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MAGNETIC CENTER LOCATION IN MULTIPOLE FIELDS*

J. K. Cobb
Stanford Linear Accelerator Center, Stanford University, Stanford, California

and

J. J. Muray
Hewlett-Packard Company, Palo Alto, California

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I. INTRODUCTION

The magnetic center of a quadrupole magnet does not necessarily correspond to the mechanical center. When a quadrupole magnet is used as a focusing lens in a high energy particle beam it is especially important to know that the magnetic centerline of the element (quadrupole) coincides with the centerline of the beam. To determine the magnetic centerline of a quadrupole, several methods are available using various properties of the magnetic field. The properties usually used are its symmetries and its lack of radial field in the center. In using the symmetries in the field, a rotating loop can be used. If it is a symmetric loop, the method is reasonably simple since the voltage induced by the quadrupole field is canceled by the symmetry of the coil and the output voltage of the coil is proportional to the magnitude of the dipole field which is only a function of distance from the magnetic center. Many variations on this technique may be used, employing coils of various geometries that are more or less sensitive to the various multipole fields. The accuracy of this method of locating the magnetic center is no better than several thousandths of an inch because of uncertainties in the coil geometry, coil vibrations and runout of the coil driving shaft.

A floating wire is an example of a method of center determination for quadrupoles that uses the property of zero field at the center. One procedure for using this method involves putting a taut wire through the magnet at the approximate center. First, the magnet is energized, then a current is passed through the taut wire, and deflection of the wire is noted as evidence that the wire is not at the magnetic center. The wire is then moved and the process repeated until no deflection of the wire is observed as the wire current is turned on. The wire is then in the magnetic center. The floating wire technique is probably good for center location to an accuracy of a few thousandths of an inch also, but requires a considerable amount of elaborate equipment. As with the rotating coil technique ther are many variations on the floating wire method that can be used to determine magnetic center.

Another method of center location, which we advocate as clearly the most satisfactory of all, involves the use of a colloidal suspension of ferrosoferric oxide particles. This technique was proposed and used by R. M. Johnson[1] to locate the magnetic center of quadrupole fields. The physical mechanism of this method was explained recently as scattering of polarized light on aligned colloidal particles in multipole fields.[2] In this system a small vial of the suspension is placed in the magnetic quadrupole field such that the mechanical center falls within the area of the vial. White plane-polarized light is directed through the vial of solution from one end of the magnet. The observer at the opposite end of the magnet then looks at the vial through a plane-polarizing analyzer which is aligned with the polarizer of incoming light such that complete cancellation of light should occur when the magnetic field is turned off. With magnetic field, complete cancellation does not occur except along two mutually perpendicular axes which cross at the magnetic center of the quadrupole. The accuracy of this type of center determination is of the order of $\pm$ 0.001 inch. (The experimental arrangement is shown in Fig. 1.)

Typical scattering patterns in multipole fields are shown in Fig. 2 for a quadrupole field, in Fig. 3 for a sextupole field, and in Fig. 4 for octupole fields.

The scattering centers in the colloidal solution are Fe_3O_4 crystallites. The preparation of such a colloidal solution is described by D. J. Craik and P. M. Griffiths.[3] The individual crystallites of the magnetite (Fe_3O_4) have been measured with an electron microscope by Craik[4] and it was found that the particles are of the order of 100 Å.

Since this is the method of center location used for the alignment of quadrupole lenses at SLAC, we have looked into the mechanism that causes the dark lines to appear in the solution. Furthermore, we have examined the location of similar sets of lines in sextupoles and octupoles and the theory is shown to predict the location and spacing of lines in these magnets also.

The alignment of these magnetite crystallites in the magnetic field might be explained by the theory of paramagnetic alignment. If N_0 is the number of crystallites per unit volume, the number of aligned scattering centers is given by the following formula

$$N = N_o L(a) = N_o \left[\coth \frac{mH}{kT} - \frac{kT}{mH}\right] ; a = \frac{mH}{kT}$$

where m is the magnetic moment of the colloidal particle, H is the applied field, T is the temperature of the solution, and $L(a)$ is the well-known Langevin function used in the classical theory of paramagnetism. In the case of very strong field or very low temperature, the Langevin function becomes unity, so

$$N = N_o = N_{sat}$$

If all the dipoles are aligned with the field, the number of scattering centers is independent of the applied field, that is, the number of scattering centers is saturated.

The dependence of the sharpness of the scattering pattern on temperature and field can be easily observed by a simple experimental setup, such as that shown in Fig. 1.

II. SYMMETRY RELATIONS IN MULTIPOLE FIELDS

The theory of anisotropic light scattering is very complicated and a rigorous solution of the problem exists only in a few special cases. In this case the symmetry properties of the magnetic multipoles allow a number of simplifications in the calculation of the intensity distribution of the scattering pattern. Such a symmetry relation in a quadrupole field is that any line passing through the center of symmetry with an angle θ, with respect to the X axis, will cross the magnetic field lines at an angle β, where

$$\beta = -\frac{\pi}{2} + 2\theta$$

In order to prove this relation, write the magnetic field in a quadrupole in the following form

$$\vec{H} = -\vec{i}\frac{\partial u}{\partial X} - \vec{j}\frac{\partial u}{\partial Y}$$

where $u = B_2XY$ is the scalar magnetic potential. Thus

$$\vec{H} = -B_2(\vec{i}Y + \vec{j}X)$$

The line which gives the direction of the magnetic field at point Q intersects the X axis with an angle γ (see Fig. 5) which is given by

$$\tan\gamma = \frac{(\vec{H})_Y}{(\vec{H})_X} = \frac{X}{Y} = \frac{r\cos\theta}{r\sin\theta} = \cot\theta = \tan(\pi/2 - \theta)$$

or

$$\gamma = \frac{\pi}{2} - \theta$$

Hence, since $\gamma + \pi - \theta + \beta = \pi$,

$$\beta = \frac{\pi}{2} + 2\theta$$

But β is defined as the angle between two vectors; therefore, one must consider β and $\beta + \pi$ as the angles between the direction of the magnetic field line at point Q and the line passing through the center. This yields

$$\boxed{\begin{aligned}\beta &= \frac{\pi}{2} + 2\theta \\ \beta &= -\frac{\pi}{2} + 2\theta\end{aligned}}$$

A similar analysis can be carried out in the case of the sextupole field. There

$$u = B_3 \left[X^2 Y - \frac{Y^3}{3} \right]$$

$$\vec{H} = -B_3 \left\{ \vec{i}(2XY) + \vec{j}(X^2 - Y^2) \right\}$$

This leads to

$$\tan \gamma = \frac{(\vec{H})_Y}{(\vec{H})_X} = \frac{Y^2 - X^2}{-2XY} = \frac{r^2 \sin^2 \theta - r^2 \cos^2 \theta}{-2r^2 \cos \theta \sin \theta}$$

$$= \frac{-\cos 2\theta}{-\sin 2\theta} = \cot 2\theta = \tan\left(\frac{\pi}{2} - 2\theta\right)$$

From this one obtains $\gamma = \frac{\pi}{2} - 2\theta$, $\gamma + \pi - \theta + \beta = \pi$ gives

$$\boxed{\begin{aligned}\beta &= -\frac{\pi}{2} + 3\theta \\ \beta &= \frac{\pi}{2} + 3\theta\end{aligned}}$$

Quite similarly, for an octupole field the magnetic scalar potential can be written as

$$u = B_4 \, XY \, [X^2 - Y^2]$$

and

$$\vec{H} = - B_4 \left[\vec{i}(3X^2Y - Y^3) + \vec{j}(X^3 - 3XY^2) \right]$$

and from this we have

$$\tan \gamma = \frac{(\vec{H})_Y}{(\vec{H})_X} = \frac{4X^3 - 12XY^2}{12X^2Y - 4Y^3} = \frac{r^3 \cos^3 \theta - 3r^3 \cos \theta \sin^2 \theta}{3r^3 \cos^2 \theta \sin \theta - r^3 \sin^3 \theta} = \frac{1}{\tan 3\theta} = \cot 3\theta$$

Then

$$\gamma = \frac{\pi}{2} - 3\theta$$

and from $\gamma + \pi - \theta + \beta = \pi$, one gets

$$\boxed{\begin{aligned} \beta &= -\frac{\pi}{2} + 4\theta \\ \beta &= \frac{\pi}{2} + 4\theta \end{aligned}}$$

It was observed that the scattering pattern does not change with a change in polarity, which means that a particle aligned parallel with the magnetic field scatters the same way in the scattering process as a particle that is aligned opposite to the field. Particles with induced magnetic moments are aligned along the field lines irrespective of the relative directions of the magnetic field $\vec{H}$ and the moment $\vec{m}$. Therefore, the relative orientations of $\vec{m}$ and $\vec{H}$ are not taken into account in further calculations. The symmetry relations used for the following calculations can be written as

$$\beta = 2\theta - \frac{\pi}{2} \quad \text{for quadrupole fields}$$

$$\beta = 3\theta - \frac{\pi}{2} \quad \text{for sextupole fields}$$

$$\beta = 4\theta - \frac{\pi}{2} \quad \text{for octupole fields}$$

III. THEORY OF LIGHT SCATTERING ON ALIGNED PARTICLES IN MULTIPOLE FIELDS

In order to explain the intensity distribution of the scattered polarized light on the aligned magnetite crystallites, one can assume anisotropy in the scattering process. One of the simplest assumptions is that the aligned magnetite has a different polarizability along the magnetic field than it does perpendicular to the field. The net effect of this anisotropy in the scattering is a rotation of the initial angle of polarization along certain lines going through the center of the multipole fields. The polarizability tensor in the coordinate system of the aligned particle (X'-Y') can then be written as

$$\alpha_{ik} = \begin{vmatrix} \alpha_{\perp} & 0 \\ 0 & \alpha_{\parallel} \end{vmatrix}$$

In order to calculate the polarizability tensor in the X-Y coordinate system, it is desirable to use the symmetry properties of the multipole fields. Figure 6 shows the relative orientation of the (X'-Y') coordinate system to the (X-Y) system in a quadrupole magnetic field.

With these relationships, the polarizability tensor in the X-Y system can be expressed using a similarity transformation (see Fig. 6)

$$|\alpha_{ik}|_{XY} = S\left(\frac{\pi}{2} - \theta + \beta\right) \begin{vmatrix} \alpha_{\perp} & 0 \\ 0 & \alpha_{\parallel} \end{vmatrix} S\left(-\frac{\pi}{2} + \theta - \beta\right)$$

where $S\left(\frac{\pi}{2} - \theta + \beta\right)$ is the transformation matrix, i.e.,

$$S\left(\frac{\pi}{2} - \theta + \beta\right) = S(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

with the use of the symmetry relation, $\beta = -\frac{\pi}{2} + 2\theta$.

Using $|\alpha_{ik}|_{XY}$, all quantities can be expressed in the X-Y coordinate system and the acattering amplitude can be calculated easily. The size of the scattering centers (100 - 1000 $\mathring{A}$) is small as compared to the polarized light, so the Rayleigh approximation can be used. In this case the total intensity of the scattered light is the sum of the scattered intensities of each of the scattering centers, and the scattering amplitude by the i-th volume element of the system at the location of the observer is given[5] as

$$A_i = K(\vec{P}_i \cdot \vec{O}) \cos k(\vec{r}_i \cdot \vec{s})$$

The induced dipole in the i-th volume element is $\vec{P}_i$, which is located a distance r_i from the origin, $k = 2\pi/\lambda$ (λ = wavelength in the medium); $\vec{s} = \vec{s}' - \vec{s}_o$ where $\vec{s}'$ and $\vec{s}_o$ are unit vectors along the scattered and incident beams; $\vec{O}$ is the unit vector perpendicular to the scattered light beam and along the polarization direction of the scattered light; K is a proportionality constant.

The dipole moment P_i is given by

$$P_i = \alpha_{ik} E_K$$

In the X-Y coordinate system the components of $\vec{E}$ are given as

$$\vec{E} = E_o \left[(\cos\phi)\vec{i} + (\sin\phi)\vec{j}\right]$$

where ϕ is the angle of polarization, ϕ being measured counterclockwise from the Y axis. The components of $\vec{O}$ can be expressed as

$$\vec{O} = \left[(\sin\phi)\vec{i} - (\cos\phi)\vec{j}\right]$$

when observation is perpendicular to the X-Y plane and along the symmetry axis of the multipoles. In this case $\vec{s}' = \vec{s}_o$ and $\cos k(\vec{r}_i \cdot \vec{s}) = 1$.

The total amplitude of the scattered light from the X-Y plane can be written

$$A = \Sigma A_i = K' \int_{r=o}^{R} \int_{\phi=o}^{2\pi} (\vec{O} \cdot \vec{P}) r dr d\phi$$

By squaring the total amplitude, the intensity is obtained.

The angle θ relative to the X axis at which the intensity is zero for a given polarization angle ϕ is given by the expression

$$(\vec{P} \cdot \vec{O}) = P_X \sin\phi - P_Y \cos\phi = 0$$

Scattering processes in different multipoles will now be considered.

A. Light Scattering on Aligned Particles in a Quadrupole Field

In a quadrupole field, the dielectric tensor in the X-Y system can be written as

$$\alpha_{ik}\big|_{XY} = \begin{pmatrix} \alpha_\perp \cos^2\theta + \alpha_\parallel \sin^2\theta & \frac{\alpha_\parallel - \alpha_\perp}{2} \sin 2\theta \\ \frac{\alpha_\parallel - \alpha_\perp}{2} \sin 2\theta & \alpha_\perp \sin^2\theta + \alpha_\parallel \cos^2\theta \end{pmatrix}$$

and the induced dipole moment as

$$\vec{P} = |\alpha|_{XY} \vec{E} = E_o \begin{pmatrix} \alpha_\perp \cos^2\theta \cos\phi + \alpha_\parallel \sin^2\theta \cos\phi + \frac{\alpha_\parallel - \alpha_\perp}{2} \sin 2\theta \sin\phi \\ \frac{\alpha_\parallel - \alpha_\perp}{2} \sin 2\theta \cos\phi + \alpha_\perp \sin^2\theta \sin\phi + \alpha_\parallel \cos^2\theta \sin\phi \end{pmatrix}$$

And with this

$$(\vec{O} \cdot \vec{P}) = \frac{\alpha_\perp - \alpha_\parallel}{2} \sin 2(\phi + \theta)$$

The scattering intensity is proportional to the square of the amplitude; consequently,

$$I_i \alpha A_i^2 = K^2 (\vec{O} \cdot \vec{P})^2 = K^2 \left(\frac{\alpha_\perp - \alpha_\parallel}{2}\right)^2 \sin^2 2(\phi + \theta)$$

The numerical value for the constant K might be obtained from the Rayleigh formula, from which

$$K^2 = \frac{8\pi^4}{\lambda^4} N_i E^2$$

where N_i is the density of scattering centers in volume element V_i , and

$$N_i = N_o \left[\coth \frac{mH}{kT} - \frac{kT}{mH}\right]$$

The location of the zero intensity lines can be obtained for a given polarization angle ϕ, in terms of ϕ , from the scattering intensity formula as

$$\theta = \frac{n\pi}{2} - \phi \quad \text{where} \quad n = 0,\ 1,\ 2\ldots$$

Along the Z axis, the scattered light intensity from volume element V_i can be expressed as

$$I = \frac{8\pi^4 N_o E^2}{\lambda^4} \left[\coth \frac{mH}{kT} - \frac{kT}{mH}\right] \left[\vec{O} \cdot \vec{P}\right]^2$$

B. Light Scattering on Aligned Particles in Sextupole and Octupole Fields

In sextupole and octupole fields, the magnetic field intensity changes as $(B_o/R_o^2)r^2$ and $(B_o/R_o^3)r^3$, respectively, where B_o is the field at the pole faces, R_o is the half-aperture, and $r^2 = X^2 + Y^2$.

Therefore, the magnetic field intensity is very low near the Z axis and is not sufficient to align the scattering centers in the field direction. This might be the reason for the unclear scattering picture near the Z axis as seen in Figs. 3 and 4.

In a sextupole field the dipole moment can be written as

$$\vec{P} = |\alpha|_{XY}\vec{E} = E_o \begin{pmatrix} (\alpha_\perp \cos^2 2\theta + \alpha_\parallel \sin^2 2\theta) \cos\phi - \dfrac{(\alpha_\perp - \alpha_\parallel)}{2} \sin 4\theta \sin\phi \\ \\ -\dfrac{(\alpha_\perp - \alpha_\parallel)}{2} \sin 4\theta \cos\phi + (\alpha_\perp \sin^2 2\theta + \alpha_\parallel \cos^2 2\theta) \sin\phi \end{pmatrix}$$

where ϕ is the angle of polarization. Using this, one finds that the intensity is

$$I = k^2 (\vec{O} \cdot \vec{P})^2 = k^2 \left(\frac{\alpha_\perp - \alpha_\parallel}{2}\right)^2 \sin^2 4(\theta + \tfrac{\phi}{2}) \quad \text{and} \quad I = O \quad \text{when}$$

$$n\pi = 4(\theta + \tfrac{\phi}{2}) \quad \text{or} \quad \theta = n\tfrac{\pi}{4} - \tfrac{\phi}{2} \quad \text{where} \quad n = 0, 1, 2, 3$$

Quite similarly, for an octupole field the dipole moment of the aligned colloidal particles can be written as:

$$\vec{P} = |\alpha|_{XY}\vec{E} = E_o \begin{pmatrix} \alpha_\perp \sin^2 3\theta \cos\phi + \alpha_\parallel \cos^2 3\theta \cos\phi - \left(\dfrac{\alpha_\parallel - \alpha_\perp}{2}\right) \sin 6\theta \sin\phi \\ \\ -\left(\dfrac{\alpha_\parallel - \alpha_\perp}{2}\right) \sin 6\theta \cos\phi + (\alpha_\perp \cos^2 3\theta + \alpha_\parallel \sin^2 3\theta) \sin\phi \end{pmatrix}$$

and with this

$$I = k^2 (\vec{O} \cdot \vec{P})^2 = k^2 \left(\frac{\alpha_\perp - \alpha_\parallel}{2}\right)^2 \sin^2 6(\theta + \tfrac{\phi}{3})$$

$$I = O \quad \text{when} \quad n\pi = 6(\theta + \tfrac{\phi}{3}) \quad \text{or} \quad \theta = n\tfrac{\pi}{6} - \tfrac{\phi}{3}$$

In both cases the observed locations of dark lines characterized by the azimuth angle θ agree with the calculated values for a given polarization angle ϕ. At zero polarization angles, as shown in Figs. 3 and 4, the dark lines are located at

$$\theta = 0^\circ, 45^\circ, 90^\circ, \text{ and } 135^\circ$$

for the sextupole field, and at

$$\theta = 0^{\circ},\ 30^{\circ},\ 60^{\circ},\ 90^{\circ},\ 120^{\circ},\ \text{and } 150^{\circ}$$

for octupole fields. It is interesting to note that the angular separation of the dark lines is 45° in a sextupole field and 30° in the octupole field (see Figs. 3 and 4).

Table I lists the calculated azimuthal location of the dark lines as a function of the polarization angle $\phi (0 > \phi > -60^{\circ})$.

REFERENCES

1. R. M. Johnson, Internal Report BeV-687, Lawrence Radiation Laboratory, Berkeley, California (1961).

2. J. J. Muray, J. of Appl. Optics, 4, 1011 (1965); J. K. Cobb and J. J. Muray, IEEE Trans. on Nuclear Science 12, 395 (1965).

3. D. J. Craik and P. M. Griffiths, Proc. Phys. Soc. (Vol. B) 70, 1000 (1957).

4. D. J. Craik, Proc. Phys. Soc. (Vol. B) 69, 647 (1956).

5. H. C. van de Hulst, Light Scattering by Small Particles, (Wiley and Sons, New York, 1957).

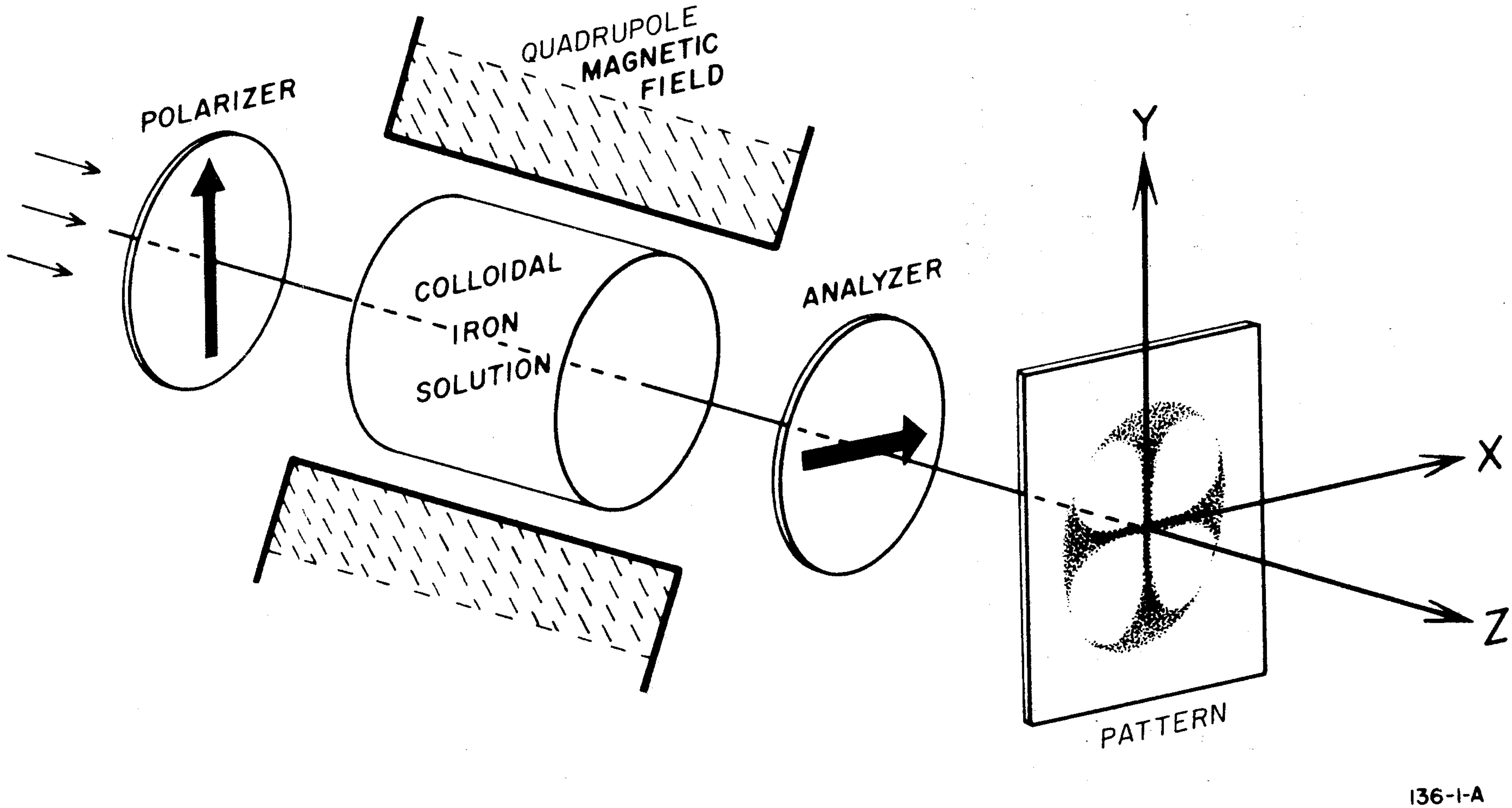

FIG. 1-SETUP FOR MAGNETIC CENTER LOCATION IN QUADRUPOLE MAGNETIC FIELD

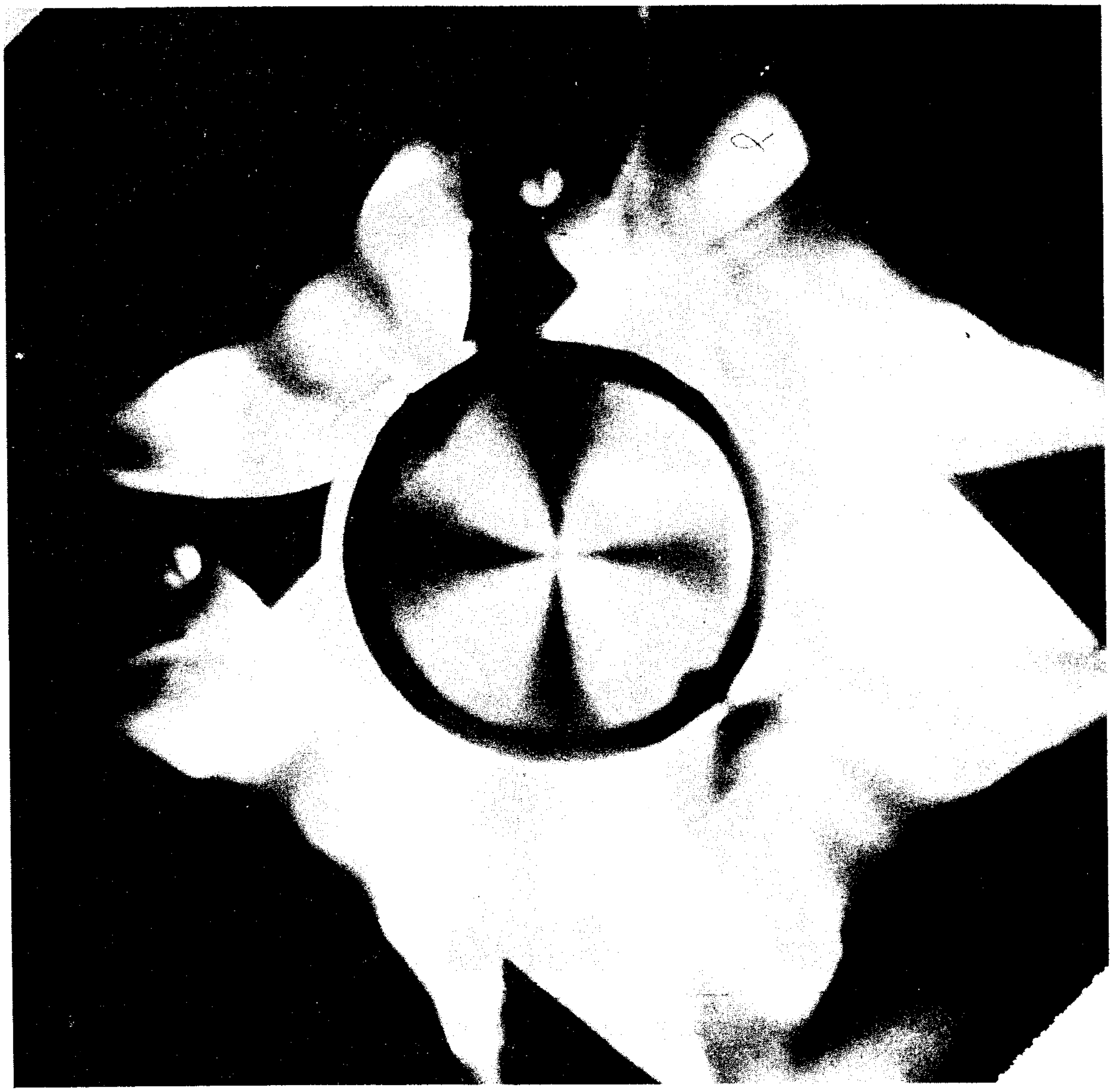

Fig. 2--Light scattering pattern in a quadrupole magnetic field ($\phi = 0$).

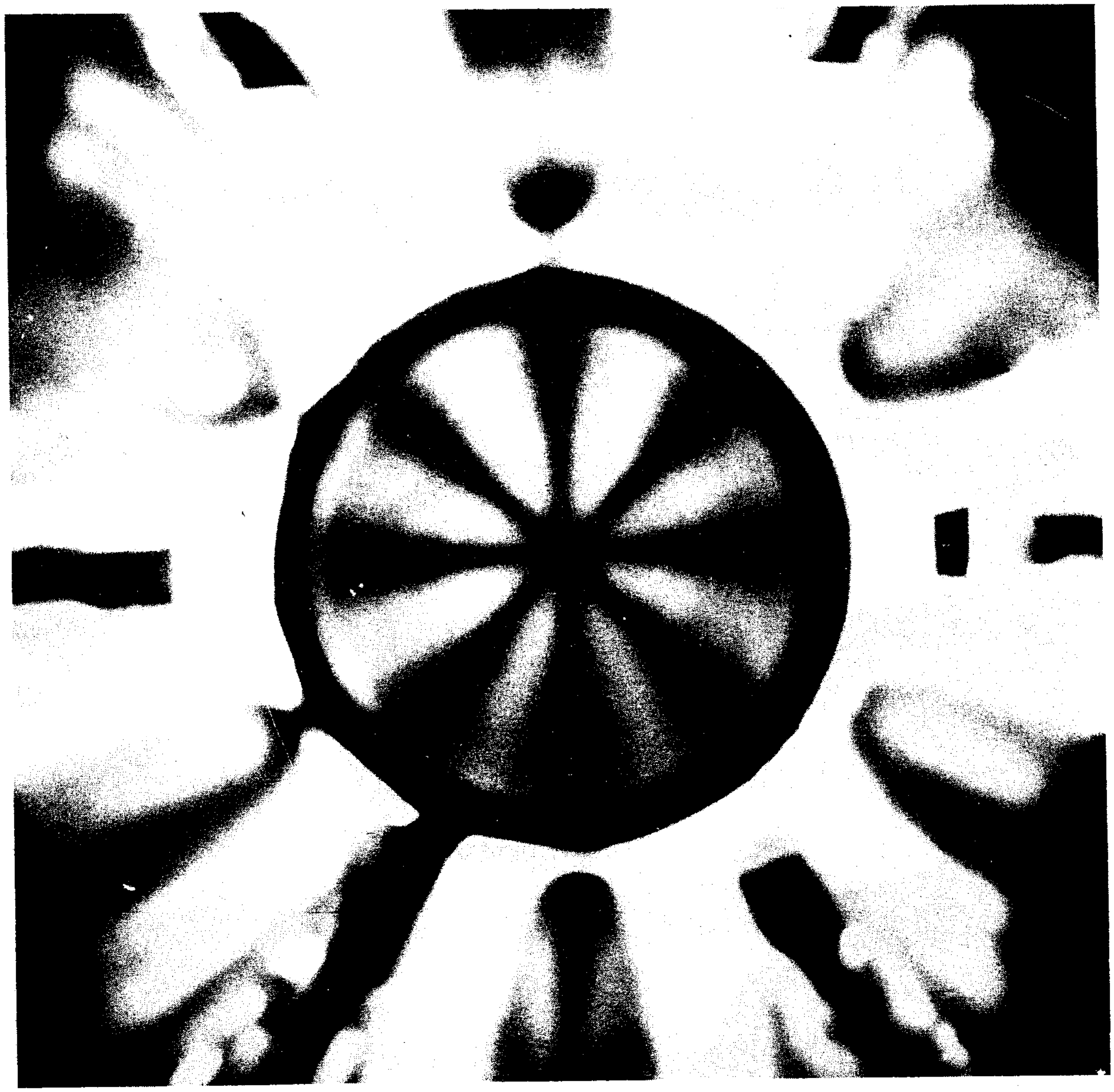

Fig. 3--Light scattering pattern in a sextupole magnetic field ($\phi = 0$).

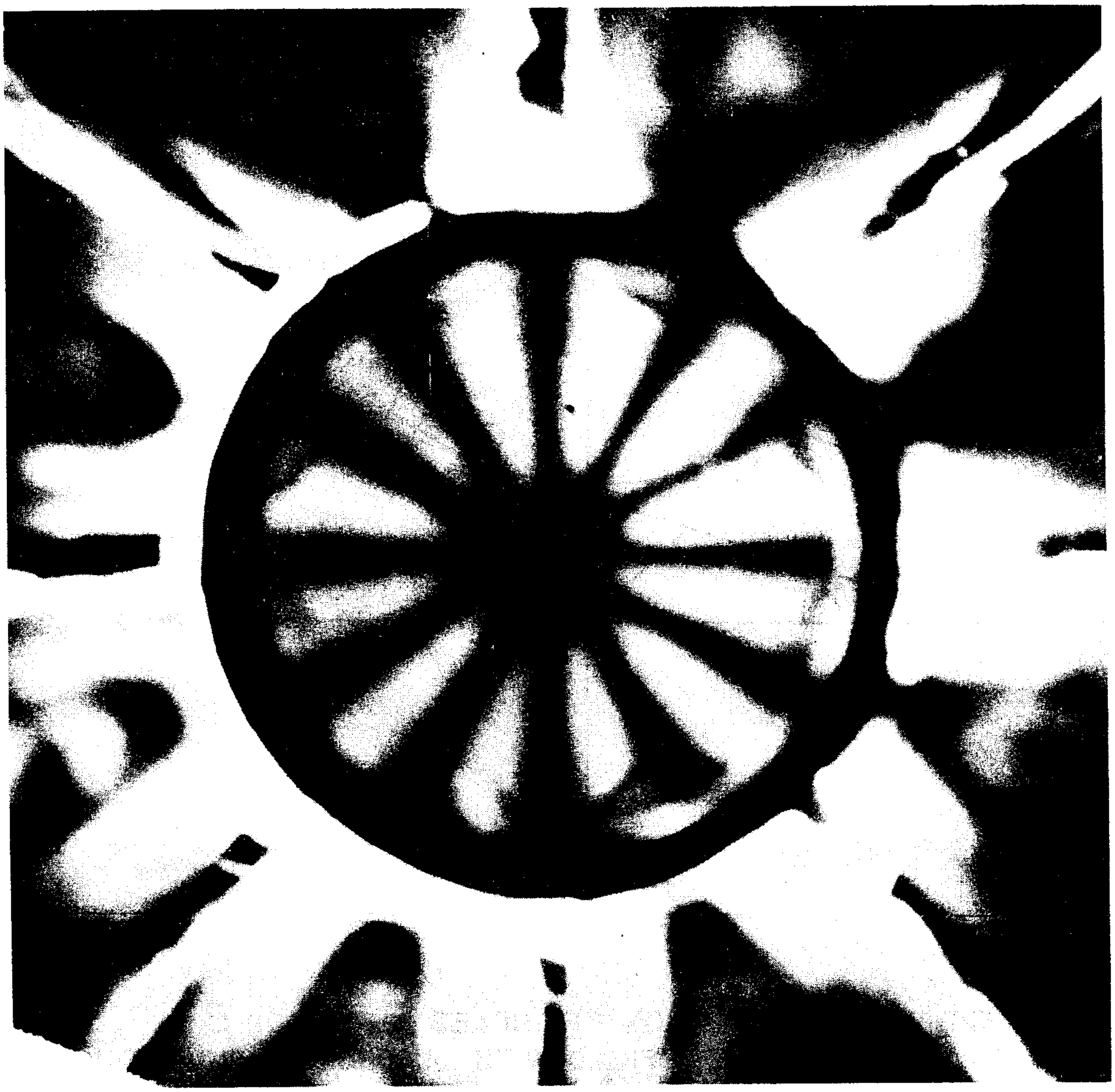

Fig. 4--Light scattering pattern in an octupole magnetic field (ϕ = 0).

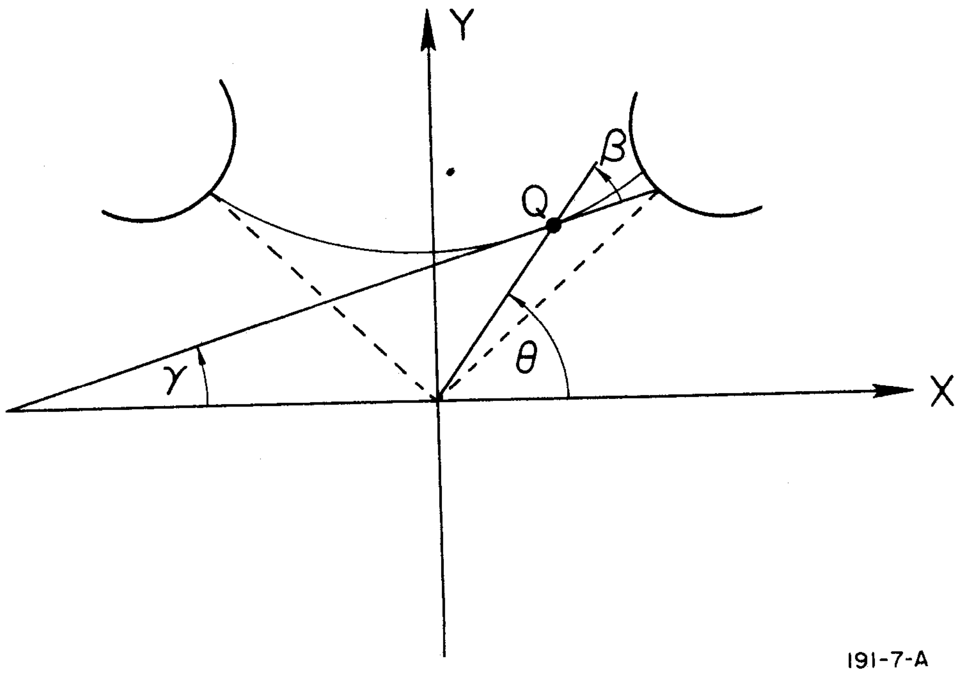

FIG 5 - INTERRELATION OF ANGLES γ, θ, AND β IN A MAGNETIC FIELD WITH QUADRUPOLE SYMMETRY.

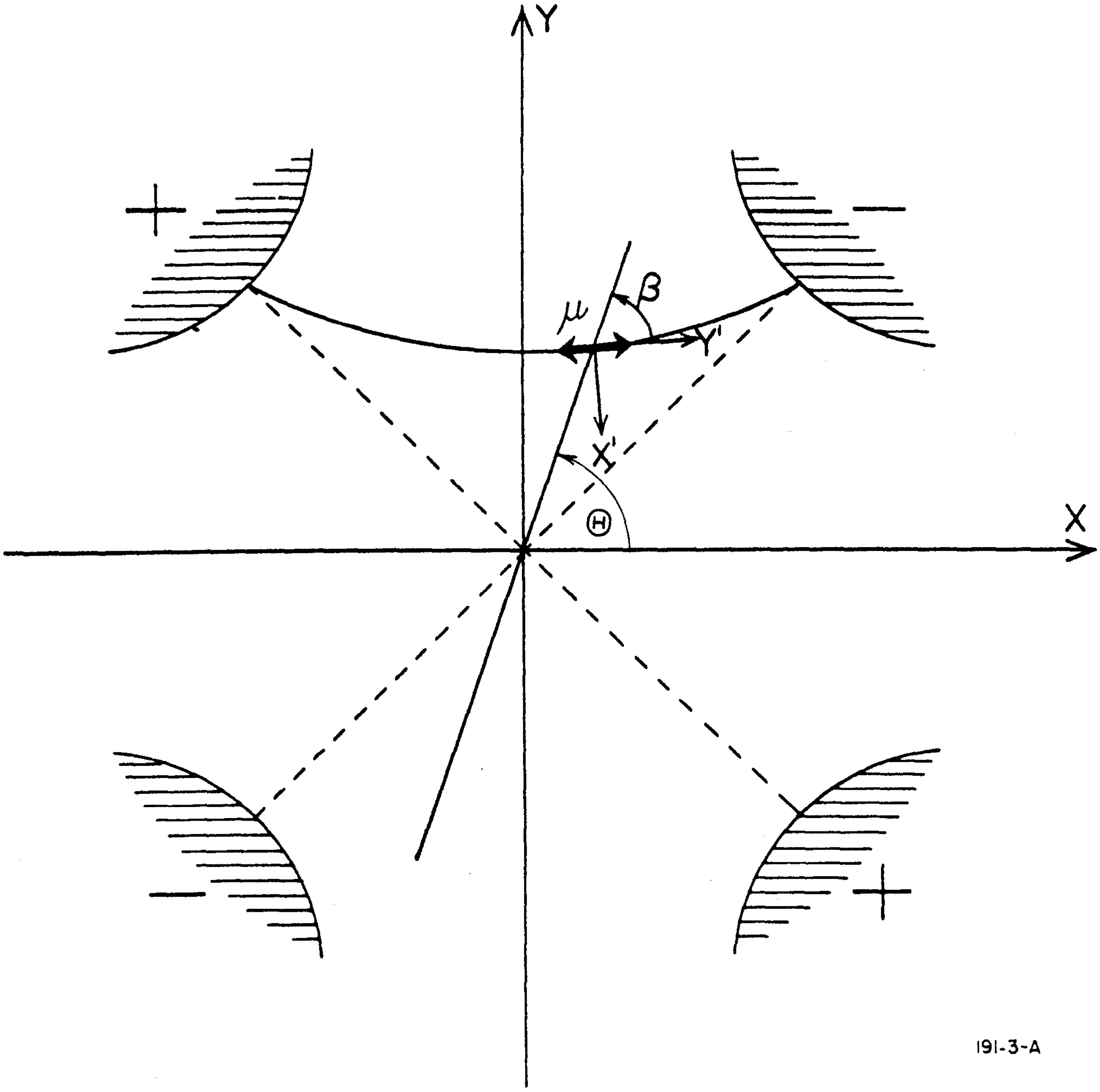

FIG. 6 ORIENTATION FOR A MAGNETIC DIPOLE μ IN A MAGNETIC FIELD WITH QUADRUPOLE SYMMETRY.

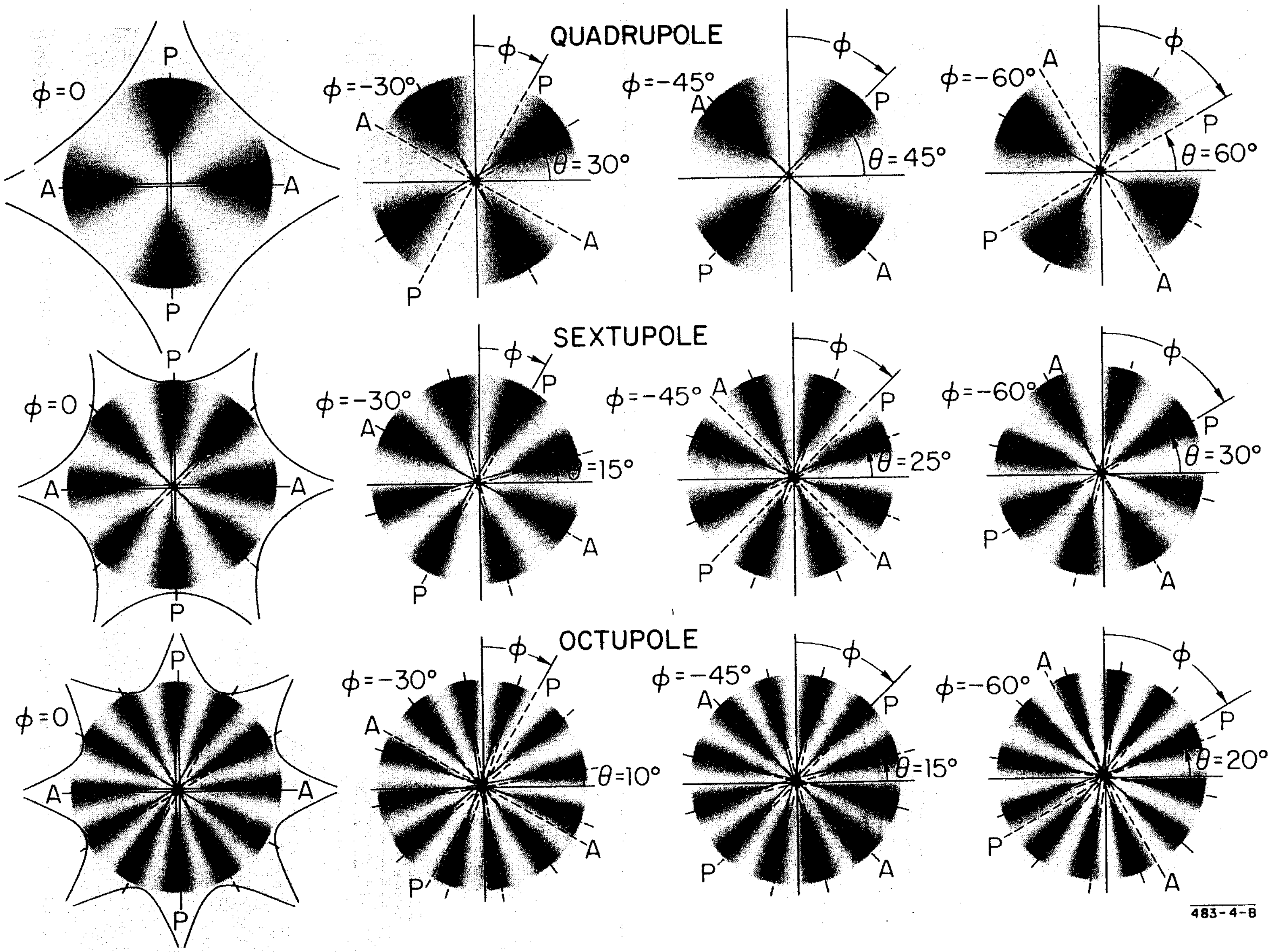

TABLE I. ANGULAR POSITION OF ZERO INTENSITY LINES