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EXAMPLE OF AN "INELASTIC" BOUND STATE *

J. B. Bronzan †

Stanford Linear Accelerator Center
Stanford, California

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ABSTRACT

An example is given of a bound state which occurs in a channel with a repulsive Born approximation. The bound state occurs due to the attraction provided at low energy by three particle intermediate states.

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† Permanent Address: Department of Physics and Laboratory for Nuclear Science, Massachusetts Institute of Technology, Cambridge, Massachusetts.

I. INTRODUCTION

For some time, physicists have speculated that inelastic channels in particle collisions might give rise to resonances or bound states. Such a mechanism is well known in nuclear physics,[1] and it has been applied to particle physics by several authors, including, for instance, Cook and Lee.[2] These authors performed a matrix N over D calculation to see if the higher nucleon resonances might be driven by the opening of the ρN channel. The interest in such a mechanism stems from the fact that particle exchange on the left is not always attractive in the channels where resonances are known to occur. Put in modern terminology there is sometimes a breakdown in naive bootstrap philosophy, which assumes that elastic unitarity and the crossing matrix are sufficient principles for the prediction of resonances. The inelasticity mechanism is invoked as a cure for the breakdown of bootstrap theory.

Unfortunately, all past attempts to assess the effects of inelasticity have been marred by an enormous number of approximations and simplifications. It has neven been clear whether it was the inelasticity or the approximations which produced the resonances. In the present paper we wish to correct this situation by presenting a model calculation in which the dynamics are carefully evaluated, without important approximations other than the exclusion of states involving more than three particles. Specifically, our scattering amplitudes will have the hallowed properties of analyticity, crossing symmetry and unitarity.

We study a reaction in which the single particle exchange poles provide a repulsive force. Corresponding to this, when we construct a scattering amplitude satisfying crossing and elastic unitarity (one meson approximation), no bound state appears. This is in agreement with naive bootstrap theory. However, when we construct a scattering amplitude satisfying crossing and two and three particle unitarity (two meson approximation), a bound state appears when the coupling is sufficiently strong. The bound state can only be a result of inelasticity because we have the elastic calculation for comparison. In addition, the development of a <u>bound</u> state indicates that inelasticity can affect the low energy properties of scattering amplitudes. Contrary to popular belief, inelastic effects are not limited to energies where three particle phase space is large.

II. THE MODEL

The model we study is the charged scalar static model. This model has a spin zero source which can exist in either positive (p) or neutral (n) charge states, and which emits charged mesons (π^+ and π^-) in S-waves with conservation of charge. There are two elastic scattering amplitudes: $A_+(\omega)$, which refers to $\pi^+ p$ and $\pi^- n$ scattering, and $A_-(\omega)$, which refers to $\pi^- p$ and $\pi^+ n$ scattering. ω is the meson energy. We are principally interested in the amplitude A_-.

We denote the one meson solutions for $A_\pm$ by $M_\pm$. They satisfy crossing and elastic unitarity, and were originally given by Castillejo, Dalitz and Dyson.[3] In the present paper we choose the one meson solutions which have no CDD poles:

$$M_-(\omega) = -\frac{g^2 \omega^{-1}}{1 - \alpha(\omega)},$$

$$\alpha(\omega) = -\frac{2\omega g^2}{\pi} \int_\mu^\infty \frac{d\omega_1 k_1 u^2(\omega_1)}{4\pi \omega_1 (\omega_1^2 - \omega^2 - i\epsilon)}, \qquad (1)$$

$$M_+(\omega) = M_-(-\omega - i\epsilon) .$$

Here g is the meson-source coupling constant, μ is the meson mass, $k = \left[\omega^2 - \mu^2\right]^{1/2}$ is the meson momentum, and $u^2(\omega)$ is the cutoff function. For sufficiently large g, $\alpha(\mu) < -1$, so that M_+ has a bound state B. This is reasonable in view of the attractive character of n exchange in $\pi^+ p$ scattering. On the other hand, M_- never has a bound state, which is the conventional conclusion from the repulsive character of the direct n pole in $\pi^- p$ scattering. (The contribution of B^{++} exchange to $\pi^- p$ scattering is attractive, but never sufficiently so to overcome the repulsive direct n pole and produce a bound state in M_-.)

At this point, we introduce the relation between $M_\pm$ and the one meson phase shifts $\delta_\pm$:

$$\frac{ku^2(\omega)}{4\pi} M_\pm(\omega) = e^{i\delta_\pm(\omega)} \sin \delta_\pm(\omega) . \tag{2}$$

We can then define the Omnes functions $\Delta_\pm(z)$:

$$\Delta_\pm(z) = \exp\left[\frac{z}{\pi} \int_\mu^\infty \frac{d\omega\, \delta_\pm(\omega)}{\omega(\omega - z)} \right] . \tag{3}$$

It is possible to represent M_- in terms of these functions.[5] When the bound state B is present, the representation is

$$M_-(\omega) = - \frac{g^2 \omega_B(\omega+\mu)}{\mu\, \omega(\omega+\omega_B)} \Delta_-(\omega + i\epsilon)\, \Delta_+(-\omega) , \tag{4}$$

where ω_B is the energy of the bound state. By examining the residue of M_- at $\omega = -\omega_B$, we can determine the meson-source-bound state coupling constant g_B:

$$g_B^2 = g^2 \frac{\mu-\omega_B}{\mu} \Delta_+(\omega_B)\, \Delta_-(-\omega_B) . \tag{5}$$

Two meson solutions for $A_\pm$ have been given by the author. There are two versions of the solutions: first, for the case that the bound state B is absent in

M_+,[4] and second, when B is present.[5] These solutions do not have free parameters analogous to CDD parameters in them, and therefore they are the two meson companions to $M_\pm$. We denote the two meson approximations to $A_\pm$ by $T_\pm$. $T_\pm$ satisfy the same crossing relations as $M_\pm$ [Eq. (1)], and they satisfy two and three particle unitarity. By this we mean that production and six point amplitudes are calculated which satisfy appropriate dispersion relations, and the amplitudes fit together with $T_\pm$ to form an unitary two and three particle scattering matrix. The distinction between the two versions of $T_\pm$ lies in the fact that $M_\pm$ are used to describe final state interactions in three particle states. Consequently, in the state $\pi^+ \pi^- n$, which is connected to $p \pi^-$ by a production amplitude, the $\pi^- n$ system can coalesce into a bound state B^-. Therefore, the version of T_- for the case that B is present in M_+ includes an inelastic two particle cut coming from the $\pi^+ B^-$ state. The new cut appears automatically when the weak coupling forms of $T_\pm$ are analytically continued in g, and the enlarged scattering matrix remains unitary when the new channel appears. In the following calculation we shall need the form of T_- which holds when M_+ has a bound state.

T_- has the form

$$T_-(\omega) = - \frac{g^2 \omega^{-1}}{[1 + \omega C(-\omega)]\,[1 - \omega C(-\omega)]^{-1} - \alpha(\omega)}, \qquad (6)$$

where

$$C(-\omega) = \frac{1}{\pi} \int_{\mu+\omega_B}^{\infty} \frac{d\omega_1 \rho_-^B(\omega_1)}{\omega_1(\omega_1 - \omega - i\epsilon)} + \frac{1}{\pi} \int_{2\mu}^{\infty} \frac{d\omega_1 \rho_-^T(\omega_1)}{\omega_1(\omega_1 - \omega - i\epsilon)} \qquad (7)$$

$$+ \frac{1}{\pi} \int_{2\mu}^{\infty} \frac{d\omega_1 \rho_+(\omega_1)}{\omega_1(\omega_1 + \omega)} .$$

The weight functions are

$$\rho_-^B(\omega) = \frac{g_B^2\,\omega_B^2\,\bar{k}\,u^2(\bar{\omega})\,\Delta_-^2(-\omega)\,\left|\Delta_-(\bar{\omega})\right|^2}{2\pi\,\omega\,\bar{\omega}^2\,\Delta_-^2(-\omega_B)},$$

$$\rho_-^T(\omega) = \frac{g^4\,\omega_B^2\,\Delta_-^2(-\omega)}{8\pi^3\,\mu^2\,\omega}\int_\mu^{\omega-\mu} d\omega_1\,k_1\,k_{-1}u^2(\omega_1)\,u^2(\omega_{-1})$$

$$\times\left|\frac{(\omega_1-\mu)}{\bar{\omega}_1\,\omega_{-1}}\;\Delta_+(\omega_1)\,\Delta_-(\omega_{-1})\right|^2, \tag{8}$$

$$\rho_+(\omega) = \frac{g^4\,\omega\omega_B^2\,(\omega+\mu)^2}{16\,\pi^3\,\mu^2\,(\omega+\omega_B)^2}\,\Delta_+^2(-\omega)\int_\mu^{\omega-\mu} d\omega_1\,k_1k_{-1}u^2(\omega_1)\,u^2(\omega_{-1})$$

$$\times\left|\frac{\Delta_-(\omega_1)\,\Delta_-(\omega_{-1})}{\omega_1\,\omega_{-1}}\right|^2,$$

where $\bar{\omega} = \omega - \omega_B$, $\bar{\omega}_1 = \omega_1 - \omega_B$, $\omega_{-1} = \omega - \omega_1$, $\bar{k} = \left[\bar{\omega}^2 - \mu^2\right]^{1/2}$ and $k_{-1} = \left[\omega_{-1}^2 - \mu^2\right]^{1/2}$. Note that when we set $C = 0$, the one meson solution is recovered. ρ_-^B gives the contribution of $\pi^+ B^-$ intermediate states to T_-, ρ_+ gives the contribution of $\pi^+\pi^-n$ intermediate states, and ρ_+ gives the contribution of $\pi^+\pi^+n$ exchange. π^-p intermediate states and π^+p exchange contribute to α, as in the one meson solution.

A bound state of T_- occurs if the denominator of Eq. (6),

$$D(\omega) = \frac{1+\omega C(-\omega)}{1-\omega C(-\omega)} - \alpha(\omega), \tag{9}$$

increases through zero between $\omega = 0$ and $\omega = \mu$. (A zero of D between $\omega = -\mu$ and $\omega = 0$ is a bound state in the T_+ channel. We know that such a bound state exists, analogous to B in M_+.) In the one meson approximation, $C = 0$, D is always greater than one between $\omega = 0$ and $\omega = \mu$, so M_- has no bound state. However, if C is large, D developes a pole, and it can then increase through zero (see Fig. 1). The conditions for T_- to have a bound state are

$$\mu\, C(-\mu) > 1\,, \tag{10}$$

$$\frac{1 + \mu\, C(-\mu)}{1 - \mu\, C(-\mu)} - \alpha(\mu) > 0 \;.$$

These inequalities can be satisfied only if $\alpha(\mu) < -1$, which is the condition that M_+ have a bound state B. Therefore, the bound state in T_- cannot develop until M_+ has developed a bound state. A sufficient condition for T_- to develop a bound state is that $C(-\mu) \xrightarrow[g^2 \to \infty]{} +\infty$. In the next section we shall show that this behavior occurs.

We point out that the pole of D (see Fig. 1) is a CDD pole, since it corresponds to a zero of T_-. It is induced by the coupling to the inelastic channels, and is an example of a dynamically determined CDD pole of the type noted by Bander, Coulter and Shaw.[6]

III. THE BOUND STATE IN T_-

We have seen that T_- has a bound state if $C(-\mu) \xrightarrow[g^2 \to \infty]{} +\infty$. To establish that this happens, we first examine the behavior of the ρ's for finite ω as $g^2 \to \infty$. We observe from Fig. 2 that $\delta_\pm(\omega)$ remain finite as $g^2 \to \infty$. Although $\delta_-(\infty)$ changes from 0 to $-\pi$ as $g^2 \to \infty$, this affects only the asymptotic form of $\Delta_-(\omega)$, and at present we are studying finite ω. We conclude that $\Delta_\pm(\omega)$ approach finite limits as $g^2 \to \infty$ for finite ω. Next, we observe that near $\omega = 0$, $\alpha(\omega) \approx -g^2\, \omega\beta$, with $\beta > 0$. Thus, for large g^2,

$$\omega_B \approx \frac{1}{g^2\beta} \quad . \tag{11}$$

From Eq. 5, we observe that for large g^2,

$$g_B^2 \approx g^2 \quad . \tag{12}$$

These remarks suffice to determine that for finite ω and large g^2, ρ_-^B decreases like g^{-2}, and ρ_-^T and ρ_+ are independent of g^2. Therefore,

$$C(-\mu) = C + \frac{1}{\pi} \int_{\omega_o}^{\infty} \frac{d\omega_1}{\omega_1} \left[\frac{\rho_-^B(\omega_1) + \rho_-^T(\omega_1)}{\omega_1 - \mu} + \frac{\rho_+(\omega_1)}{\omega_1 + \mu} \right], \tag{13}$$

where ω_o is a large energy chosen so that we may use high energy forms of the quantities appearing in the integrand. C is a positive constant which is independent of g^2 for large g^2. Evidently, if $C(-\mu)$ is to increase with g^2, this increase is to be found in the high energy integral of Eq. (13).

For large positive or negative ω,

$$\mathrm{Re}\ \alpha(\omega) \approx + \frac{g^2 \gamma}{\omega}, \quad \gamma > 0 . \tag{14}$$

We assume that $u^2(\omega) \approx \eta\, \omega^{-n}$ for large ω, where $n > 1$. Examination of Fig. 2 verifies that $\Delta_+(\infty)$ is a finite positive number when $g^2 \rightarrow \infty$, so from Eq. (4) we have

$$\Delta_-(\omega) \xrightarrow[g^2 \rightarrow \infty]{} - \frac{\omega \mu \beta M_-(\omega)}{\Delta_+(\infty)} \approx \frac{g^2 \mu \beta}{\Delta_+(\infty) \left[1 - g^2 \left(\frac{\gamma}{\omega} - \frac{i u^2(\omega)\, \theta(\omega)}{4\pi}\right)\right]} . \tag{15}$$

This form is valid for large g^2 and ω, and demonstrates that the asymptotic behavior of $\Delta_-(\omega)$ changes as $g^2 \rightarrow \infty$.

We are now able to evaluate the remaining integral in Eq. (13). We first examine the ρ_-^B term, replacing all the terms in the definition of ρ_-^B by their values at $\omega = \infty$, except $\Delta_-(\omega)$, for which we use Eq. (15). We let $x = \omega_1/g^2$ be the variable of integration.

$$\begin{aligned} C_-^B &\equiv \frac{1}{\pi} \int_{\omega_o}^{\infty} \frac{d\omega_1\, \rho_-^B(\omega_1)}{\omega_1 (\omega_1 - \mu)} \\ &\approx \frac{\mu^4 \beta^2 \eta}{2\pi^2 g^{2n} \Delta_+^4(\infty)} \int_{\omega_o/g^2}^{\infty} \frac{x^{n-2}\, dx}{[x + \gamma]^2 \left[(x - \gamma)^2 x^{2n-2} + \eta^2/16\pi^2 g^{4n-4}\right]} . \end{aligned} \tag{16}$$

For large g, the dominant contribution to the integral comes from the region around $x = \gamma$, and C_-^B vanishes like g^{-2}. Thus, the contribution of the $\pi^+ B^-$ state vanishes for large g^2, and if T_- is to have a bound state, it is solely a three particle effect.

We next examine the contribution of ρ_-^T. The integral in Eq. (8) may be bounded from below by a finite positive constant λ when ω and g^2 are large. Thus, for large ω and g^2,

$$\rho_-^T(\omega) > \frac{\lambda g^4}{8\pi^3 \Delta_+^2(\infty)\ \omega \left[1 + \frac{\gamma g^2}{\omega}\right]^2} . \tag{17}$$

Using x as variable again, we have

$$C_-^T \equiv \frac{1}{\pi} \int_{\omega_o}^{\infty} \frac{d\omega_1\, \rho_-^T\,(\omega_1)}{\omega_1\,(\omega_1 - \mu)} > \frac{\lambda}{8\pi^3\, \Delta_+^2(\infty)} \int_{\omega_o/g^2}^{\infty} \frac{dx}{x\,[x+\gamma]^2} . \tag{18}$$

This integral has a logarithmic singularity at the lower limit. Since the contribution of ρ_+ is positive, we have for large g^2

$$C(-\mu) > C_o + C_1 \log g^2;\quad C_o,\ C_1 > 0 . \tag{19}$$

Therefore, T_- has a bound state for large g^2.

It is worth mentioning that for large g^2, the bound state B moves to the origin and nearly cancels the source pole there (see Eqs. (11) and (12)). This is the way unitarity is maintained when g^2 is large. However, it would be erroneous to conclude that the net effect of the direct n pole and exchanged B^{++} pole is no longer repulsive when g^2 is large. This is evident from the fact that M_- never can have a bound state. Even in the limit $g^2 \to \infty$, the bound state in T_- must be interpreted as three particle effect which occurs despite repulsive Born terms.

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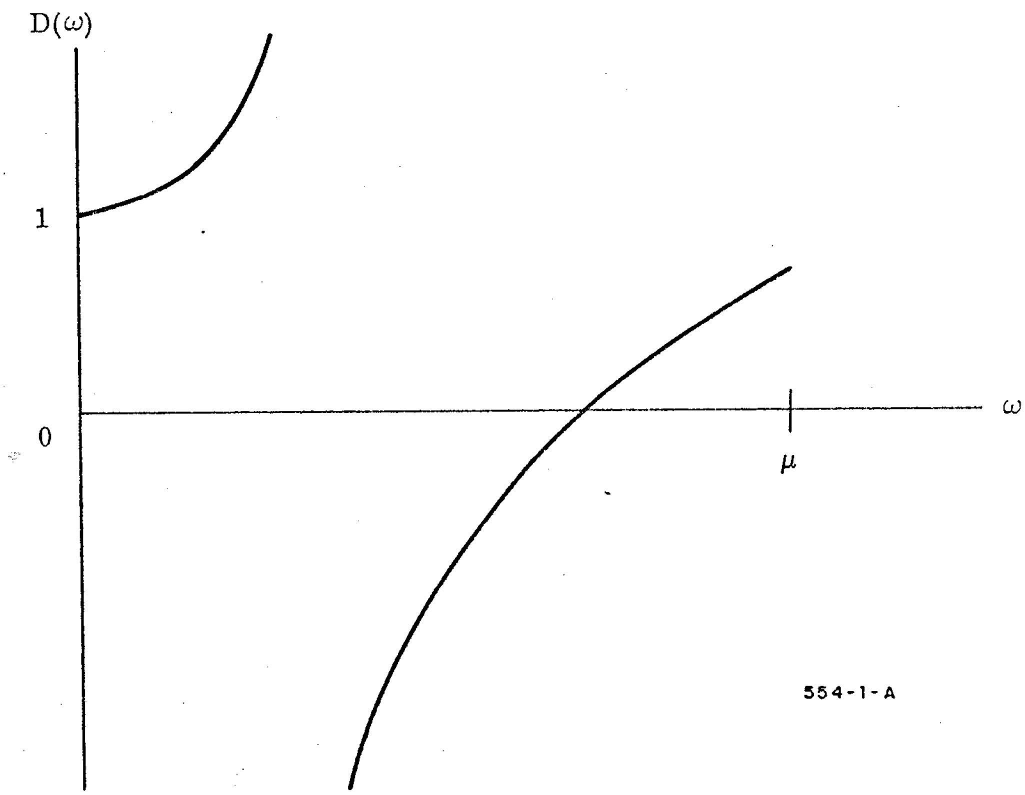

Fig. 1

The denominator function $D(\omega)$ when T_- has a bound state. $\omega C(-\omega)$ is a monotonically increasing function of ω, so D has at most one pole.

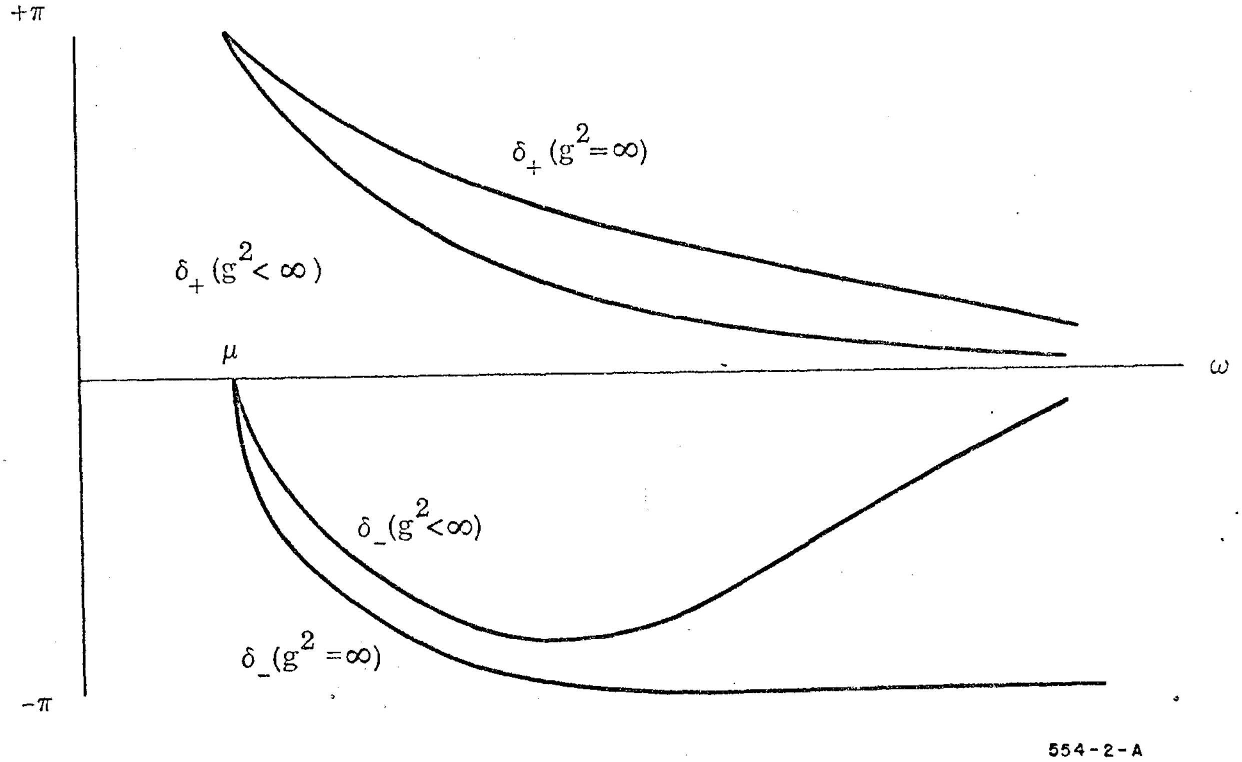

Fig. 2

The one meson phase shifts for finite and infinite g^2.