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HIGH ENERGY TRIDENT PRODUCTION WITH DEFINITE HELICITIES*

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ABSTRACT

The high-energy large-angle limit of the completely differential cross section for the process $e^- + Z \rightarrow e^- + e^+ + e^- + Z$ (or for the process with all or some of the particles replaced by μ-mesons) is computed for arbitrary helicities of incident and final particles. This process is interesting as a test of electrodynamics and of the statistics of the μ-meson. The formula is short and perhaps suitable for numerical integrations.

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I. INTRODUCTION

Trident production, i.e., electron-positron pair production by incident electrons of hydrogen (or the analogous process for μ-mesons), has been discussed in the past by different authors as a test of quantum electrodynamics at small distances.[1,2] For the case of the μ-meson, this process also may provide a test of the statistics of the μ-meson. In this article we present the differential cross section for the trident process for definite helicities of incident and outgoing leptons, in the limit of vanishing lepton mass. Although such a process is unlikely to be measured directly, the expression for the cross section is reasonably compact and amenable to machine integration and summation over the unobserved variables.

In Section II, the calculation is described and the result is given in Section III.

II. THE MATRIX ELEMENT

The eight diagrams are shown in Fig. 1. The matrix element of diagram (1) is, by the usual Feynman rule,[3]

$$\mathcal{M}_1 = - \frac{i}{(2\pi)^2} \sqrt{\frac{m^4}{EE_1E_2E_+}} \; \frac{1}{q^2} \; \frac{1}{(p-p_1)^2}$$

$$\left[\bar{u}(p_2,\lambda_2)\, \gamma^\mu \, \frac{1}{\not p_1 + \not p_2 - \not p - m} \, \gamma^0 \, v(p_+,\lambda_+) \right] \cdot \left[\bar{u}(p_1,\lambda_1)\, \gamma^\mu \, u(p,\lambda) \right] . \quad (1)$$

All the notations are summarized in the Appendix. The method we use is to calculate directly the amplitude for each graph, sum the amplitudes, and then square. To calculate the amplitude explicitly, we multiply and divide the first square bracket by

$$\bar{v}(p_+,\lambda_+)\, \gamma^0\, u(p_2,\lambda_2) = \left| A_1 \right| e^{i\theta_1} = A_1 \; e^{i\theta_1} \quad (2)$$

and the second square bracket by

$$\bar{u}(p,\lambda)\,\gamma^{o}\,u(p_1,\lambda_1) = \left|A_2\right| e^{i\theta_2} = A_2\, e^{i\theta_2} \,. \tag{3}$$

The numerators and denominators after multiplication can be put into invariant form by the use of spin projection operators[4] and the usual trace method.

The invariant spin projection operator O_λ for positive energy states, defined by $O_\lambda u(p,\lambda') = \delta_{\lambda\lambda'}\, u(p,\lambda')$, is reduced to $O_\lambda = \frac{1}{2}(1+\lambda\gamma^5)$ in the limit of zero mass. It is then easily verified that for diagrams of any order, so long as mass is neglected, the matrix element vanishes unless the fermion lines entering and leaving the diagram have the same chirality.[5] This asserts that only one spin projection operator is needed in summing the spins of the spinors. Thus the first bracket may be written

$$\frac{1}{A_1 e^{i\theta_1}} \sum_{\lambda_2'=\pm 1} \bar{u}(p_2,\lambda_2')\,\gamma^\mu \frac{1}{\not{p}_1+\not{p}_2-\not{p}-m}\,\gamma^{o} \sum_{\lambda_+'=\pm 1} v(p_+,\lambda_+)\,\bar{v}(p_+,\lambda_+)\,O_{\lambda_2}\,u(p_2,\lambda_2')\,.$$

This will be simplified by the usual energy state projection operator and trace method, and then becomes

$$\frac{1}{A_1 e^{i\theta_1}} \mathrm{Tr}\left[\frac{\not{p}_2}{2m}\,\gamma^\mu \frac{1}{\not{p}_1+\not{p}_2-\not{p}-m}\,\gamma^{o}\,\frac{\not{p}_+}{2m}\,\gamma^{o}\,\frac{1}{2}\,(1+\lambda_2\gamma^5)\right]\epsilon_\mu$$

$$= \frac{1}{A_1 e^{i\theta_1}}\,\frac{1}{2m^2}\,\frac{1}{(p-p_1-p_2)^2-m^2}\Big\{\tilde{p}_+\cdot\epsilon\, p_2\cdot(p-p_1-p_2)\cdot\epsilon\, p_2\cdot\tilde{p}_+$$

$$-\,\tilde{p}_+\cdot(p-p_1-p_2)\,p_2\cdot\epsilon + i\lambda_2\,(p-p_1)\cdot\epsilon p_2\,\tilde{p}_+\Big\} \tag{4}$$

where $|ABCD|$ is a short notation for the determinant formed by the components of the four vectors:

$$\left|ABCD\right| = \begin{vmatrix} A^o & B^o & C^o & D^o \\ A^1 & B^1 & C^1 & D^1 \\ A^2 & B^2 & C^2 & D^2 \\ A^3 & B^3 & C^3 & D^3 \end{vmatrix}$$

and $\tilde{p}_+ = (E, \ -\vec{p}_+)$.

The quantity A_1 may be computed explicitly by multiplying by the complex conjugate and then using spin and energy projection operators as before; thus

$$A_1^2 = \sum_{\lambda_+' = \pm 1} \bar{v}(p_+, \lambda_+')\, \gamma^o\, O_{\gamma_2} \sum_{\lambda_2' = \pm 1} u(p_2, \lambda_2')\, \bar{u}(p_2, \lambda_2') \cdot \gamma^o = \frac{1}{2m^2}\, p_+ \cdot \tilde{p}_2 \,. \quad (5)$$

The second bracket of (1) may be treated in the same manner, and found to be

$$\bar{u}(p_1 \lambda_1)\, \gamma^\mu\, u(p_1 \lambda)\, \epsilon_\mu \times$$

$$\times \frac{1}{2m^2} \frac{1}{A_2 e^{i\theta_2}} \left\{ E_1 p \cdot E + E p_1 \cdot E - \epsilon_o p \cdot p_1 + i\lambda_1 \left| p_1 p \eta \epsilon \right| \right\}, \quad A_2^2 = \frac{1}{2m^2}\, p_1 \cdot p \quad (6)$$

It is then only a matter of putting (4), (5), and (6) into (1) to obtain the full matrix element. The matrix elements of the remaining seven diagrams may be obtained from M_1 by interchange of parameters. The results are summarized in the next section.

We note that the phase factor θ_1 is the same for diagrams (1) to (4) and θ_2 the same for diagrams (5) to (8). If $\lambda = \lambda_1 \neq \lambda_2$, only diagrams (1) to (4) contribute, and the phase factors have no effect to the cross section. If, on the other hand, $\lambda = \lambda_2 \neq \lambda_1$, only diagrams (5) to (8) contribute, and the phase factors have no effect either. The phase factors are important only if $\lambda = \lambda_1 = \lambda_2$. In this case all eight diagrams contribute, but only the relative phase θ need to be computed. From the definitions of (2) and (3), θ can again be evaluated by means of trace techniques. The result is given in the next section.

III. THE CROSS SECTION

We may write down the differential cross section

$$d\sigma = \frac{\alpha^4}{4\pi^4} \frac{|\underset{\sim}{p}_1|\,|\underset{\sim}{p}_2|\,|\underset{\sim}{p}_+|}{E} \frac{1}{q^4} \cdot \left| \delta_{\lambda\lambda_1} \frac{F_1 + F_2 + F_3 + F_4}{\sqrt{p_+ \cdot \tilde{p}_2\, p_1 \cdot \tilde{p}}} \right.$$

$$\left. + \delta_P \delta_{\lambda\lambda_2} \frac{F_5 + F_6 + F_7 + F_8}{\sqrt{p_+ \cdot \tilde{p}_1\, p_2 \cdot \tilde{p}}}\, e^{i\theta} \right|^2 \cdot dE_1\, dE_2\, d\Omega_1\, d\Omega_2\, d\Omega_+ . \qquad (7)$$

In the above expression, δ_P is the signature of identical particles in the final state, and in our case $\delta_P = -1$ if the fermions obey Fermi-Dirac statistics and $\delta_P = +1$ were they to obey Bose-Einstein statistics. The function F_1 is given by

$$F_1(p, p_1, p_2, p_+, \lambda_1, \lambda_2) = \frac{1}{(p-p_1)^2\left[(p-p_1-p_2)^2 - m^2\right]}$$

$$\cdot \Big[2p_+ \cdot \tilde{p}_2(-E_2 p \cdot p_1 + E_1 p \cdot p_2 + E p_1 \cdot p_2) - \tilde{p}_+ \cdot p(E + E_1 p_1 \cdot p_2 - E_2 p \cdot p_1)$$

$$+ \tilde{p}_+ \cdot p_1 (E + E_1 p \cdot p_2 - E_2 p \cdot p_1) + E_+ p \cdot p_1 (\not{p}_1 \cdot \not{p}_2 - \not{p} \cdot \not{p}_2)$$

$$- i\lambda_1 \left\{ \tilde{p}_+ \cdot (p - p_1 - 2p_2) \left| p_1 p \eta\, p_2 \right| + p_2 \cdot (p - p_1) \left| p_1 p \eta\, \tilde{p}_+ \right| \right\}$$

$$+ i\lambda_2 \left\{ (E + E_1) \left| p p_1 p_2 \tilde{p}_+ \right| + p \cdot p_1 \left| (p - p_1)\eta\, p_2 \tilde{p}_+ \right| \right\}$$

$$- \lambda_1 \lambda_2 \begin{vmatrix} E_+ & p_1 \cdot \tilde{p}_+ & p \cdot \tilde{p}_+ \\ E_2 & p_1 \cdot p_2 & p \cdot p_2 \\ E - E_1 & p \cdot p_1 & -p \cdot p_1 \end{vmatrix} \Big] \qquad (8)$$

The other functions F_2 to F_8 are obtained from $F_1(p\, p_1, p_2, p_+, \lambda_1, \lambda_2)$ by the following exchange:

$$F_2 = -F_1(p, p_1, p_+, p_2, \lambda_1, -\lambda_2) ,$$

$$F_3 = -F_1(-p_+, p_2, -p, p_1, \lambda_2, -\lambda_1) ,$$

$$F_4 = F_1(-p_+, p_2, p_1, -p, \lambda_2, -\lambda_1) ,$$

and F_5, F_6, F_7 and F_8 are obtained from F_1, F_2, F_3, and F_4 respectively by the interchange $p_1 \longleftrightarrow p_2$, $\lambda_1 \longleftrightarrow \lambda_2$. The relative phase θ is given by

$$\theta = \tan^{-1} \frac{\lambda \left| p_1 \tilde{p}_+ p_2 \tilde{p} \right|}{p_1 \cdot \tilde{p}_+ p_2 \cdot \tilde{p} + p_1 \cdot \tilde{p}\, p_2 \cdot \tilde{p}_+ - p_1 \cdot p_2 \tilde{p} \cdot \tilde{p}_+} \tag{9}$$

Our calculation assumes a static Coulomb field with $Z = 1$. Practical considerations will probably force targets of $Z > 1$, e.g., carbon. For the elastic trident production (no nuclear excitation) Eq. (7) need by modified only by inclusion of an extra factor $[ZF(q^2)]^2$, with $F(q^2)$ the charge form factor measured in elastic electron scattering. Higher orders in Z are of course neglected in this Born approximation calculation.

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APPENDIX

p, p_1, p_2 and p_+ are four-momenta of incoming μ^- (μ^+), two outgoing $\mu^-(\mu^+)$ and one $\mu^+(\mu^-)$. E and p are energy and three momentum, etc. λ, λ_1, λ_2, and λ_+ are the chiralities of these particles, and take the value +1 when the particle is right handed or antiparticle left handed. A four vector with a tilde is related to its original vector by changing the sign of the space momentum, e.g., if $p = (E, \underset{\sim}{p})$, then $\tilde{p} = (E, -\underset{\sim}{p})$. η^μ is a unit four vector with o-th component only. $p = \underset{\sim}{p}$ = magnitude of space momentum. $d\Omega_1$, $d\Omega_2$, and $d\Omega_+$ are solid angles of corresponding particles.

$$\alpha = \frac{1}{137}$$

$$q = p - p_1 - p_2 - p_+$$

REFERENCES

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2. M. C. Chen, Phys. Rev., 127, 1844 (1962).
3. We use $\hbar = c = 1$, $\gamma_5^2 = 1$. Our notation, metric, etc., is that of J. D. Bjorken and S. D. Drell, "Relativistic Quantum Mechanics," McGraw Hill (1964).
4. L. Michel and A. S. Wightman, Phys. Rev., 98, 1190 (1955).
5. Right-handed particles and left-handed antiparticles have positive chirality.

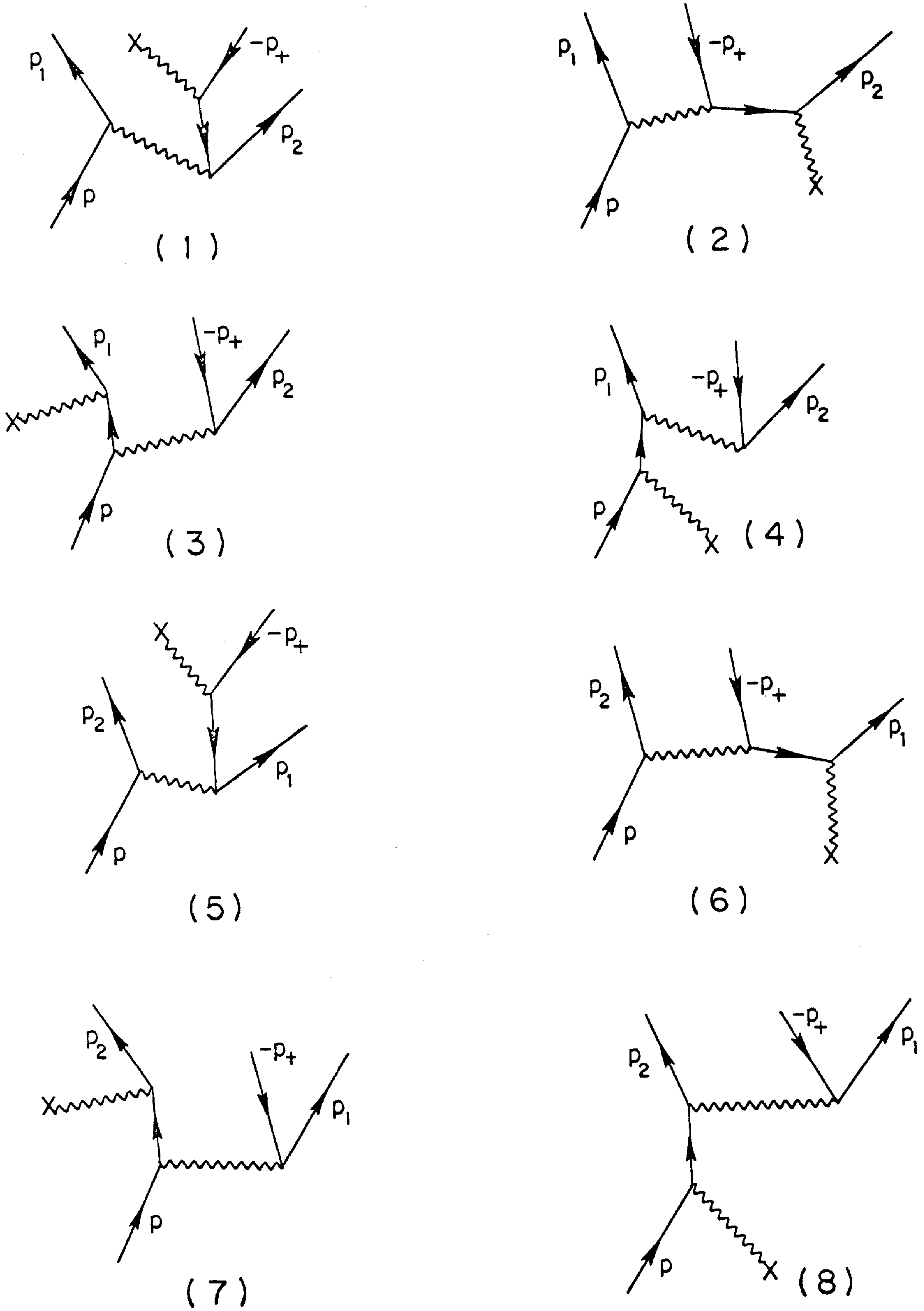

FIG. 1--Diagrams for trident production.

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