SLAC-PUB-187
ITP-208

EXACT SUM RULE FOR NUCLEON MAGNETIC MOMENTS*

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(Submitted to Phys. Rev. Letters)

*Work supported in part by the U. S. Atomic Energy Commission and in part by the U. S. Air Force through Air Force Office of Scientific Research Contract AF 49(638)-1389.

A sum rule is constructed on very general assumptions which relates experimental quantities and thus can be tested in the laboratory. Define $\sigma_P(\nu)$ $\left[\sigma_A(\nu)\right]$ as the total cross-section for the absorption of a circularly polarized photon of laboratory energy ν by a proton polarized with its spin parallel (anti-parallel) to the photon spin. The sum rule then reads

$$\int_0^\infty \frac{d\nu}{\nu} \left[\sigma_P(\nu) - \sigma_A(\nu)\right] = + \frac{2\pi^2\alpha}{M_p^2} \kappa_p^2 \approx 205 \ \mu b \tag{1}$$

where $\alpha = 1/137$, M_p is the proton mass, and $\kappa_p = 1.79$ is the anomalous magnetic moment of the proton in nucleon magnetons. A similar rule exists for the neutron magnetic moment involving the coresponding neutron quantities. Equation (1) follows immediately from the dispersion relation for forward Compton scattering derived by Gell-Mann, Goldberger, and Thirring[1] and from the low energy theorem for Compton scattering proved by Low[2] and by Gell-Mann and Goldberger,[3] together with the assumption that the left hand side of Eq. (1) converges. We demonstrate this as follows.

The forward Compton scattering amplitude may be written in terms of two scalar invariant functions of the squared energy ν^2

$$f(\nu) = f_1(\nu^2) \, \underline{e}'^* \cdot \underline{e} + \nu \, f_2(\nu^2) \, i\underline{\sigma} \cdot \underline{e}'^* \times \underline{e} \tag{2}$$

where $\underline{e}$ and $\underline{e}'$ are the transverse polarization vectors of the incident and forward scattered photon, respectively. The dispersion relation for the spin flip amplitude may be written with the assumption of no subtraction as[4]

$$\mathrm{Re}\ f_2(\nu^2) = \frac{+1}{4\pi^2} P \int_0^\infty \frac{\left[\sigma_A(\nu') - \sigma_P(\nu')\right] d\nu'^2}{\nu'^2 - \nu^2} \tag{3}$$

Since the low energy theorem[2,3] informs us that

$$f_2(0) = -\frac{1}{2}\frac{\alpha}{M_p^2}\kappa_p^2 \tag{4}$$

we see that Eq. (1) follows immediately.

The contribution of this letter is very simply that of joining the dispersion relation [Eq. (3)] and the low energy theorem [Eq. (4)] with the no subtraction assumption in constructing Eq. (1). It is of interest because of its experimental as well as theoretical implications.[5]

On the experimental side, Milburn[6] has shown that it is possible to produce high energy circularly polarized photon beams by back scattering of a laser beam from the electron beam in a high energy synchrotron or linear accelerator. The laser beam is converted to circularly polarized light by passage through a quarter-wave plate and its incident frequency ν is increased to (mc^2 = .51 MeV)

$$\nu' = \nu \frac{(2E/mc^2)^2}{\left[1 + \frac{4E(h\nu)}{(mc^2)^2}\right]} \tag{5}$$

for the back scattered radiation from the electron beam of energy E. In this way high energy circularly polarized photons can be obtained, up to 7.8 GeV at SLAC for incident ruby laser light.[6] Thus although the left hand side of Eq. (1) presents a formidable experimental challenge it is not generally thought to be insurmountable.

On the theoretical side the generality of the input assumptions suggests very strongly that Eq. (1) should be verified. The no subtraction assumption which permits us to "calculate" the anomalous Pauli moment of the proton by this sum rule is the only step in the derivation open to "reasonable" question. Since an analogous no subtraction hypothesis underlies many other recent sum rules based on more restrictive assumptions on the algebra of current components we would like direct confirmation of its validity.

To see how close the low energy photoproduction data comes to satisfying Eq. (1) we have carefully integrated fits to photoproduction over the threshold and 3,3 resonance regions and made further estimates of contributions up to ~ 1 GeV. A simple approximation of photoproduction by a 3,3 resonance and integrated from threshold up to 500 MeV already gives a fairly good approximation to the magnetic moment, contributing ~ 200 μb. This number is based on the assumption that the $\pi^0 p$ photoproduction cross-section is 270 μb at the peak of the resonance and is pure (3/2, 3/2), and neglects any non-resonant background. In this approximation $\kappa_p^2 = \kappa_n^2$ since contributions to the integrals are all isovector.

In order to make a more detailed estimate including isoscalar contributions we have used the full Gourdin-Salin[7] model of photoproduction which parametrizes the data up to 500 MeV in terms of a 3,3 isobar along with contributions from single particle pole terms and a phenomenological s-wave subtraction constant. With this model, the integral yields 180 μb. However in spite of the success of Gourdin and Salin in fitting the total and differential cross-sections of $\pi^0 p$ and $\pi^+ n$ photoproduction it must be remembered that the relevant quantity here, $\sigma_p - \sigma_A$, may be very sensitive to terms which are relatively unimportant in the unpolarized cross-section. Our number should thus be taken as a good guide but not as an accurate determination.

The situation is much more muddled if we attempt to include contributions from the energy region 500 MeV to ~ 900 MeV. The Gourdin-Salin[7] model uses a phenomenological p-wave background term as well as a d_{13} isobar to fit the data. From pion nucleon scattering analyses it is known, however, that in this energy region the isobar structure is more complicated. On the basis of this simple parametrization there is an additional contribution to the sum rule of ≈ + 90 μb from 500 MeV to 900 MeV for a total of 270 μb as shown in Fig. 1. To this we must add the contribution from multi-pion production, with a cross-section of the order of 100 μb over this energy region.[8] It is unclear at this stage whether this contributes to σ_p or σ_A, but it is possible that near threshold it is mainly p_{11} production[8] and hence contributes only to σ_A and thus with a minus sign to the sum rule. It is not inconsistent with present data then that the sum rule is well satisfied by energies of the order of 1 GeV, but the final answer can only be found by experiment.[9]

The above discussion shows however that data in the $\gtrsim 1$ GeV region will play a crucial role in the verification or denial of Eq. (1). Beyond its implications for this sum rule there is strong interest in measurement of σ_P and σ_A since individually they are sensitive to terms that must be known if a complete parametrization of the photo-pion amplitude is to be achieved.

It is also instructive to compare Eq. (1) with other recently derived rules based on the commutator algebra of currents components proposed by Gell-Mann.[10] One exact rule[11] derived from the electric dipole moment operators relates the difference of the neutron and proton moments and the nucleon's isovector charge radius to an integral over total cross-sections $\sigma_{1/2}$, $\sigma_{3/2}$ for the production of I = 1/2, I = 3/2 states respectively by isovector photons absorbed on nucleons. Specifically

$$\left[2\pi^2\alpha \; \frac{\left\{1+\kappa_P-\kappa_N\right\}^2-1}{4M^2} - 2\,\frac{dG_E^V(q^2)}{dq^2}\bigg|_{q^2=0}\right] = \int_0^\infty \frac{d\nu}{\nu}\left[\sigma_{3/2} - 2\sigma_{1/2}\right] \qquad (6)$$

Inserting experimental values for the nucleon moments and isovector electric radii into the left hand side of Eq. (6) we find a negative number showing that the 3,3 resonance cannot dominate the sum rule on the right hand side which would then be positive.

Another sum rule has been derived by Fubini, Segre and Walecka,[12] who apply the equal time commutation rules to quark charges generating the group U(12). They obtain

$$\frac{2\pi^2\alpha}{M^2}\left\{\kappa_V^2 - 3\,\kappa_S\kappa_V\left[\frac{\kappa_S}{\kappa_V}+2R\right]\right\} = \int_0^\infty \frac{d\nu}{\nu}\left(\sigma_P^V - \sigma_A^V\right) \qquad (7)$$

where κ_v, κ_s are the isovector and isoscalar magnetic moments, R is related to the f/d ratio of the weak interactions and is experimentally of the order of 1/9, and $\sigma^v_{P,A}$ are the isovector projections of the cross-sections in Eq. (1). In the derivation of Eq. (7) an extrapolation must be made from the mass of the ρ meson to zero mass for a real photon. It is also not clear whether $\kappa_{v,s}$ should represent total or anomalous Pauli moments. If the anomalous moment is used the second term in Eq. (7) is small since then $\kappa_s = 0.06$. Neglecting this term, we can write

$$\frac{2\pi^2\alpha}{M^2}\,\kappa_v^{\,2} = \int_0^\infty \frac{d\nu}{\nu}\left(\sigma^v_P - \sigma^v_A\right) \tag{8}$$

which is similar in form to Eq. (1) and is well satisfied if one assumes that the 3,3 resonance dominates the right hand side. If the full moment is used, however, the agreement is not as good.[13]

It will be of great interest if experiment can verify directly the validity of Eq. (1) by proving that the difference $\sigma_P(\nu) - \sigma_A(\nu)$ either drops smoothly to zero or has big contributions of different signs and compensating magnitudes before settling down to zero as predicted by Regge pole analyses.

We thank our colleagues at Stanford for discussions.

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4. The sign of the right hand side of Eq. (3) is opposite to that written in Ref. 1. The need for a subtraction in the dispersion relation for the spin independent amplitude $f_1(\nu)$ is clear from the sign of the classical Thomson amplitude which is its low energy limit; c.f. Ref. 1.

5. This sum rule also allows part of the unknown contribution of two virtual photon exchange between the electron and proton in the hyperfine structure analysis (i.e. of the polarizability contribution) to be normalized against the real photon amplitude. (S. D. Drell and J. D. Sullivan, to be published.)

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8. J. V. Allaby, H. L. Lynch, D. M. Ritson, Phys. Rev. 142, 887 (1966). This experiment does not include neutral pion modes, which are expected to contribute another 30% or so.

9. The good agreement with observed moments obtained when only low energy photopion production data (< 500 MeV) is used in Eq. (1) is similar to the result of Drell and Pagels. (S. D. Drell and H. Pagels, Phys. Rev. 140, B397 (1965)) They computed the nucleon

moments with dispersion theory by continuing the electromagnetic vertex in the nucleon mass and found that the low energy contribution of the absorptive amplitude as given by the Kroll-Ruderman theorem approximately reproduced the magnitude and isovector character of the nucleon moments. The 3,3 channel did not contribute in their analysis but, in common with what is found above, the low energy region gave approximately a correct result.

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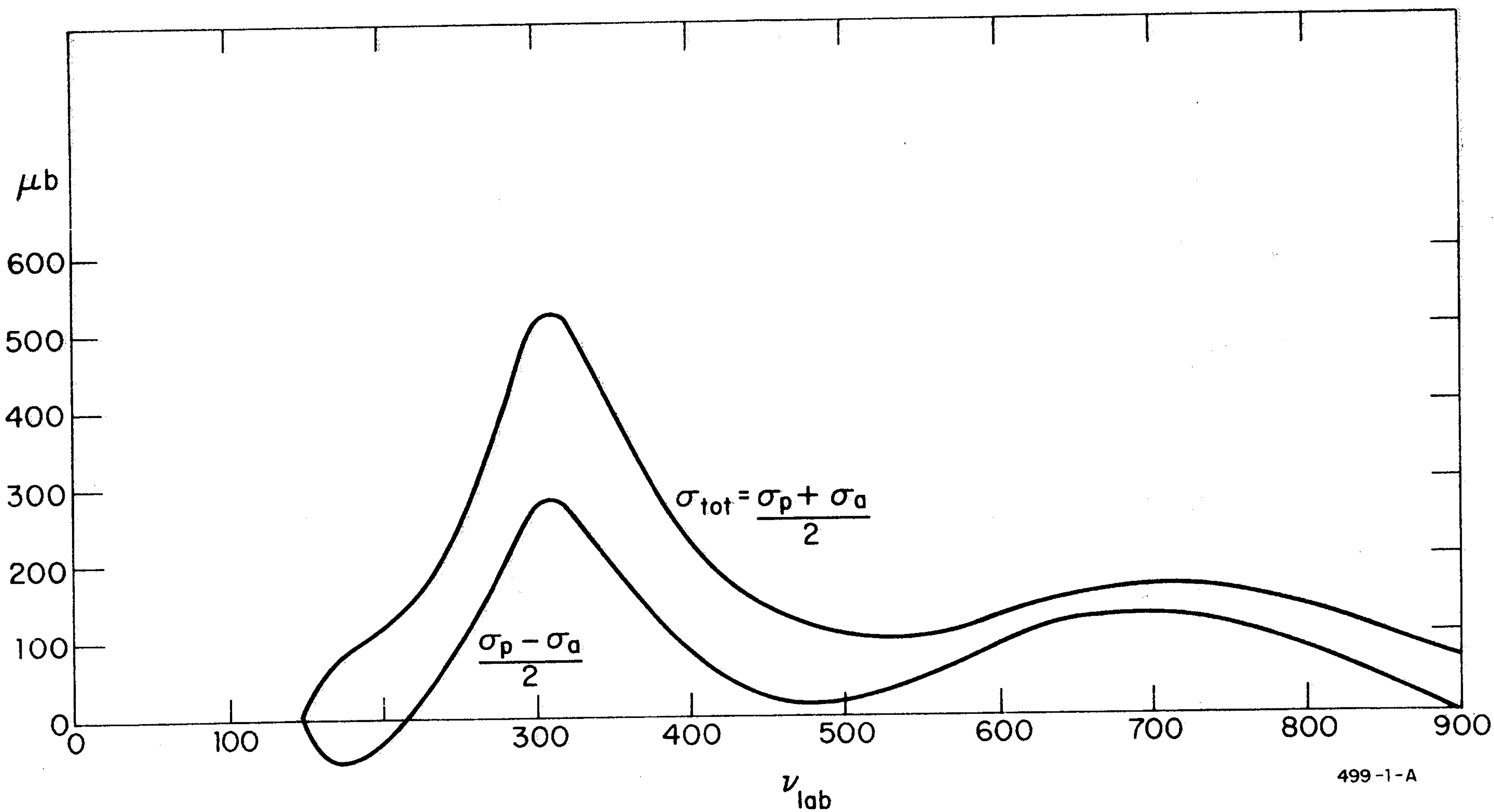
13. Magnetic moment sum rules have also been derived from commutators of the electromagnetic current and axial charge components. (S. Fubini, G. Furlan and C. Rossetti, to be published; S. Gasiorowicz, to be published.) In this case the integrand is proportional to the imaginary part of photoproduction amplitudes and cannot be related directly to physical cross-sections.

FIGURE CAPTION

1. Fit of the Gourdin-Salin model to $\sigma_{tot} = \frac{1}{2}(\sigma_P + \sigma_A)$ and to $\frac{1}{2}(\sigma_P - \sigma_A)$ for single pion photoproduction.


μb
600
500
400
300
200
100
0
0
100
300
400
500
600
700
800
900
$\sigma_{tot} = \frac{\sigma_p + \sigma_a}{2}$
$\frac{\sigma_p - \sigma_a}{2}$
ν_{lab}
499-1-A