [\phi(X), \phi(Y)] - [\phi(X), B\phi(Y)] \\
&\quad - [B\phi(X), Y] - [B\phi(X), \phi(Y)] + [B\phi(X), B\phi(Y)]. \\
&= [X, Y] + [\phi(X), \phi(Y)] - [\phi(X), B\phi(Y)] \\
&\quad + [\phi(X), \phi(Y)] + [\phi(X), \phi(Y)] - [\phi(X), B\phi(Y)] \\
&\quad - [B\phi(X), \phi(Y)] - [B\phi(X), \phi(Y)] + [B\phi(X), B\phi(Y)] \cdot
\end{aligned}$$

Thus we have:

Theorem 6.3. A given automorphism B of $\underset{\sim}{K}$ extends to an automorphism of G' which satisfies

$$BX - X \in \underset{\sim}{K} \quad \text{for} \quad X \in \underset{\sim}{G}$$

if, and only if for X, Y $\in \underset{\sim}{G}$:

$$[\phi(X), \phi(Y)] + B[\phi(X), \phi(Y)] - [\phi(X), B\phi(Y)] - [B\phi(X), \phi(Y)] = 0 .$$

These conditions are, of course, automatically satisfied if $B\phi(X) = \phi(X)$ for all $X \in \underset{\sim}{G}$.

VII. FINAL REMARKS

In summary, we have presented in this paper several comments that prepare the way for an attack by the methods of Lie algebra theory on some of the perplexing problems concerning the role of the discrete symmetries in elementary particle physics. These remarks are not essentially new (they follow L. Michel's idea that the extensions of Lie algebras are the useful objects to study) but hopefully they might serve to point the way toward new problems that may be of physical interest. We have in mind the following problems, which we will discuss in later papers :

a. Study from both a mathematical and physical point of view the non-semidirect product extensions of the Poincaré algebra, and their automorphisms.

b. A more detailed analysis of the semidirect product extensions. (Much of this is probably contained in a different language in the work in the physics literature on invariant wave equations.)

c. An algebraic formulation and proof of the PCT theorem.

d. Study of the infinite dimensional extensions of the Poincaré algebra, particularly with the aim of isolating the algebraic aspects of the work in the physics literature on "gauge invariance."

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