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RANDOM PHASE K-MATRIX APPROACH TO ABSORPTION IN PERIPHERAL INTERACTIONS*

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ABSTRACT

Initial- and final-state corrections to single-particle exchange amplitudes are studied. Two multichannel K-matrix formalisms, which are expected to be valid for different energy regions, are developed by means of a random phase approximation, and two similar absorption formulas are derived. Both reduce to the Gottfried-Jackson form for weak absorption, but give more absorption in the lower partial waves. The models are applied to the reactions $\pi^{\pm} + p \rightarrow \pi^{\pm} p$ for scattering angles near 180°, and to the reaction $\pi^{-} p \rightarrow \pi^{0} n$ for small momentum transfer. It is found that the improved absorption models do not solve the problem found in earlier work of failing to fit the energy dependence of the cross sections. However, the increased absorption makes it possible to fit the data in the 4-10 BeV/c region by an arbitrary choice of absorption parameters.

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Section I

There has been a great deal of interest lately in so called absorption models of high energy interactions. The first of these models[1] was based on an extrapolation of the low energy distorted wave Born approximation (DWBA), and has the form

$$A_{12} = \sqrt{S_{11}}\, B_{12}\, \sqrt{S_{22}} \tag{1.1}$$

In Fig. 10, we give a simple physical interpretation of this equation. The two interacting particles first scatter elastically by means of an initial state interaction (the first "blob"), then scatter inelastically by some process (second blob), and the two emerging particles scatter elastically through a final state interaction (third blob). The idea of the absorption model is to treat the initial and final state interaction exactly by putting in the experimentally observed amplitudes for S_{11} and S_{22} and to treat the inelastic scattering in some approximate way (usually by using the Born amplitude).

In this paper we shall derive two similar absorption models on the basis of a multichannel K matrix formalism. We shall see that the simple interpretation given above will be valid only for high partial waves (i.e., for high impact parameters), and that at lower partial waves results differing somewhat from Eq. (1.1) will hold.

We shall treat the two interactions

$$\pi^{\pm} + p \rightarrow \pi^{\pm} + p$$

at scattering angles near 180°, and the reaction

$$\pi^{-} p \rightarrow \pi^{\circ} n$$

in the forward direction. In all that follows, we shall work in the center of mass frame of the πN system.

We shall find that the model of Gottfried and Jackson fails to fit the experimental data in both interactions, although it still represents a tremendous improvement over the unmodified Born amplitude. The K matrix models, because they introduce more absorption into the lower partial waves, can be made to give fairly good fits to the experimental data for both reactions. However, as we shall see, they can be made to do so only by a rather arbitrary choice of the absorption parameter.

The greatest problem with the absorption models will be seen to be their complete failure to predict the observed drop of $\frac{d\sigma}{d\Omega}$ as the energy is increased. They can be made to follow this drop by increasing the amount of absorption in the S wave, but this procedure can be justified only in a rough way. In Section VI we discuss the possibility of improving this aspect of the model by Reggeization of the πN inelastic amplitude.

The general plan of the paper will be as follows: in Section II we shall work out the helicity amplitudes for the above interactions. In Sections III and IV, we shall derive our absorption equations, and in Sections V and VI we shall present and discuss numerical results.

Section II

Consider a πN reaction of the type as in Fig. 11. We then write the scattering matrix as

$$S_{Fi} = \delta_{fi} + \frac{(2\pi)\ \delta\ (P_f - P_i)}{V^2\ \sqrt{2wE\ 2w'\ E'}}\ A_{fi} \tag{2.1}$$

so that the cross section is

$$\frac{d\sigma}{d\Omega} = \frac{1}{32\pi^2 S}\left(\sum_{\text{spins}} |A_{fi}|^2\right) \tag{2.2}$$

and the partial wave elastic unitarity condition reads

$$\text{Im}\ A^{\ell}_{fi} = \frac{1}{4\pi}\ \frac{|\vec{p}|}{\sqrt{s}}\ |A^{\ell}_{fi}|^2 \tag{2.3}$$

It is well known[2] that the most general πN amplitude can then be written in the form

$$A = F_s\ \bar{u}(p')\ u(p) + F_v\ \bar{u}(p')\ \not{q}\ u(p) \tag{2.4}$$

where F_s and F_v depend only on the scalars of the problem. The usual way to write these amplitudes is in terms of the helicity expansion,[3] so that we will need the following results:[2]

$$\begin{aligned} \bar{u}(p'+)\ u(p+) &= (1 - rr')\ e^{-i\varphi/2}\ \cos\theta/2 \\ \bar{u}(p'+)\ u(p-) &= (1 + rr')\ e^{i\varphi/2}\ \sin\theta/2 \end{aligned} \tag{2.5a}$$

and

$$\bar{u}(p'+) \not{q} u(p+) = \cos \theta/2 \; e^{-i\varphi/2} \left[w(1 + r') + 2qr\right]$$

$$\bar{u}(p'+) \not{q} u(p-) = \sin \theta/2 \; e^{i\varphi/2} \left[w(1 - rr')\right] \tag{2.5b}$$

where we have set

$$r = \frac{|\vec{p}|}{E + M}$$

Since we shall usually be considering scattering which is kinematically elastic, we can set $r = r'$ in the above expressions. We shall also define the scattering plane such that $\varphi = 0$.

We expect that the cross section for back scattering will be dominated by central collisions, that is, by those pions which penetrate to the core of the nucleon. Consequently, we expect that the "nucleon exchange" graph, shown in Fig. 12, will play an important role.

A standard calculation[4] easily gives

$$F_s = 0$$

$$F_v = g^2_{\pi N} \frac{M}{\mu^2 - 2E_w - 2p^2 \cos \theta} T_I \tag{2.6}$$

where T_I is the isotopic factor.

The other Born diagram, containing the uncrossed diagram, gives

$$F_s = 0$$

$$F_v = \frac{g_{\pi N}^2 M}{(E + w)^2 - M^2} T_I$$

which is seen to be a pure S wave contribution. Since the idea of the absorption models is to remove most or all of the lower partial waves, we do not expect this diagram to contribute much to our final cross section.

The cross section for the nucleon exchange diagram can easily be shown to have the following asymptotic form:

$$\left.\frac{d\sigma}{d\Omega}\right|_{\theta=180^\circ} \simeq \left(\frac{g^2}{4\pi}\right) \frac{1}{M^2} \simeq 80\text{mb/st} \qquad (2.7)$$

which is a factor of $\sim 10^3$ above the experimental value.

One possible improvement of the Born term would be to realize that the accepted value of

$$g_{\pi N/4\pi}^2 = 15$$

is derived for the case where the nucleon is on the mass shell. In our case, however, the nucleon is virtual, and in fact, has zero mass. This problem has been investigated,[5] and the net result seems to be that if we include the modifications in $g_{\pi N}$ due to the fact that the nucleon is off-the-mass-shell, and, in addition, include the propagator modification, the amplitude is reduced by a factor $\sim 1/4$. While this is still not enough to bring Eq. (2.7) into agreement with experimental results, we shall see that it plays an important part in fitting the data with our absorption models.

We shall also want the amplitudes for reactions such as those shown in Fig. 13. The amplitude for the crossed diagram is proportional to

$$\left(\frac{\lambda}{m_\pi}\right)^2 \bar{u}\, p'^{\alpha} \left\{ g_{\alpha\beta} - \frac{4\xi_\alpha \xi_\beta}{3M^{*2}} + \frac{\xi_\alpha \not{\xi} \gamma_\beta}{3M^{*2}} + \frac{\gamma_\alpha \not{\xi} \xi_\beta}{3M^{*2}} - \frac{1}{3}\gamma_\alpha \gamma_\beta \right\} p^\beta \frac{1}{\not{p} - \not{q}' - M^*} u \qquad (2.8)$$

so that some algebra gives

$$F_v = \frac{M}{D}\left(\frac{\lambda}{m_\pi}\right)^2 \left\{ - p\cdot p' + \frac{4}{3M^{*2}}(\xi\cdot p)(\xi\cdot p') + \frac{\xi\cdot p'}{3M^{*2}}\left(- M^2 - MM^* + 2p\cdot q'\right) \right.$$

$$\left. - \frac{\xi\cdot p}{3M^{*2}} M^2 - \frac{M^2}{3} \right\} \qquad (2.9)$$

$$F_s = M\left(\frac{\lambda}{m_\pi}\right)^2 \frac{1}{D}\left\{ (M + M^*)\left(p\cdot p' - \frac{4}{3M^{*2}}(\xi\cdot p)(\xi\cdot p')\right) + \frac{\xi\cdot p'}{3M^{*2}}\left(M^2(M + M^*) - M\mu^2\right) \right.$$

$$\left. + \frac{M(\xi\cdot p)}{3M^{*2}}\left((M + M^*)M - 2p\cdot q'\right) + \mu^2 \right) + \frac{M}{3}\left(2p\cdot q' - (M + M^*)M\right) \Big\}$$

where we have set

$$D = (p - q')^2 - M^{*2}$$

$$\xi = (p - q')$$

The uncrossed amplitude can be obtained from this by the simple substitution

$$-q \rightleftarrows q'$$

and so the uncrossed diagram contributes only to the S and P wave amplitudes, and, as with the simple nucleon exchange term, will not contribute very much to the cross section.

The coupling constant is calculated in the usual way to be

$$\frac{\lambda^2}{4\pi} = \left[\frac{3M^* m_\pi^2}{p^3(E + M)}\right] \Gamma \qquad (2.10)$$

which, if we take

$$\Gamma = 125 \text{ MeV}$$

gives

$$\lambda = 2.18 \qquad (2.11)$$

Of course, the remarks about virtual effects made earlier apply here, too. In the absence of any theoretical work on the problem for the N*, however we have assumed that the reduction due to virtual effects was the same for the N and N* exchange.

The reaction

$$\pi^- p \rightarrow \pi^\circ n$$

we expect to be dominated by Fig. 14. This diagram has already been investigated in the DWBA.[6,7] We can write the amplitude as

$$A = \frac{Mg_{\rho\pi\pi}}{\xi^2 - m_\rho^2}\,\bar{u}\left\{\left[G_v\gamma_\mu + \frac{G_T}{2M}\,\sigma_{\mu\nu}\,\xi^\nu + \frac{G_s}{2M}\,\xi_\mu\right]\left(g_{\mu\nu} - \frac{\xi_\mu\xi_\nu}{m_\rho^2}\right)(q+q')^\nu\right\}u \qquad (2.12)$$

where G_v, G_T, and G_s represent the vector, tensor, and scalar coupling constants at the lower vertex.

The coupling constant at the $\rho\pi\pi$ vertex can be evaluated[8] to give

$$\frac{g^2_{\rho\pi\pi}}{4\pi} = \frac{12\Gamma}{m_\rho}\left(1 - \frac{4m_\pi^2}{m_\rho^2}\right)^{-3/2} \qquad (2.13)$$

so that if $\Gamma = 100$ MeV,

$$\frac{g^2_{\rho\pi\pi}}{4\pi} = 2.0 \qquad (2.14)$$

Following the usual universal coupling scheme,[9] we set

$$G_v = g_{\rho\pi\pi} \qquad (2.15)$$

and we use the ρ - photon anology[10] to give

$$K_V = \frac{G_T}{G_v} = 3.7 \qquad (2.16)$$

Some simple algebra leads to the result

$$F_s = \frac{2M}{\xi^2 - m_\rho^2} (G_V + G_T)\, T_I$$

$$F_v = \frac{-M}{\xi^2 - m_\rho^2} \left(\frac{G_T}{2M}\right) (p + p')\cdot(q + q')\, T_I \qquad (2.17)$$

Unlike the nucleon exchange amplitude, the asymptotic form of this graph can be shown to be

$$\left.\frac{d\sigma}{d\Omega}\right|_{\theta=0} \simeq \left[\frac{g_{\rho\pi\pi}^2}{4\pi} \frac{G_V^2}{4\pi}\right] \cdot \frac{4p^2}{m_\rho^4} \qquad (2.18)$$

As we shall see later, this highly unphysical asymptotic form $\left(\frac{d\sigma}{d\Omega} \sim p^2\right)$ will cause serious problems in our absorption calculations.

Section III

In this section we shall present a derivation of a multichannel K matrix absorption model which is somewhat unphysical, but which is algebraically simple enough to allow explicit calculation of the absorption effect due to the fact that the real part of the elastic scattering amplitude is not zero.

Before going on, however, it will be useful to consider some general remarks on the K matrix. If we restrict our attention to two particles intermediate states, then the unitarity condition for each partial wave in our normalization reads

$$\text{Im}\ \underset{\sim}{A} = \underset{\sim}{A}^{+}\ \underset{\sim}{r}\ \underset{\sim}{A} \tag{3.1}$$

where

$$r_{ij} = r_i\ \delta_{ij}$$

and

$$r_i = \frac{1}{4\pi}\frac{P_i}{\sqrt{s}}$$

and where we have suppressed the partial wave index on the matrices. We shall do this in all that follows, and every equation must be understood as applying to each partial wave.

If we write the amplitude as

$$\underset{\sim}{A} = \underset{\sim}{K}\ (\underset{\sim}{I} - i\underset{\sim}{r}\underset{\sim}{K})^{-1} \tag{3.2}$$

then Eq. (3.1) is automatically satisfied so long as K is real and Hermitian.

It is also known that[11] in the N/D formalism,

$$\underset{\sim}{K} = \underset{\sim}{N}\left[\mathrm{Re}\ \underset{\sim}{D}\right]^{-1} \tag{3.3}$$

so that to a first approximation we can set

$$K_{ij} = B_{ij}$$

where B_{ij} is the Born amplitude for the process in question. We see, then, that the K matrix gives a way of constructing unitary amplitudes from Born amplitudes, which are not in themselves unitary.

However, as indicated in the above discussion, the K matrix in the form of Eq. (3.2) takes account of two particle intermediate states only, and does not consider the effect of 3 or more particle intermediate states on the unitarity condition. In addition, in the first approximation given here, the virtual effects are not included, although we shall see that they must be put in to match the data for πN back scattering.

Even with these shortcoming, however, the K matrix formalism is extremely useful in that no matter what approximations we make in the elements of $\underset{\sim}{K}$, the scattering amplitude will be unitary so long as $\underset{\sim}{K}$ is real and Hermitian.

Let us consider a system in which we wish to have two important channels, and in which we desire to estimate the effect of all other channels in an approximate way. Now we know that the Born amplitude near threshold behaves like

$$B_\ell \sim p^\ell$$

Suppose that the two channels which are important and let us take them to be the entrance and exit channel. Let us also suppose they can be

considered to be well past threshold at the energy we are considering, while all other open channels can be considered to be near threshold. Then two points are clear:

$$K_{1j} \gg K_{mj} \qquad m \neq 1,2$$

$$K_{2j} \gg K_{mj}$$

and

$$r_1 \gg r_i \qquad i \neq 1,2$$

$$r_2 \gg r_i$$

It seems reasonable, then to approximate (1 - irK) by

$$\begin{pmatrix} 1-ir_1\alpha & -ir_1\beta & -ir_1C_3 & \cdots\cdots & -ir_1C_K & \cdots\cdots \\ -ir_2\beta & 1-ir_2\beta & -ir_2D_3 & \cdots\cdots & -ir_2D_K & \cdots\cdots \\ -ir_3C_3 & -ir_3D_3 & 1 & & & \\ \vdots & \vdots & & 1 & & \\ -ir_KC_K & -ir_KD_K & & & & \\ \vdots & \vdots & & & & \end{pmatrix} \tag{3.4}$$

We would expect this approximation to be valid in a situation where the first two channels contain particles of much lighter mass than the others, and when we are at energies which are well above the first two

thresholds, but just barely past the rest of them, if such a system existed.

There are two comments which should be made at this point. First, this approximation is probably not very good for S waves. However, since the S wave will be almost completely subtracted out of the amplitude, we do not expect this to present serious problems.

Second, the matrix given in Eq. (3.4), while it does not correspond exactly to any physical system, nevertheless is of a form which can be inverted explicitly, which is not the case in more realistic models. It will also be seen that because of the relative simplicity of the assumed K matrix, it will be possible to allow the elastic scattering amplitude to have a real part, something which we shall not be able to do easily in more realistic models. Therefore the reader may, if he wishes, regard Eq. (3.4) as an ad hoc form chosen for simplicity, but which allows a more exact fitting of the elastic scattering parameters than other models.

The actual procedure of inverting the matrix is straightforward, though tedious, and is given in detail in Appendix I. We should note that this inversion is the most difficult part of the calculation, and consequently it is here that approximations are usually made. In order to achieve a simple answer, we introduce a "random phase" assumption on the matrix elements D_i and C_i. The general logic of the procedure is to solve for A_{11}, A_{22}, and A_{12} in terms of a small number of "statistical" parameters which represent the effect of channels other than initial and final ones. We then introduce the values of A_{11} and A_{22} from experiment, and obtain an expression for A_{12} which involves only B, the Born amplitude, and experimentally determined parameters.

If we assume that A_{11} and A_{22} are pure imaginary, then the expression for A_{12} becomes

$$A_{12} = B\left\{\frac{2(1-r\eta)^2}{1+\sqrt{1-4B^2\, r^2(1-r\eta)^2}}\right\} \tag{3.5}$$

On the other hand, if we allow the elastic amplitudes to have small real parts, then the expression becomes

$$A_{12} = A_{12}^{D}\left\{1 + i\epsilon\,\frac{r+\sqrt{1-4B^2\, r^2(1-r\eta)^2}}{\sqrt{1-4B^2\, r^2(1-r\eta)^2}\,(r\eta-1)}\right\}^{-1} \tag{3.6}$$

where A_{12}^{D} is the result of the diffraction picture used to obtain Eq. (3.5).

We see, then, that the net effect of the real part of the elastic amplitude is an increase in absorption.

We next ask whether Eq. (3.5) reduces to any familiar form in the weak absorption limit. The answer to this question requires the evaluation of r and η, which will be done next.

From the optical theorem, we can write

$$\sqrt{\frac{d\sigma}{d\Omega}(0)} = \left(\frac{1}{16\pi^2 S}\;\eta(0)^2\right)^{1/2} = \frac{\sigma_T\, P}{4\pi}$$

but the differential cross section at high energy is given by

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_o e^{-Bt}$$

where t is the momentum transfer. We see, then, that

$$\eta_\ell = 4\pi \sqrt{S} \sqrt{\frac{d\sigma}{d\Omega}(0)} \qquad \frac{1}{2} \int_{-1}^{1} P_\ell(x)\, e^{-B/2\, t}\, dx$$

Carrying out the integral we find that

$$\eta_\ell = \frac{\sqrt{S}\ \sigma_T}{2K\ B}\, e^{-\sigma \ell^2}$$

while from Eq. (2.3), we get that

$$r\eta_\ell = \frac{\sigma_T}{8\pi B}\, e^{-\sigma \ell^2}$$

so that the absorption factor in the numerator of Eq. (3.5) is just

$$\left(1 - \frac{\sigma_T}{8\pi B}\ e^{-\sigma \ell^2}\right)^2$$

which, for high ℓ, reduces to

$$\left(1 - \frac{\sigma_T}{4\pi B}\, e^{-\sigma \ell^2}\right)$$

Now at high ℓ, we expect B_ℓ to be small, also. Consequently, Eq. (3.5) reduces, in the weak absorption (or high partial wave) limit to

$$A_{12} \simeq B \left(1 - \frac{\sigma_T}{4\pi B}\, e^{-\sigma \ell^2}\right) \tag{3.7}$$

which is the DWBA result.

In what follows, I shall refer to the results in Eqs. (3.5) and (3.6) as the "Winged K Matrix" model (WKM).

Section IV

In this section we present an absorption model which is expected to be more realistic at high energies. At high energies, a large number of channels are open for a given reaction, and unless we can calculate in detail the effect of each of these channels on the reaction which we are investigating, it becomes necessary to make some sort of statistical assumption about the unknown channels. The first work of this sort was done by Squires,[12] who considered a multichannel N/D equation where the elements of $\underset{\sim}{N}$ and $\underset{\sim}{D}$ were all equal in magnitude and random in sign. He got

$$A_{12} = N_{12}\left(1 - \frac{A}{2}\, e^{-\sigma l^2}\right)^2 \tag{4.1}$$

which, for weak absorption, reduces to the DWBA form. However, Eq. (4.1) suffers from a serious defect. If we let the Born amplitude increase without bounds, the scattering amplitude, which is supposed to be unitary, also increases without bounds.

This is a difficult problem, since the whole point of going to multichannel N/D or K matrix formalisms is to insure unitarity no matter what we put in for the initial amplitude. A resolution of this difficulty, similar to the one presented here, has recently been proposed by Squires.[12]

It was suggested by Ross[13] that a better approximation might be to put in the explicit values of the K matrix in the channels in which we are interested, and then impose the random phase condition on the rest of the matrix elements.

If we write the amplitude as

$$\underset{\sim}{A}' = \underset{\sim}{K}'(1 - ir\underset{\sim}{K}')^{-1} \tag{4.2}$$

and if we assume that we are at sufficiently high energies that

$$\sqrt{\frac{P}{S}} \sim \frac{1}{2}$$

For all channels, the matrix r can be taken to be

$$\underset{\sim}{r} = r\underset{\sim}{I}$$

If, in addition, we define

$$irK' = K \tag{4.3}$$

then the amplitude takes the form

$$r\underset{\sim}{A}' = \underset{\sim}{A} = \underset{\sim}{K}(1 - \underset{\sim}{K})^{-1} \tag{4.4}$$

The redefinition of K and A is simply a device to help the bookkeeping, and our final answers will be in terms of the original definition in Eq. (4.2).

Let us now define our statistical assumption in the following way: consider an enxemble of matrices which have the following properties:

1. The element $K_{12} = K_{21} = B$ in all matrices

2. The equivalent diagonal elements are the same in all matrices.

3. All other elements are distributed according to some distribution law, which we could take to be a normal distribution. That is, the ensemble average of K_{ℓ_m} ($\equiv \langle K_{\ell_m} \rangle$) is zero.

We could think of this as a system in which each matrix represents the amplitudes for a given infinitesimal energy slice, and in which the ensemble average represents the average amplitude over a small energy range. Such models are well known in nuclear physics.[14] Alternatively, we could simply think of this as a restatement of the random phase model in which certain matrix elements are specifically given, and are not included in the random phase hypothesis.

The question naturally arises as to what the diagonal elements of K ought to be. We can take a hint from the WKM and suppose that in the diffraction model $K_{ii} = 0$. It has been shown[13] that this is indeed the case, and we shall reproduce this previous work later.

Even with these assumptions, it is difficult to invert the matrix. The inversion procedure is carried out in Appendix II, together with a discussion of the random phase hypothesis. The net results is the expression

$$A_{12} = \frac{K_{12}(1 - r\eta)\left(1 + \frac{1 - \sqrt{1 + 4r\eta}}{2}\right)}{1 + r^2 K_{12}^2 \left(1 + \frac{1 - \sqrt{1 + 4r\eta}}{2}\right)^2} \tag{4.5}$$

Now for higher partial waves, $r\eta$ becomes small, and this reduces to

$$A_{12} = K_{12}(1 - r\eta)^2 \tag{4.6}$$

which is the same limiting form as the WKM model. Consequently we see that the Gottfried-Jackson result is the limiting case of both of our models, and this fact may be taken as another "proof" of Eq. (1.1).

The problem of allowing the elastic amplitude to develop a real part is much more difficult here. From Eq. (B.23) we see that Re $A_{ii} = 0$ means that $K_{ii} = 0$.

We note that the derivation of Eq. (4.6) depends very heavily on this assumption. The basic point is that if we consider a term like

$$\sum K_{2\ell}(1-\delta_{\ell 2})\, K_{\ell m}(1-\delta_{m2})\, K_{m\ell}\, K_{\ell 2} = \sum_{\ell m} (1-\delta_{\ell 2})\, K_{2\ell}^{\,2}(1-\delta_{m2})(1-\delta_{\ell m})\, K_{m\ell}^{\,2} +$$

$$+ \sum_{\ell} (1-\delta_{\ell 2})\, K_{\ell 2}^{\,2}\, k_{\ell\ell}^{\,2}$$

then for Eq. (4.5) to be true, we must be able to drop the second term in the above. The only way to do this is to say that the energy is high enough to make the diffraction model reasonable for all channels.

The problem of putting in a small real part to the amplitude is thus seen to be quite complicated, and was not attempted. However, the results of the WKM seem to indicate that its net effect is to increase the amount of absorption.

Another point to note about the random phase approximation (RPA) is that the approximation in which we set

$$\underset{\sim}{r} = r\underset{\sim}{I}$$

is essential to the derivation, since otherwise the matrix

$$\underset{\sim}{r}\cdot\underset{\sim}{K}$$

is not symmetric.

Section V

Before giving the detailed results of numerical calculations, it will be useful to discuss the parameterization of the absorption equation. The general type of term we want to look at are of the form[1]

$$\frac{\sigma_T}{4\pi B} e^{-\frac{1}{2Bp^2}\ell^2} = A e^{-\sigma\ell^2}$$

Now the total cross section is given approximately by[16]

$$\sigma_T = 22.6 + \frac{25.8}{p_L}$$

while the half width of the peak at 4.0 Bev/c is just[17]

$$B = 8.33 \text{ Bev}^{-2}$$

Consequently, we can calculate the following table

P_L	A	σ
4.0	.765	-0.038
5.9	.70	-0.025
8.0	.671	-0.018
10.0	.66	-0.014

The value of B given above is not the value given by some experimentors,[18] but the final results are not very sensitive to variations in B. The value of the ratio of the real to imaginary part of the forward amplitude is taken to be[19]

$$\alpha_{\pi^+P \to \pi^+P} = 0.33$$

$$\alpha_{\pi^-P \to \pi^-P} = 0.23$$

The main points of interest $\frac{d\sigma}{d\Omega}$ for backward π^+P scattering are[20]

1. A sharp exponential fall-off with scattering angle.
2. A drop in $\frac{d\sigma}{d\Omega}$ of a factor of 3 between $P_L = 4$ and $P_L = 8$.
3. The value of $\frac{d\sigma}{d\Omega}$ at 180° is smaller than the simple Born term by a factor of at least 10^{-3}.

It might be objected that the reactions

$$\pi^\pm p \to \pi^\pm p$$

represent elastic scattering, and therefore ought not to be treated in an absorption model. However, in terms of helicity amplitudes, we have

$$A_{+-} = 0$$

at zero angle scattering, and

$$A_{++} = 0$$

at back scattering angles. Consequently, if we label the channels of our K matrix by helicities, it is possible to make an approximate separation between the forward and back scattering channels on the bases of helicity flip arguments.

The calculation of this peak by the DWBA has been reported previously[21] and has been done elsewhere in a somewhat different form.[22] The DWBA with A = 1 can bring the cross section down to $\sim$ 1-3 mb, but no further. To match the data, it is necessary to include the virtual effects. In Fig. 1, we show the results of the DWBA for N exchange alone. We see that while it is possible to get a reasonable fit at P_L = 4.0 Bev/c, there are two major disparities between the DWBA and the experimental points.

1. The DWBA predicts a rise of cross section with energy, instead of the observed drop.
2. The DWBA does not reproduce the sharpening of the slope as energy is increased.

In Fig. 2, we present the results for the RPA model. While it is possible, by increasing A, to correct point 1 above, the second objection still holds. In Fig. 3, we present the same results for the WKM, and the comments made about the RPA model hold here as well.

We see, then, that the simple nucleon exchange term, even if we include virtual effects, fails to reproduce the sharpening of the peak. We might ask, then, about the effect of including an N^* exchange in the unsubtracted amplitude. This, in N/D language, would correspond to starting the calculation with two poles instead of one. For the reaction

which we are considering, this means we will sum the three diagrams in Fig. 15.

Since the diagrams add constructively, it is clear that the DWBA, which was pushed to it's limit to get the results of Fig. 1, it simply will not be able to provide enough absorption when the extra terms are added. Although it is not so obvious, the same results are found to hold true for the RPA model. The WKM, however, can be made to match the data quite well, as is shown in Fig. 4.

Now it must be realized that when we talk of "fitting" the data, we are, in fact, choosing A on the basis of best fit, and not according to any theoretical criterion. We will discuss later whether the increase of A with energy can be justified. It should also be noted that the calculation of the virtual effects in Ref. 5 is highly approximate, and consequently, the absolute magnitudes of the calculated cross sections is brought into question. It is conceivable that if the effective scaling factor due to virtual effects were changed, that a situation could arise in which none of the models could be made to fit the data, or in which all of them could.

The relevant facts about the reaction

$$\pi^- P \rightarrow \pi^- P$$

are that[20]

1. The experimental curves are almost flat in the region of interest.
2. The cross section drops slightly as the energy is increased.

This reaction goes predominantly by N^* exchange, and is dominated by the diagrams in Fig. 16.

In Fig. 5 we present the DWBA and RPA results for this amplitude. We see that while both predict the relatively flat curve, both also predict a rise in the cross section with energy, which is not observed. In Fig. 6, the results of the WKM are presented, and we see that with the additional absorption due to deviations from the diffraction picture, it is again possible to fit the data reasonable well.

The data for the forward charge exchange scattering[19] reveals that the cross section differs from elastic scattering in that instead of peaking in the forward direction, it begins to level off as $\theta \to 0$.

In Fig. 7 and 8, we present the best fits for lab momenta of 5.9 and 10 Bev/c for the different models. Once more we see that the DWBA and the RPA models fail to follow the slight energy drop, but that the WKM can be made to fit the data near zero scattering angle quite well. On the other hand, for larger angles, it is impossible to match the slope. In an attempt to improve this situation, we tried to vary K_v. The results of these calculations for the WKM model are shown in Fig. 9. The general result seems to be that K_v governs both the turnover at $\theta = 0$ and the magnitude of the slope, and that if we match one of these, we must lose

the other. Since the peripheral model is expected to be best for small momentum transfers, it would seem reasonable to choose our original value of K_v, and thereby match the turnover.

Section VI

The following points emerge from an analysis of the numerical calculations:

1. The DWBA fails to reproduce the drop in cross section with energy for any of the processes considered.
2. The absorption constant in the RPA can be adjusted to match the energy drop for N exchange, in which case it fails to match the shrinking of the peak. In the other cases, the energy drop cannot be matched.
3. If we include the effect of the real part of the elastic amplitude, it is possible to choose A such that the WKM gives a reasonable fit to the data in all three cases.
4. In considering the reaction

$$\pi^+ P \rightarrow \pi^+ P$$

in the backward direction, it was necessary to take a more complicated expression than the simple Born amplitude for the undistorted inelastic amplitude.

This last point leads to the conjecture that it might be possible to do a better job of matching the slope of the forward charge exchange cross section if we included more than the simple ρ exchange diagram. It also points out the fact that as the energy of the interaction is increased, the replacing of the inelastic amplitude with the Born term becomes less and less justified. In fact, the identity

$$K_{12} = B$$

is the weakest point in the entire absorption procedure.

It must be noted, however, that the main usefulness of the absorption models is that it allows a simple calculation of high energy cross sections which takes into account the many channel aspects of the problem. Clearly, a point of diminishing returns will be reached when the calculation of the inelastic terms will become a forbidding problem in itself, and at this point other methods of attack will have to be found. Our calculations seem to indicate that this point will not lie very far above

$$P_L \sim 10 \text{ Bev/c}$$

Another point which requires some attention is the question of whether or not the increase of A with energy which was required to fit the data can be justified in any way. The elastic unitarity condition in our notation is

$$\text{Im } A^j_{\lambda\mu} = \frac{P}{16\pi^2\sqrt{S}} \sum_m A^\lambda_{\lambda m} A^j_{m\mu} \tag{6.1}$$

so that if we write

$$A^j_{++} = K e^{i\delta} \sin\delta$$

$$A^j_{+-} = N e^{i\eta} \sin\eta$$

Eq. (6.1) becomes

$$K \sin^2\delta = \frac{P}{16\pi^2\sqrt{S}} \left\{ N^2 \sin^2\eta + K^2 \sin^2\delta \right\} \tag{6.2}$$

from which it follows that

$$\frac{16\pi^2\sqrt{S}}{P} \geq K \geq A_{++}$$

On the other hand, we know that for π-N back scattering we can write

$$A^j_{++} = \pi g^2_{\pi N} (E + M) I_j T_I$$

where

$$I_j = \int d^j_{\lambda\mu} \left\{ F_v \bar{X} \not{q} X + F_s \bar{X} X \right\} d(\cos\theta)$$

and where we've defined

$$u(p,s) = \sqrt{\frac{E + M}{2M}} X(p,s)$$

so that the unitarity condition reads

$$I_j \leq \frac{2\sqrt{s}}{P\left(\frac{g^2_{\pi N}}{4\pi}\right)} (E + M) \tag{6.3}$$

A similar calculation for the forward charge exchange scattering yields

$$I_j \leq \frac{\sqrt{s}}{E + M} \frac{2\sqrt{2}}{P\left(\frac{g^2_{\rho\pi\pi}}{4\pi}\right)} \tag{6.4}$$

Let us consider just the S wave for a moment. The S wave in the DWBA can be written

$$(1 - A) I_o = A_o$$

where a typical value of I_o is

$$I_o \simeq .5$$

so that at $P_L = 8$, Eq. (6.4) says that

$$A \geq 0.86$$

which is somewhat higher than the value of A which was calculated in Section V.

In the RPA and WKM models, we write for the S wave

$$(1 - A)^2 I_o = A_o$$

which yields

$$A \gtrsim 0.65$$

While this is smaller than the parameters actually used, it is considerable higher than the theoretical absorption parameter, which is half of the Gottfried-Jackson result.

We might say that although the K matrix approach guarantees unitarity if carried out exactly, the approximations which we have made seem to leave the amplitude nonunitarity for some choices of A. Consequently, we can think of the use of a higher absorption parameter as a partial correction for the approximations which were made to derive simple formulae.

As for the increase of A with energy, the charge exchange scattering unitarity condition leads to the expression

$$\frac{(1 - A_1) I_1}{(1 - A_2) I_2} = \frac{S_1}{S_2} \frac{P_2}{P_1} \frac{E_2 + M}{E_1 + M} \tag{6.5}$$

where the subscript 1 refers to $P_L = 5.9$ and 2 to $P_L = 10$. Clearly, since in this case $I_2 > I_1$, the above equation requires

$$A_2 > A_1$$

In fact, if we let

$$A_1 = 0.9$$

then we find

$$A_2 = 0.95$$

While this increase is not large enough to justify the parameters which we picked to match the data, at least it shows that increasing A as energy is increased is not totally unreasonable.

Finally, we must ask why the WKM seems to give results which are so much better than the RPA, since one would suppose that the latter is the more reasonable model.

One reason might lie in the fact that although we are at energies where many channels are open, it is not clear that we can ignore threshold effects, as we did in the RPA. For example, in our energy range the threshold factor for the πN channel is virtually at its asymptotic value, while the same factor for the ρN channel is only within 60% of this value. Consequently, it may be that the random phase method does not become valid until much higher energies are reached, at which point the use of the simple Born term in the K matrix is certainly not justified.

Another difficulty with the RPA is that we know that $\alpha \neq 0$ in our energy range, although we assumed that it was when we derived the RPA. If the WKM is any guide, the inclusion of this effect would lead to an increase in absorption, which might bring the RPA results more into line with the experimental data.

As was mentioned above, one way to improve the results of our calculations would be to include more poles, or perhaps some cuts, in Eq. (3.3). However, a more useful approach has been suggested by Arnold,[23] who pointed out that one can use a Regge pole for the undistorted amplitude rather than the dispersion theory pole. It has been pointed out[24] that the use of Regge poles might overcome the problem of the energy dependence of the backscattering cross section, and this problem, as we have seen, is the main weakness of the absorption model as developed this far. The authors hope to present calculations of the Reggeization of the absorption model in the near future.

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APPENDIX I

In this Appendix the detailed derivation of the Winged K matrix (WKM) formulae is given. The first step is the inversion of (1 - irK). We will require only the first two columns of the inverse, as will be seen below. The first column is

$$\begin{matrix} \beta + r_2 \sum_k r_k D_k^2 \\ ir_2 B - r_2 \sum_k r_k D_k C_k \\ \vdots \\ - \left\{ -ir_k C_k(\beta + r_2 \sum_{i\neq k} r_i D_i^2) + ir_k D_k (- ir_2 B + r_2 \sum_{i\neq k} r_i C_i D_i) \right\} \\ \vdots \end{matrix} \tag{A1}$$

and the second column is

$$\begin{matrix} ir_1 B + r_1 \sum_k r_k C_k D_k \\ \alpha + r_1 \sum_k r_k C_k \\ \vdots \\ \left\{ - ir_k C_k(- ir_1 B + r_1 \sum_{i\neq k} r_k C_k D_k) + ir_k D_k (1 - ir_1 \alpha + r_1 \sum_{i\neq k} r_i C_i^2) \right\} \\ \vdots \end{matrix} \tag{A2}$$

Where in all of the above I have left out the factor $\frac{1}{\Delta}$, where Δ is the determinant of the matrix (1 - irK).

Up to now, except for our initial assumption, everything has been perfectly general. We now make a "random phase" approximation on D and C.

If there are a large number of channels open, and if the individual C_i and D_i are small, and if it is reasonable to assume that they have random signs, we can drop terms like

$$\sum_i C_i D_i, \quad \sum C_i, \quad \text{and} \quad \sum D_i$$

with respect to terms like

$$\sum D_i^2 = D \qquad \sum C_i^2 = C \tag{A3}$$

since the error involved is of order $1/n$, where n is the number of open channels.

Now we started with a K matrix of the form

$$K = \begin{pmatrix} \alpha & B & C_3 \cdots C_k \cdots \\ B & \beta & D_3 \cdots D_k \cdots \\ C_3 & D_3 & \\ \vdots & \vdots & \\ C_k & D_k & \\ \vdots & \vdots & \end{pmatrix} \tag{A4}$$

where the submatrix ϵ is presumed to have small entries. To calculate A_{11}, A_{22}, and A_{12}, however, we don't need to know ϵ at all.

If we calculate the elements of A which are of interest, we find that

$$A_{12} = A_{21} = \frac{B}{\Delta} \tag{A5}$$

and

$$A_{11} = \frac{1}{\Delta}\left\{\alpha(1 - ir_2\beta + r_2D) + ir_2B^2 + iC + r_2\beta C + ir_2 CD\right\} \tag{A6}$$

and

$$A_{22} = \frac{1}{\Delta}\left\{ir_1B^2 + \beta(1 - ir_1\alpha + r_1C) + iD + r_1\alpha D + ir_1 CD\right\} \tag{A7}$$

The expression for Δ turns out to be

$$\Delta = r_1r_2B^2 + (1 - ir_1\alpha + r_1C)(1 - ir_2\beta) + r_2D(1 - ir_1\alpha + r_1C) \tag{A8}$$

Let us now see what the requirements of the diffraction model are for this case. Set

$$\begin{aligned} A_{11} &= iX \\ A_{22} &= i\xi \end{aligned} \tag{A9}$$

Then combining Eqs. (A6), (A7), and (A8), we get the following two equations:

$$\begin{aligned} &\alpha\beta(-ir_1r_2X + ir_2) + \alpha(-1 + r_1X + r_1r_2XD - r_2D) \\ &\quad + \beta(r_2X + Xr_1r_2C - r_2C) + iXr_2D + C(iXr_2 - i) \\ &\quad + B^2(iXr_1r_2 - ir_2) + iX + DC(iXr_1r_2 - ir_2) = 0 \end{aligned} \tag{A10}$$

and

$$\alpha\beta(-ir_1r_2\,\xi + ir_1) + \alpha(r_2\,\xi + r_1r_2\,D\,\xi - r_1\,D)$$

$$+ \beta(r_2\,\xi + r_1r_2\,C\text{-}1\text{-}r_1\,C) + D(i\xi\,r_2 - i) + C(ir_1\,\xi) \tag{A11}$$

$$+ B^2(ir_1r_2\,\xi - ir_1) + i\xi + iD(i\xi\,r_1r_2 - ir_1) = 0$$

Now unitarity requires that α and β be real, and the diffraction assumption imposes two conditions--namely that

$$\text{Re } A_{11} = \text{Re } A_{22} = 0 \tag{A12}$$

Therefore, if we can find an α and β which satisfy Eq. (3.16), we can be sure that they are unique.

Suppose we try

$$\alpha = \beta = 0 \tag{A13}$$

we find, by inspecting Eqs. (A6) and (A7), that they do indeed produce imaginary diagonal amplitudes.

For the sake of simplicity, let

$$C = D$$

and write

$$\eta = \frac{X + \xi}{2}$$

Then adding the imaginary parts of Eqs. (A10) and (A11) gives

$$D = \frac{1 - 2r\eta \pm \sqrt{1 - 4B^2\,r^2(1 - r\eta)^2}}{2r(r\eta - 1)} \tag{A14}$$

We note that the requirement of unitarity on the entries in the K matrix constrains D to be real. Therefore the unitarity condition can be cast in the form

$$4B^2 r^2(1 - r\eta)^2 < 1 \tag{A15}$$

which means that the Born term cannot grow without limits and still satisfy unitarity (this should not be surprising).

If we insert Eq. (A14) into Eq. (A8) we find

$$\Delta = \frac{2 \mp 2\sqrt{1 - 4B^2 r^2(1 - r\eta)^2}}{4(1 - r\eta)^2} \tag{A16}$$

so that

$$A_{12} = B \left\{ \frac{2(1 - r\eta)^2}{1 \mp \sqrt{1 - 4B^2 r^2(1 - r\eta)^2}} \right\} \tag{A17}$$

The problem of the choice of the sign in Eq. (A17) can be solved by requiring that as r and B approach zero (i.e. as we approach threshold) $A_{12} \to B$. This requires the choice of the positive sign, so that, finally, we have

$$A_{12} = \frac{2(1 - r\eta)^2 B}{1 + \sqrt{1 - 4B^2 r^2(1 - r\eta)^2}} \tag{A18}$$

The same sign choice could have been made by noting that for high partial waves, where B is small, the denominator of Eq. (A17) would approach zero for the other sign choice, thus giving an infinite amplitude.

Suppose we now ask what happens if we allow A_{11} to develop a small real part -- i.e. let

$$A_{11} = \epsilon + iX \qquad \alpha = \frac{\epsilon}{X} << 1$$

If for the sake of simplicity we allow $\xi = X$, $C = D$, and $r_1 = r_2$ then the analogue of Eqs. (A10) and (A11) is just

$$\begin{aligned}(ir - iX\,r^2 - \epsilon r^2)\,\alpha^2 + \alpha\left[2X\,r(1 + rD) - 2rD - 1 - 2i\epsilon r(1 + rD)\right] \\ + D(2iX\,r - i + 2r\epsilon) + B^2(iX\,r^2 - ir + \epsilon r^2) \\ + D^2(iX\,r^2 - ir + \epsilon r^2) + \epsilon + iX = 0\end{aligned} \tag{A19}$$

In the spirit of treating Re A_{11} as a pertubation, we can also let

$$\begin{aligned}\alpha &= \alpha_o + \delta \\ D &= D_o + \omega\end{aligned} \tag{A20}$$

and drop terms which are second order in the pertubation parameters. If we put these approximations into Eq. (A19), and use Eqs. (A13) and (A14), we can solve for δ and ω. After the above substitutions, the imaginary part of Eq. (A19) is just

$$\omega = 0 \tag{A21}$$

In a similar way, the real part of Eq. (A19) yields

$$\delta = -\epsilon \left\{ \frac{\Delta_o}{2X\,r(1 + r\,D_o) - 1 - 2r\,D_o} \right\} \tag{A22}$$

where Δ_o is defined from Eq. (A8).

We can now proceed to calculate the new absorption factor. Plugging Eq. (A21) and (A22) into the definition of Δ, we get the final expression for A_{12} to be

$$A_{12} = \frac{2B(1 - r\eta)^2}{1 + \sqrt{1 - 4B^2 r^2(1\ r\eta)^2}} \quad \frac{1}{1 + i\epsilon \dfrac{r\left(1 + \sqrt{1 - 4B^2 r^2(1 - r\eta)^2}\right)}{\sqrt{1 - 4B^2 r^2(1 - r\eta)^2}\ (r\eta - 1)}} \tag{A23}$$

and the net effect of allowing small deviations from the diffraction model is seen to be an increase in absorption.

APPENDIX II

In this Appendix, the inversion of the Random Phase Approximation (RPA) matrix is carried out by means of writing $(1 - irK)^{-1}$ as a power series, then evaluating the general term by the random phase hypothesis, and then summing the resulting series. From Eq. (4.4) we can see that

$$\underset{\sim}{A} = \underset{\sim}{K} + \underset{\sim}{K}^2 + \underset{\sim}{K}^3 + \ldots\ldots + \underset{\sim}{K}^n + \ldots\ldots \tag{B1}$$

The problem, then, is to evaluate $< K^n >$ under assumptions 1 through 3. Now

$$(K^n)_{12} = \sum_{\ell,m,r} K_{1\ell} K_{\ell m} K_{mn} \cdots K_{r2}$$

$$= \sum_{\ell,m,\ldots r} (1 - \delta_{\ell 2} + \delta_{\ell 2}) K_{1\ell} K_{\ell m} \ldots K_{r2}$$

$$= K_{12} \sum_{m,n,\ldots r} K_{2m} K_{mn} \ldots K_{r2} + \sum_{\ell,m,\ldots r} (1 - \delta_{\ell 2}) K_{1\ell} K_{\ell m} \ldots K_{r2}$$

$$= K_{12} (K^{n-1})_{22} + \sum_{\ell,m,\ldots r} (1 - \delta_{\ell 2})(1 - \delta_{m2} + \delta_{m2}) K_{1\ell} K_{\ell m} \ldots K_{r2}$$

$$= K_{12} (K^{n-1})_{22} + \left(\sum_{\ell} (1 - \delta_{\ell 2}) K_{1\ell} K_{\ell 2} \right) \left(\sum_{n,r,\ldots} K_{2n} \ldots K_{r2} \right)$$

$$+ \sum_{\ell \ldots r} (1 - \delta_{\ell 2})(1 - \delta_{m2}) K_{1\ell} \ldots K_{r2}$$

Clearly, by carrying this process out further, we eventually arrive at the expression

$$(K^n)_{12} = \sum_{m=1}^{n} (K^m)'_{12} (K^{n-m})_{22} \tag{B2}$$

where

$$(K^m)'_{12} = \sum_{\ell,m,\ldots r} K_{1\ell}\,(1 - \delta_{\ell 2})\,K_{\ell m}\,(1 - \delta_{m2})\ldots\ldots(1 - \delta_{r2})\,K_{r2}$$

The advantage of this procedure is that it extracts the K_{12} dependence from the power series in K. It is obvious that the term $(K^m)'_{12}$ contains no dependence on K_{12} for $m > 1$ because for each index which is summed over, there is a term like $(1 - \delta_{\ell,2})$ which cancels the single term in the sum which could contain a K_{12}.

In an exactly analagous manner to the above derivations, we can show that

$$(K^n)_{22} = \sum_m (K^m)'_{21}\,(K^{n-m})_{12} \qquad (B3)$$

Equations (B2) and (B3) together will give us a way of evaluating A_{12} and A_{22}, and hence will lead to an absorption model. Before we can do this, however, we must apply our random phase approximation to these equations which, as they stand, are perfectly general. I shall use the language of ensemble averages in proceeding, but everything I shall say is also true in the random phase picture.

In the statistical picture, it is reasonable to assume that the distribution of one element of the K matrix is independent of the distribution of other elements, so that if $i \neq m$

$$\langle K_{1\ell}\,K_{\ell m}\rangle = \langle K_{1\ell}\rangle \langle K_{\ell m}\rangle = 0$$

but if $i = m$, then

$$\langle K_{i\ell}\,K_{\ell i}\rangle = \langle K^2_{1\ell}\rangle \neq 0$$

In both of the above, of course, we are assuming that none of the K matrix elements involved is K_{12}. We can summarize the above by saying

$$< K_{1\ell}\, K_{\ell m} > = \delta_{mi}\, \alpha_{\ell i} \tag{B4}$$

from which it follows that

$$< \sum_{\ell \neq m} K_{i\ell}\, K_{\ell m} > = \sum_{\ell \neq m} \alpha_{\ell i}\, \delta_{mi} \equiv \alpha_i \tag{B5}$$

Now as far as single matrix elements go, the only one which has a non-zero ensemble average is K_{12}, from which it follows that

$$< \sum_{\ell,m} (1 - \delta_{\ell 2})\, K_{1\ell}\, (1 - \delta_{m2})\, K_{\ell m}\, K_{m2} >$$

$$= < \sum_{\ell} (1 - \delta_{\ell 2})\, K_{1\ell}\, K_{\ell 1} > < K_{12} > = \alpha_i\, B$$

From this, we can formulate the following rules concerning ensemble averages over $(K^m)'_{22}$ and $(K^m)'_{12}$. If m is even, then $(K^m)'_{12}$ can be non-zero only if the individual terms in it can be paired--for example, a possible 4^{th} order term would be

$$\sum_{\ell,m} K_{2\ell}\, K_{\ell m}\, K_{m\ell}\, K_{\ell 2}\, (1 - \delta_{\ell 2})(1 - \delta_{m2})$$

If m is odd, however, then it is impossible to pair all of the terms, and the answer will always be proportional to

$$< K_{\ell m} > = 0 \qquad\qquad m \text{ odd}$$

since none of the interior indeces in the sum can be = 2.

For $(K^m)'_{12}$, on the other hand, exactly the opposite is true. For odd m, it is possible to let the last term in the sum be K_{12}, and then pair all of the rest. A fifth order term, for example, would be

$$\sum_{\ell,m} K_{1\ell} K_{\ell m} K_{m\ell} K_{\ell 1} (1 - \delta_{\ell 2})(1 - \delta_{m2}) K_{12}$$

For even m, however, this cannot be done, and

$$(\Sigma K^m)'_{12} = 0 \qquad \text{m even}$$

We can see more clearly now the nature of the difficulty found in previous models.[12] Consider a third order term.

$$\sum_{\substack{\ell\neq 2\\ m\neq 2}} K_{1\ell} K_{\ell m} K_{m2} = \sum_{\ell\neq 2} (K_{1\ell})^2 K_{12} + \sum_{\substack{m\neq 1,2\\ \ell\neq 1}} K_{1\ell} K_{\ell m} K_{m2} \tag{B6}$$

and the second term can be further expanded into

$$K_{11} \Sigma K_{1m} K_{m2} + \sum_{\substack{\ell\neq 1,2\\ m\neq 1,2}} K_{1\ell} K_{\ell m} K_{m2} \tag{B7}$$

the diffraction hypothesis allows us to drop the first of these terms, but then we are in trouble. While each term in the sum over m in the second term in Eq. (4.11) is of order 1/n, the fact that m runs from 3 to n means that there are about n such terms, and dropping these terms is certainly not justified.

The ensemble formulation sidesteps this problem by defining the troublesome term to be zero, but the difficulty can be overcome in the random phase picture also by noting that the spirit of the statistical approach demands that the elements to be treated statistically be (1) small in magnitude, and (2) large in number. A simple comparison

shows that the error made in dropping the terms in Eq. (B7) is of order

$$\frac{K_{\ell m}}{K_{12}}$$

which is small by assumption. This also points out a difficulty in the original Squires approach to the absorption problem. By failing to distinguish the entrance and exit channels from other channels, he essentially let

$$\frac{K_{12}}{K_{\ell m}} \sim 1$$

which, as he has pointed out,[25] leads at once into difficulty in calculating A_{12}.

In view of the above, we see that

$$(K^n)'_{12} = \begin{cases} K_{12}\,(K^{n-1})'_{22} & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \tag{B8}$$

We are left, then, with the problem of evaluating

$$(K^{n-1})'_{22} = \Sigma\, K_{2\ell}\,(1 - \delta_{\ell 2})\, K_{\ell m}\,(1 - \delta_{m2}).....(1 - \delta_{r2})\, K_{r2}$$

This sum does not include terms where the interior indeces are = 2. But since all terms must pair, and since only the first and last terms have an index 2, it is clear that

$$(K^{n-1})'_{22} = K_{12} \sum_{\ell} (1 - \delta_{2\ell})\, K_{1\ell}^2\, (K^{n-3})'_{\ell\ell}$$

If, in addition we make the simplifying assumption that

$$\sum_{\ell} K_{i\ell}^2 (1 - \delta_{\ell 2}) = R \qquad \text{for all } i$$

then

$$(K^{n-1})'_{22} = K_{12} \cdot R \cdot R^{\left\lfloor \frac{n-3}{2} \right\rfloor} B(n-3) \tag{B9}$$

where $B(n)$ is the number of surviving terms in the n^{th} order sum.

It is important to note that in evaluating $(K^{n-3})'_{\ell\ell}$ no interior index is prevented from assuming the value of the first or last index, and we are therefore dealing with a well studied problem.[12,15,26]

With this introduction, we can now turn to Eqs. (B2) and (B3) and write

$$(K^n)_{12} = K_{12} \left\{ (K^{n-1})_{22} + R \sum_{m=3,5\ldots}^{n \text{ or } n-1} R^{\left\lfloor \frac{m-3}{2} \right\rfloor} B(m-3)(K^{n-m})_{22} \right\} \tag{B10}$$

and

$$(K^n)_{22} = K_{12} \left\{ (K^{n-1})_{21} + R \sum_{m=3,5\ldots}^{n \text{ or } n-1} R^{\left\lfloor \frac{m-3}{2} \right\rfloor} B(m-3)(K^{n-m})_{21} \right\} \tag{B11}$$

Suppose that now we sum Eq. (B10) over odd n, starting from n = 3.

$$\sum_{n=3,5\ldots} (K^n)_{12} = K_{12} \left\{ \sum_{\ell=2,4\ldots} (K^4)_{22} + R \sum_{n=3,5\ldots}^{\infty} \sum_{m=3\ldots}^{n} R^{\left\lfloor \frac{m-3}{2} \right\rfloor} B(m-3)(K^{n-m})_{22} \right\} \tag{B12}$$

which can be written

$$\sum_{n=3,5\ldots} (K^n)_{12} = K_{12} \left\{ \sum_{\ell=2,4\ldots} (K^\ell)_{22} + R\left(\sum_{\ell} R^{\ell/2} B(\ell) \right) \left(1 + \sum_{r=2,4\ldots} (K^r)_{22} \right) \right\} \tag{B13}$$

Now the expression

$$\beta = \sum_{\ell=0,1\ldots} R^{\ell/2} B(\ell)$$

has been fully investigated elsewhere[12,15,26] and we will just quote the result here.

$$\beta = \frac{1}{2R} \left(1 - \sqrt{1 - 4R} \right) \tag{B14}$$

Going through the same procedure on Eq. (B11) by summing over even n from n = 4, we get

$$\sum_{n=2,4\ldots} (K^n)_{22} - R - K_{12}^2 = K_{12} \left\{ \sum_{\ell=3,5\ldots} (K^\ell)_{21} + R\beta \left(K_{12} + \sum_{\ell=3,5\ldots} (K^\ell)_{21} \right) \right\} \tag{B15}$$

Now suppose we define

$$C = \sum_{n=2,4\ldots} (K^n)_{22} \qquad D = \sum_{n=3,5\ldots} (K^n)_{12}$$

using the symmetry of the K matrix, we see that we have two equations in two unknowns:

$$D = K_{12} \left\{ C + R\,\beta(1 + C) \right\}$$
$$C = R + K_{12}^2 + K_{12} \left\{ D + R\,\beta(K_{12} + D) \right\} \tag{B16}$$

which can be solved to give

$$K_{12} + D = \frac{K_{12}\left[(1 + R\,\beta)(1 + R)\right]}{1 - K_{12}^2\,(1 + R\,\beta)^2} \tag{B17}$$

and

$$C = \frac{R + K_{12}^2\,(1 + R\,\beta)^2}{1 - K_{12}^2\,(1 + R\,\beta)^2} \tag{B18}$$

Recalling Eqs. (4.3 and (4.5) we see that

$$A_{12} = \frac{K_{12}\,(1 - R)\left(1 + \frac{1 - \sqrt{1 + 4R}}{2}\right)}{1 + r^2\,K_{12}^2\left(1 + \frac{1 - \sqrt{1 + 4R}}{2}\right)^2} \tag{B19}$$

and

$$r\,A_{11} = \frac{R + r^2\,K_{12}^2\left(1 + \frac{1 - \sqrt{1 + 4R}}{2}\right)^2}{1 + r^2\,K_{12}^2\left(1 + \frac{1 - \sqrt{1 + 4R}}{2}\right)^2} \tag{B20}$$

It is these expressions, then, which must be used to obtain our absorption equation. As usual, the procedure will be to solve for R in terms of the diagonal matrix element, and then put this into the expression for the off diagonal element.

As it stands, the problem is quite formidable, since Eq. (B20) is a quartic expression in R. We look, therefore, for a reasonable approximate solution. Now at high energies,

$$r^2 \sim \left(\frac{1}{8\pi}\right)^2 = \frac{1}{64\pi^2} \sim 5 \times 10^{-3}$$

In the reactions which we shall consider later, the value of K_{12}^2 is usually $\lesssim 10$. Consequently, a reasonable approximation is to let

$$r A_{11} = r \eta = R \tag{B21}$$

If we do this, we see that for lower partial waves, when K_{12}^2 can be expected to be large, the expression

$$\left(1 + \frac{1 - \sqrt{1 + 4r\eta}}{2}\right)$$

will be at its minimum, while when the above approaches 1 (as it does for high partial waves), K_{12}^2 is approaching zero. We expect, then, that Eq. (B21) will be a good approximation to use, and indeed, numerical studies for the amplitudes we actually calculated bear this out completely.

Our final absorption equation then becomes

$$A_{12} = \frac{K_{12}\left[1 + \frac{1 - \sqrt{1 + 4r\eta}}{2}\right](1 - r\eta)}{1 + r^2 K_{12}^2 \left(1 + \frac{1 - \sqrt{1 + 4r\eta}}{2}\right)^2} \tag{B22}$$

In closing, we would like to outline the procedure developed by Ross[13] which must be followed to justify our earlier assertion that the diffraction model demands that K_{ii} be set equal to zero. Using the procedure outlined above to extract the K_{ii} dependence of $(K^n)_{ii}$, we get

$$(K^n)_{ii} = \sum_{m=1}^{n} (K^m)'_{ii} (K^{n-m})_{ii}$$

where now the ' means that interior indeces cannot be equal to i. Summing this equation first over even n and then over odd n, we get two coupled equations, as before, which are

$$\sum_{n=2,4\ldots} (K^n)_{ii} = K_{ii} \sum_{n=1,3\ldots} (K^n)_{ii} + R\beta\left(1 + \sum_{n=2,4\ldots} (K^n)_{ii}\right)$$

and

$$\sum_{n=1,3\ldots} (K^n)_{ii} = K_{ii}\left(1 + \sum_{n=2,4\ldots} (K^n)_{ii}\right) + R\beta \sum_{n=1,3\ldots} (K^n)_{ii}$$

from which we can get

$$A_{ii} = \sum_{n=1,3\ldots} (K^n)_{ii} = \frac{K_{ii}}{(1 - R\beta)^2 - K_{ii}^2} \tag{B23}$$

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FIGURE CAPTIONS

1. Backward $\pi^+ p$ scattering, DWBA model, N exchange:
   (a) Curve for P_L = 4 BeV/c, A = 1
   (b) Curve for P_L = 8 BeV/c, A = 1
2. Backward $\pi^+ p$ scattering, RPA model. N exchange:
   (a) Curve for P_L = 4 BeV/c, A = 0.88
   (b) Curve for P_L = 8 BeV/c, A = 1.0
3. Backward $\pi^+ p$ scattering, WKM model, N exchange:
   (a) Curve for P_L = 4.0 BeV/c, A = 0.7, α = 0.33
   (b) Curve for P_L = 8 BeV/c, A = 0.9, α = 0.33
   (c) Curve for P_L = 8 BeV/c, A = 0.95, α = 0
4. Backward $\pi^+ p$ scattering, WKM model, N + N^* exchange:
   (a) Curve for P_L = 4 BeV/c, A = 0.7, α = 0.33
   (b) Curve for P_L = 8 BeV/c, A = 0.95, α = 0.33
   (c) Curve for P_L = 8 BeV/c, A = 1.0, α = 0
5. Backward $\pi^- p$ scattering:
   (a) DWBA curve for A = 1, P_L = 4 BeV/c
   (b) DWBA curve for A = 1, P_L = 8 BeV/c
   (c) RPA curve for A = 0.95, P_L = 4 BeV/c
   (d) RPA curve for A = 1, P_L = 8 BeV/c
6. Backward $\pi^- p$ scattering, WKM model:
   (a) Curve for P_L = 4 BeV/c, A = 0.94, α = 0
   (b) Curve for P_L = 8 BeV/c, A = 1, α = 0
   (c) Curve for P_L = 4 BeV/c, A = 0.8, α = 0.23
   (d) Curve for P_L = 8 BeV/c, A = 1, α = 0.23

7. $\pi^- p \rightarrow \pi^\circ n$ at $P_L = 5.9$ BeV/c, $K_V = 3.7$:

   (a) DWBA, A = 0.7

   (b) DWBA, A = 1.0

   (c) RPA, A = 1.0

   (d) WKM, A = 0.9, $\alpha = 0$

   (e) WKM, A = 0.9, $\alpha = 0.23$

8. $\pi^- p \rightarrow \pi^\circ n$ at $P_L = 10$ BeV/c, $K_V = 3.7$:

   (a) DWBA, A = 1

   (b) RPA, A = 1

   (c) WKM, A = 1, $\alpha = 0$

   (d) WKM, A = 1, $\alpha = 0.23$

9. $\pi^- p \rightarrow \pi^\circ n$ at $P_L = 5.9$ BeV/c:

   (a) DWBA, A = 0.7, $K_V = 0$

   (b) DWBA, A = 1, $K_V = 1.85$

   (c) WKM, A = 0.9, $\alpha = 0$, $K_V = 1.85$

   (d) DWBA, A = 1, $K_V = 0$

   (e) WKM, A = 0.95, $\alpha = 0$, $K_V = 0$

10. Schematic interpretation of the Gottfried-Jackson model.

11. Kinematics of π - N scattering.

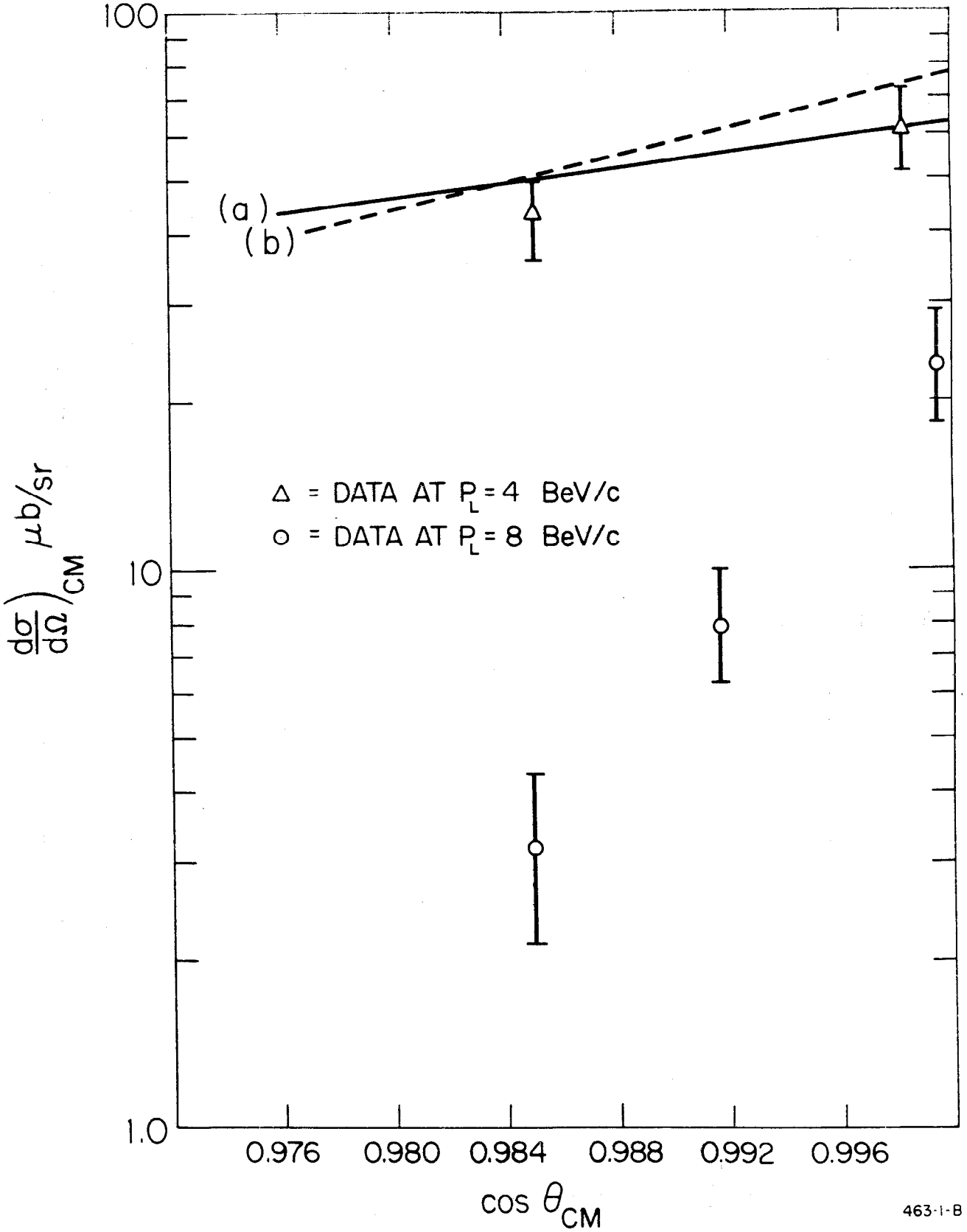
12. Single nucleon exchange diagram.

13. N^*-exchange diagrams.

14. ρ-exchange diagram.

15. $\pi^+ p$-scattering diagrams.

16. $\pi^- p$-scattering diagrams.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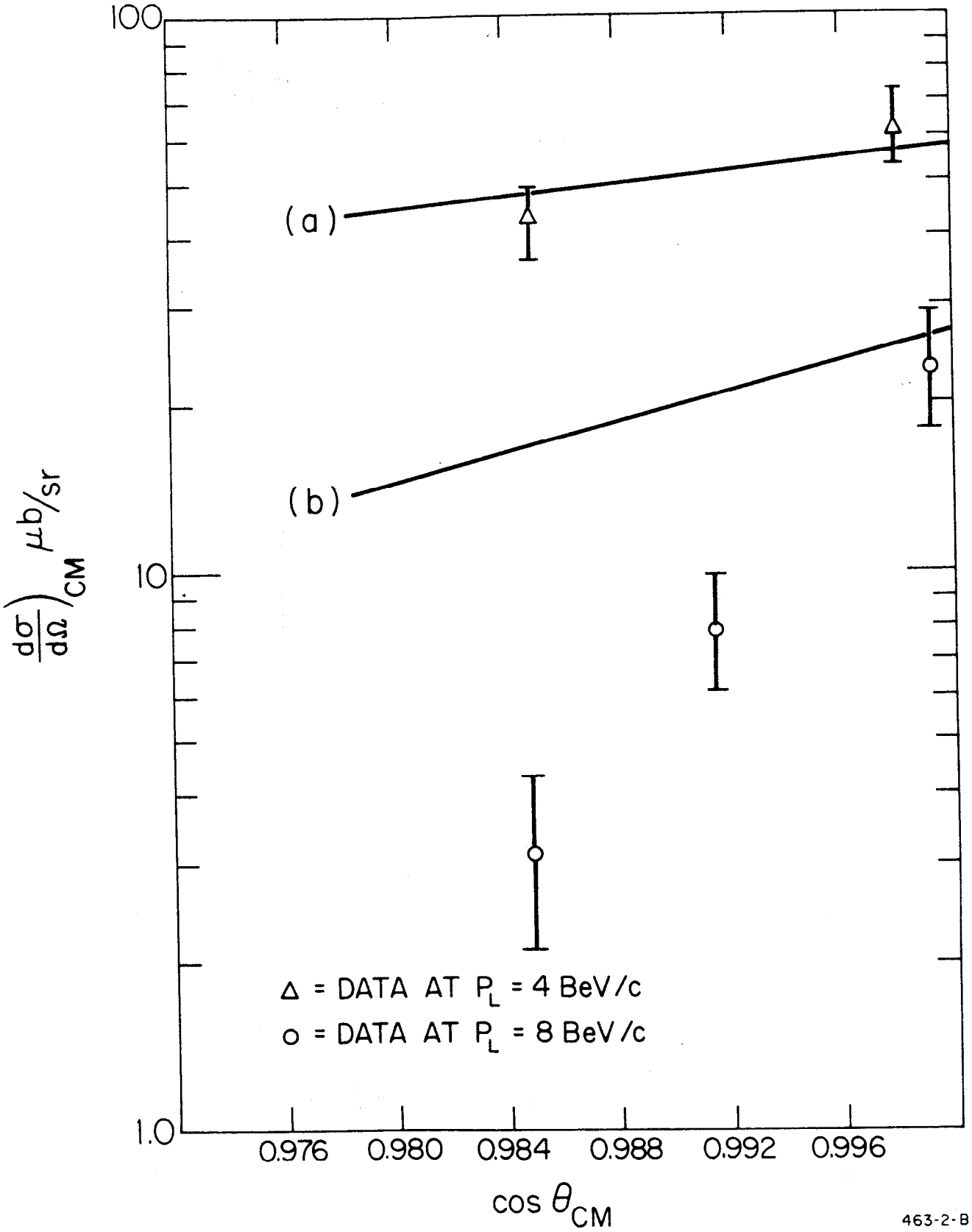
(a)
(b)
Δ = DATA AT P_L = 4 BeV/c
o = DATA AT P_L = 8 BeV/c
$\frac{d\sigma}{d\Omega}\Big)_{CM}$ μb/sr
0.976
0.980
0.984
0.988
0.992
0.996
cos θ_{CM}
463-1-B

FIG. 1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10
1.0
(a)
(b)
$\left(\frac{d\sigma}{d\Omega}\right)_{CM}$ μb/sr
Δ = DATA AT P_L = 4 BeV/c
o = DATA AT P_L = 8 BeV/c
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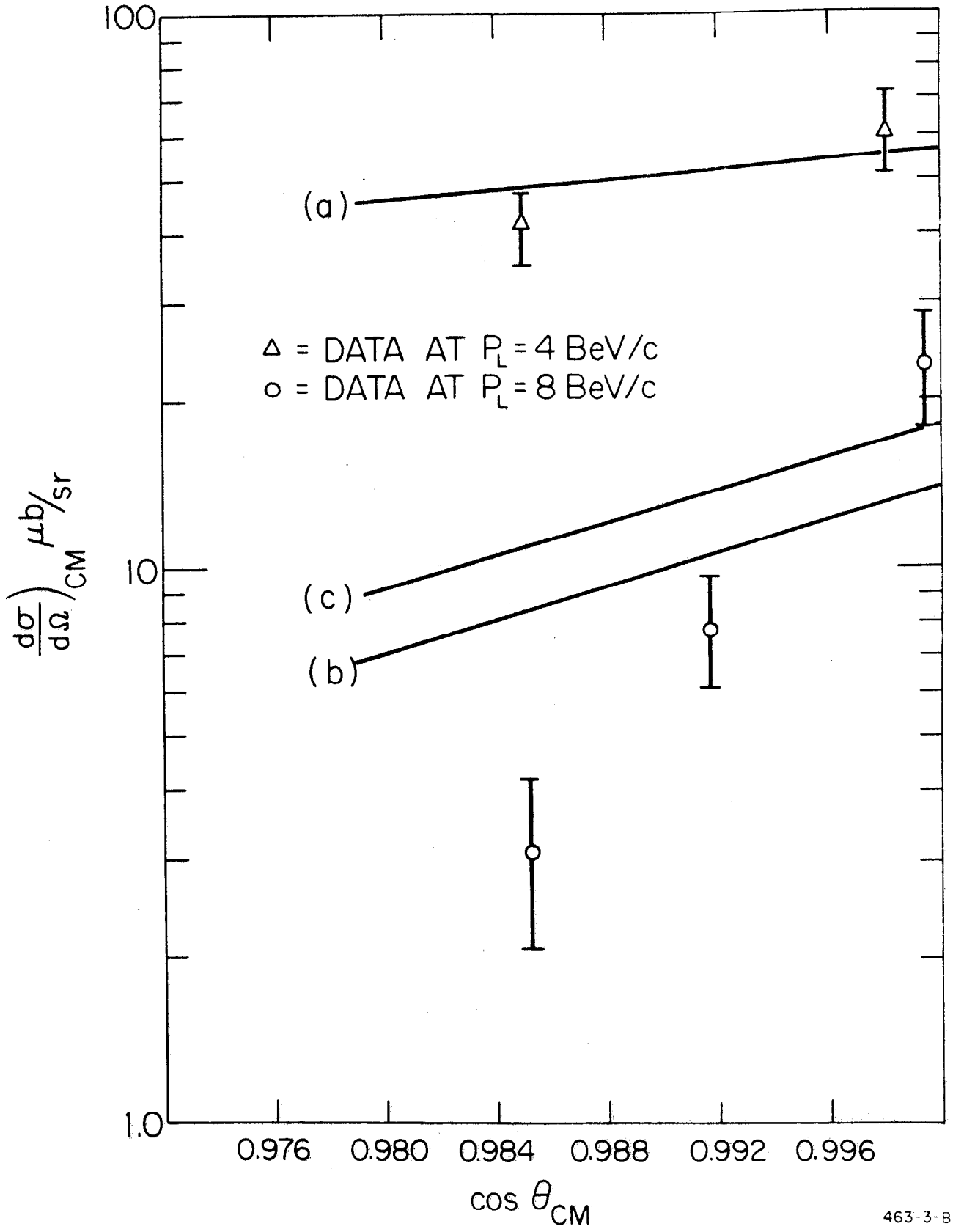
0.980
0.984
0.988
0.992
0.996
cos θ_{CM}
463-2-B

FIG. 2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10
1.0
(a)
(b)
(c)
Δ = DATA AT P_L = 4 BeV/c
o = DATA AT P_L = 8 BeV/c
$\left(\frac{d\sigma}{d\Omega}\right)_{CM}$ μb/sr
0.976
0.980
0.984
0.988
0.992
0.996
cos θ_{CM}
463-3-B

FIG. 3

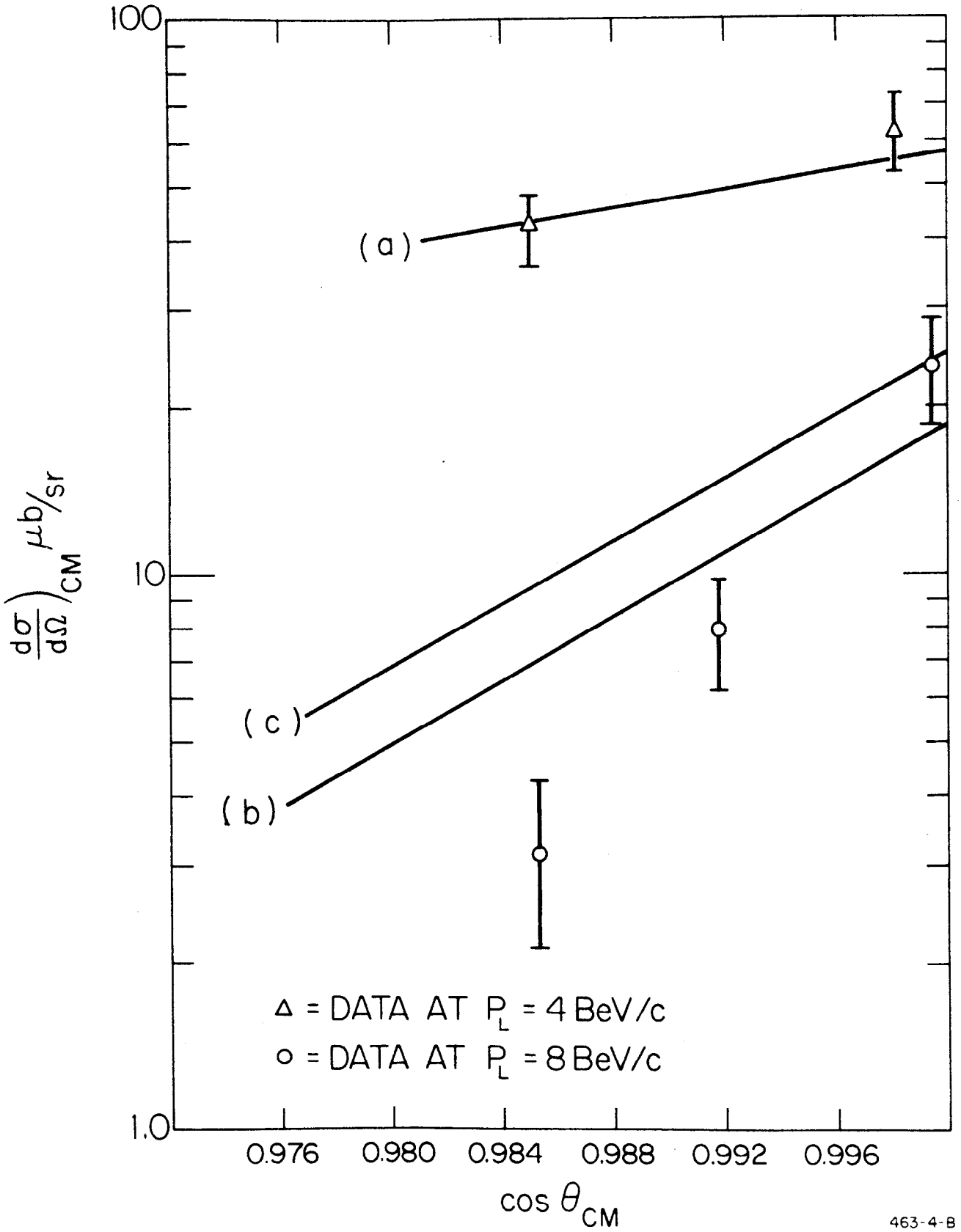

100
10
1.0
$\frac{d\sigma}{d\Omega}\Big)_{CM}$ $\mu b/sr$
(a)
(c)
(b)
Δ = DATA AT P_L = 4 BeV/c
o = DATA AT P_L = 8 BeV/c
0.976
0.980
0.984
0.988
0.992
0.996
$\cos\theta_{CM}$
463-4-B

FIG. 4

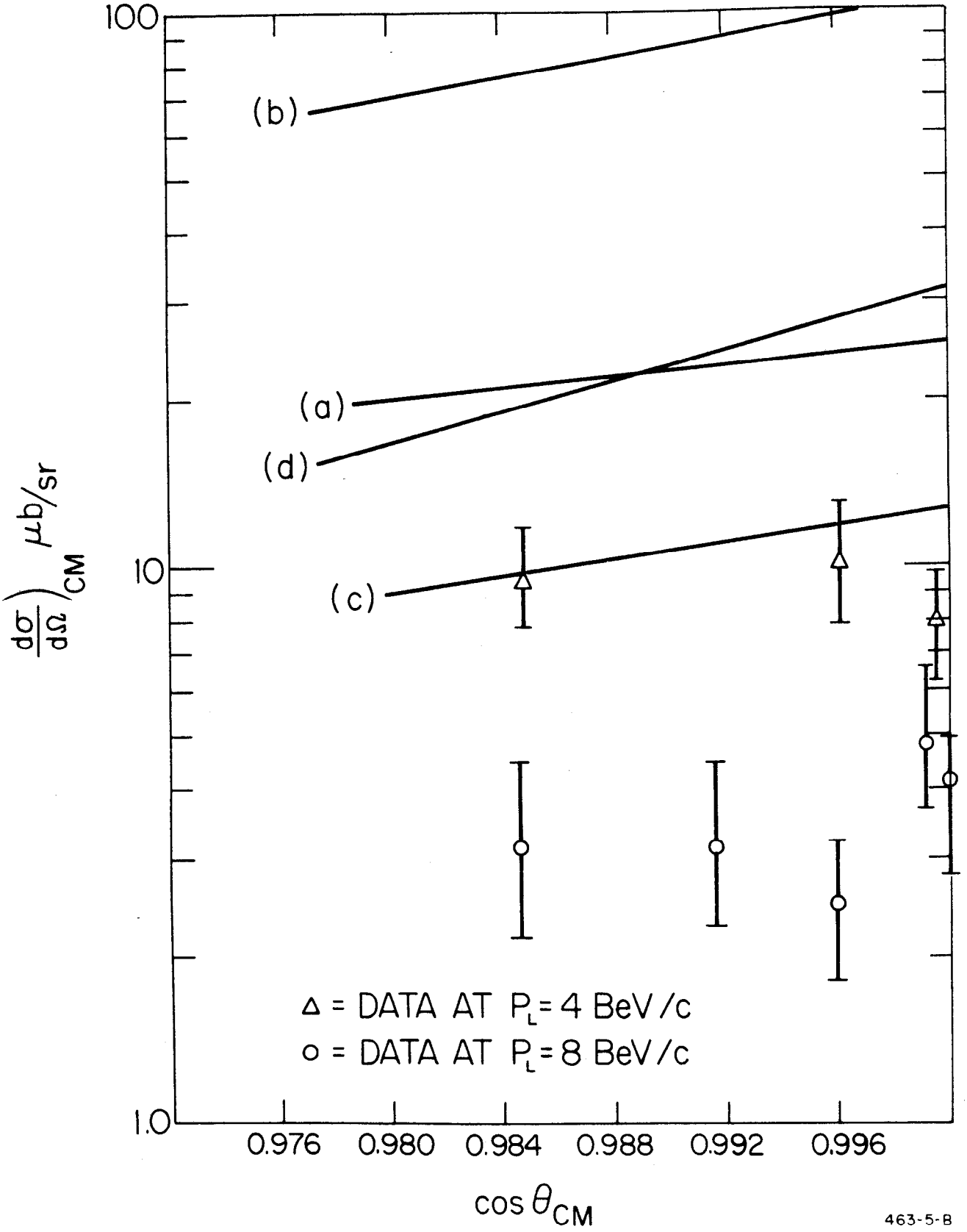
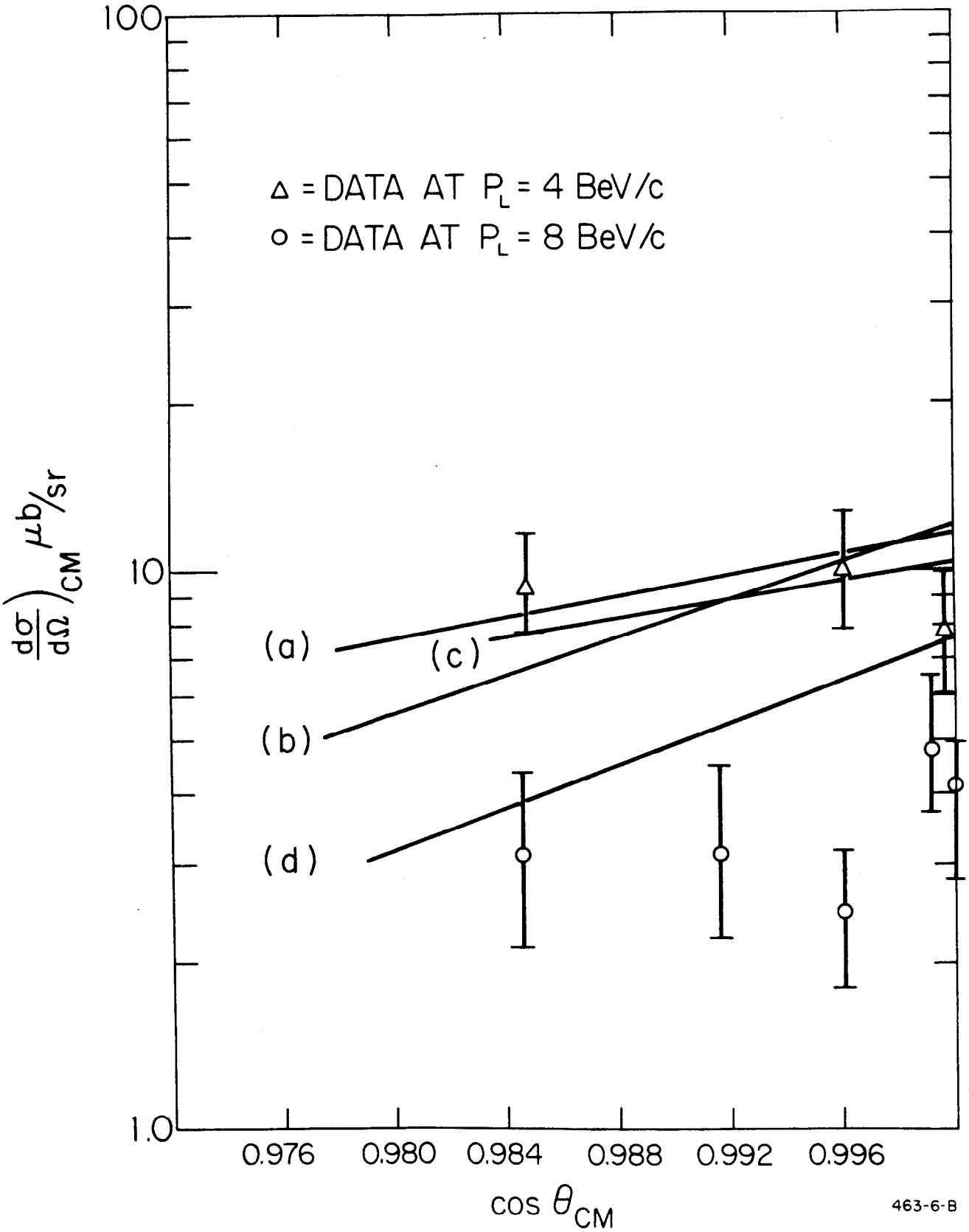

100
10
1.0
(b)
(a)
(d)
(c)
$\frac{d\sigma}{d\Omega}\Big)_{CM}$ μb/sr
Δ = DATA AT P_L = 4 BeV/c
o = DATA AT P_L = 8 BeV/c
0.976
0.980
0.984
0.988
0.992
0.996
cos θ_{CM}
463-5-B

FIG. 5


100
10
1.0
Δ = DATA AT P_L = 4 BeV/c
o = DATA AT P_L = 8 BeV/c
(a)
(b)
(c)
(d)
$\frac{d\sigma}{d\Omega})_{CM}$ μb/sr
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0.980
0.984
0.988
0.992
0.996
cos θ_{CM}
463-6-B

FIG. 6

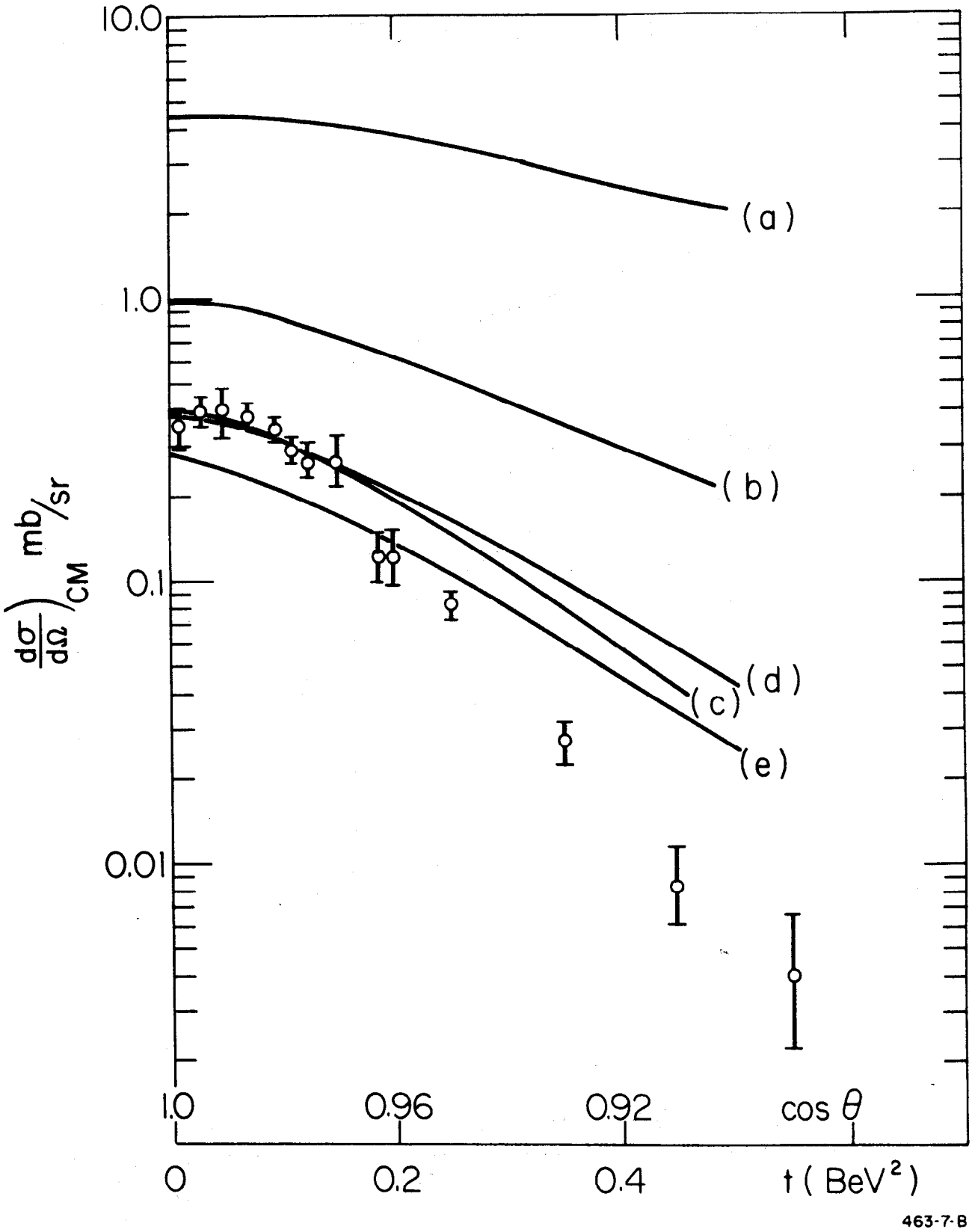

10.0
1.0
0.1
0.01
(a)
(b)
(c)
(d)
(e)
$\left(\frac{d\sigma}{d\Omega}\right)_{CM}$ mb/sr
1.0
0.96
0.92
cos θ
0
0.2
0.4
t (BeV²)
463-7-B

FIG. 7

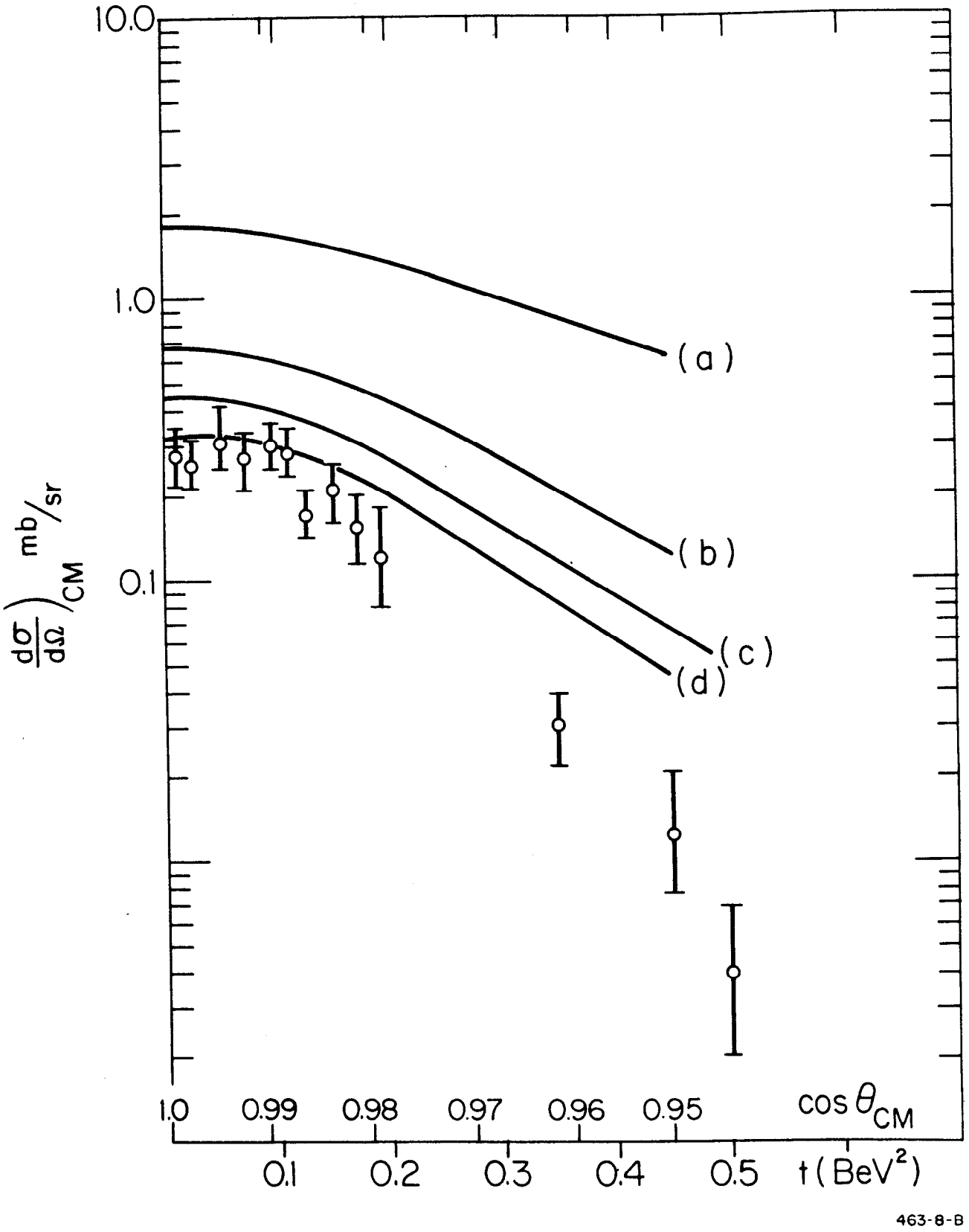

10.0
1.0
0.1
$\frac{d\sigma}{d\Omega}\Big)_{CM}$ mb/sr
(a)
(b)
(c)
(d)
1.0
0.99
0.98
0.97
0.96
0.95
$\cos\theta_{CM}$
0.1
0.2
0.3
0.4
0.5
t (BeV^2)
463-8-B

FIG. 8

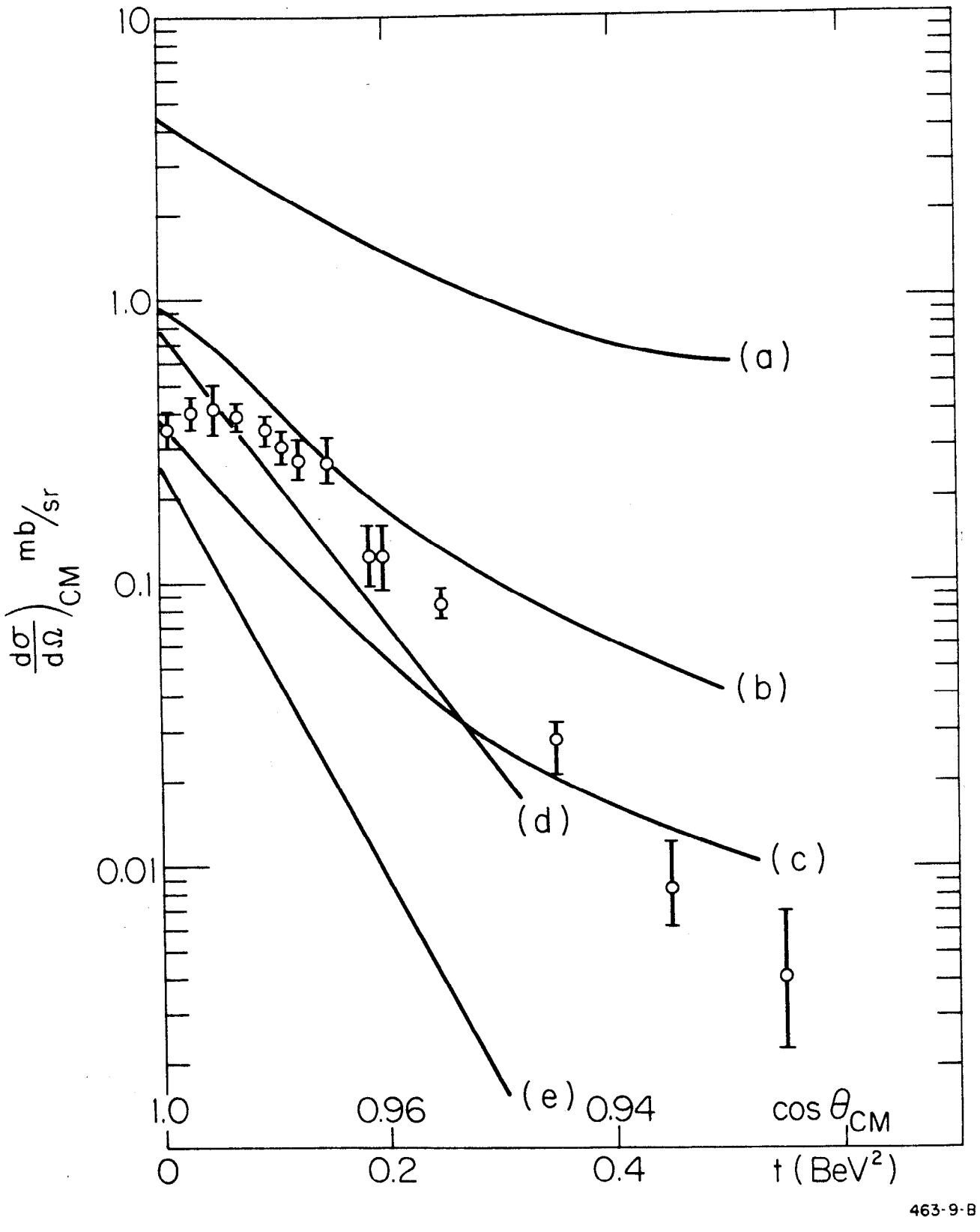

10
1.0
0.1
0.01
$\frac{d\sigma}{d\Omega}\Big)_{CM}$ mb/sr
(a)
(b)
(c)
(d)
(e)
1.0
0.96
0.94
cos θ_{CM}
0
0.2
0.4
t (BeV^2)
463-9-B

FIG. 9

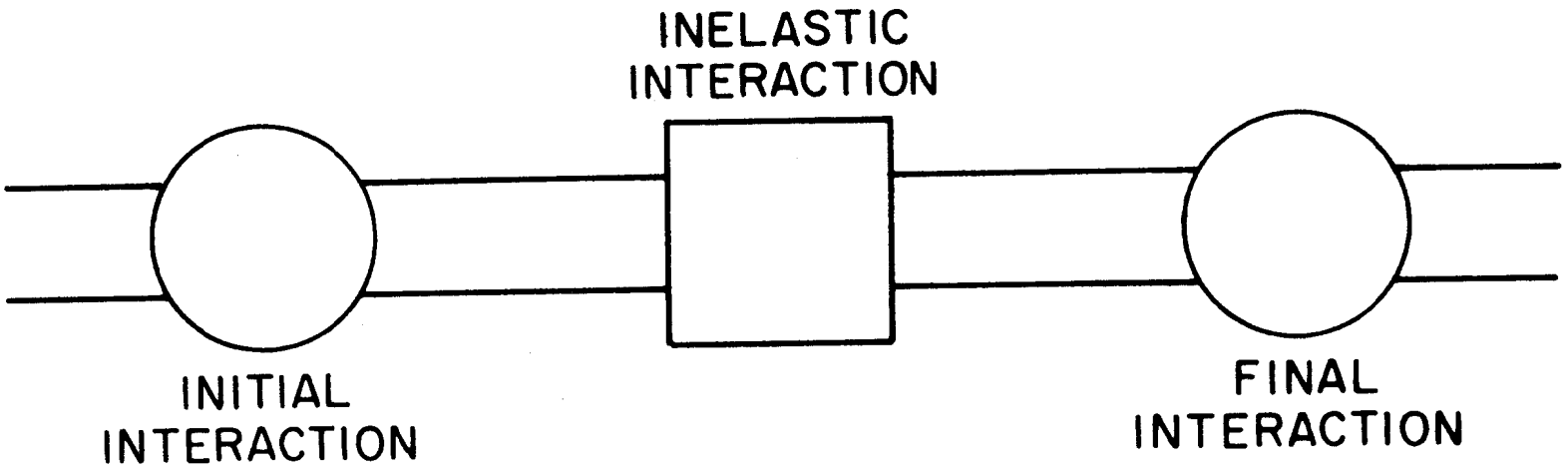
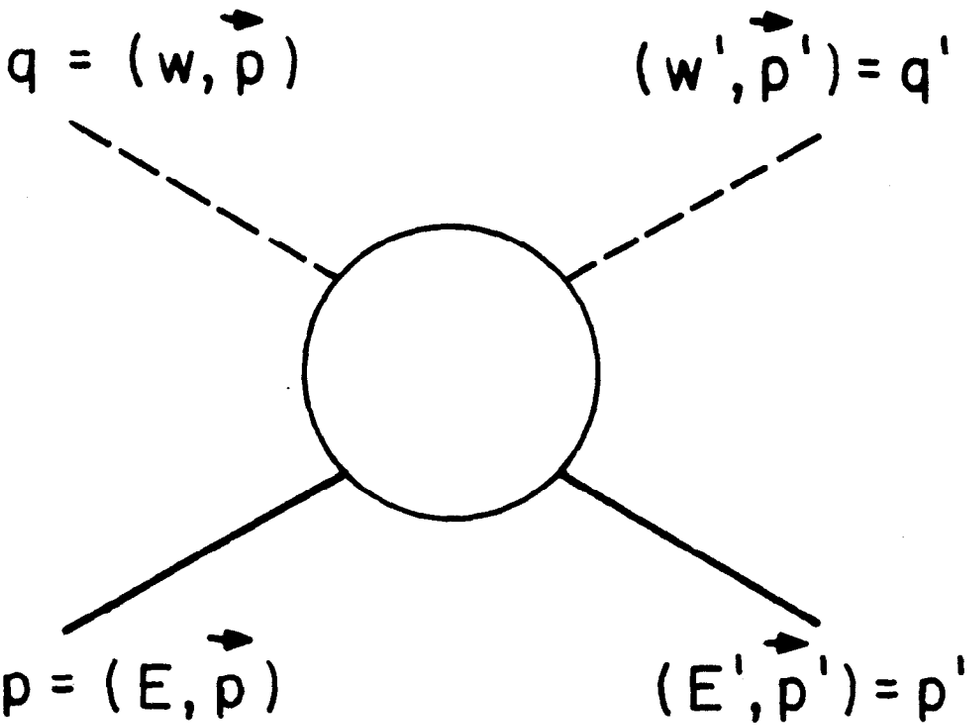
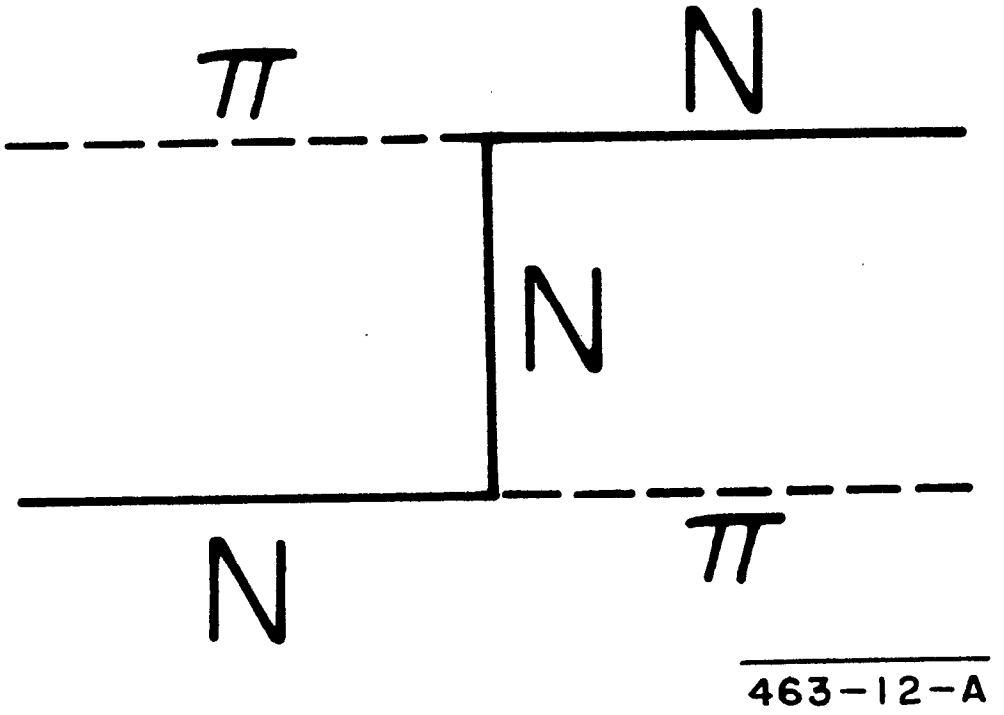

INELASTIC
INTERACTION
INITIAL
INTERACTION
FINAL
INTERACTION
463-10-A

FIG. 10.



463-11-A

FIG. 11.



463-12-A

FIG. 12

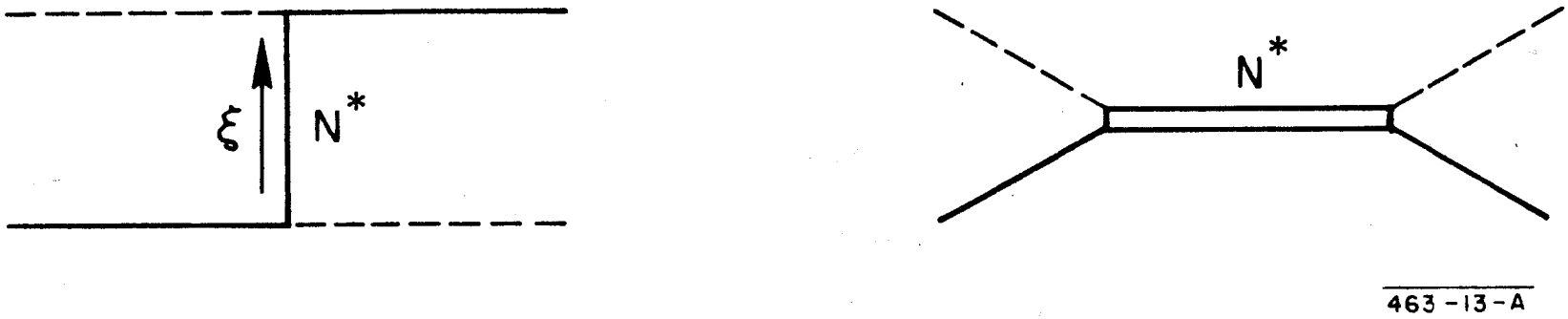


FIG. 13.

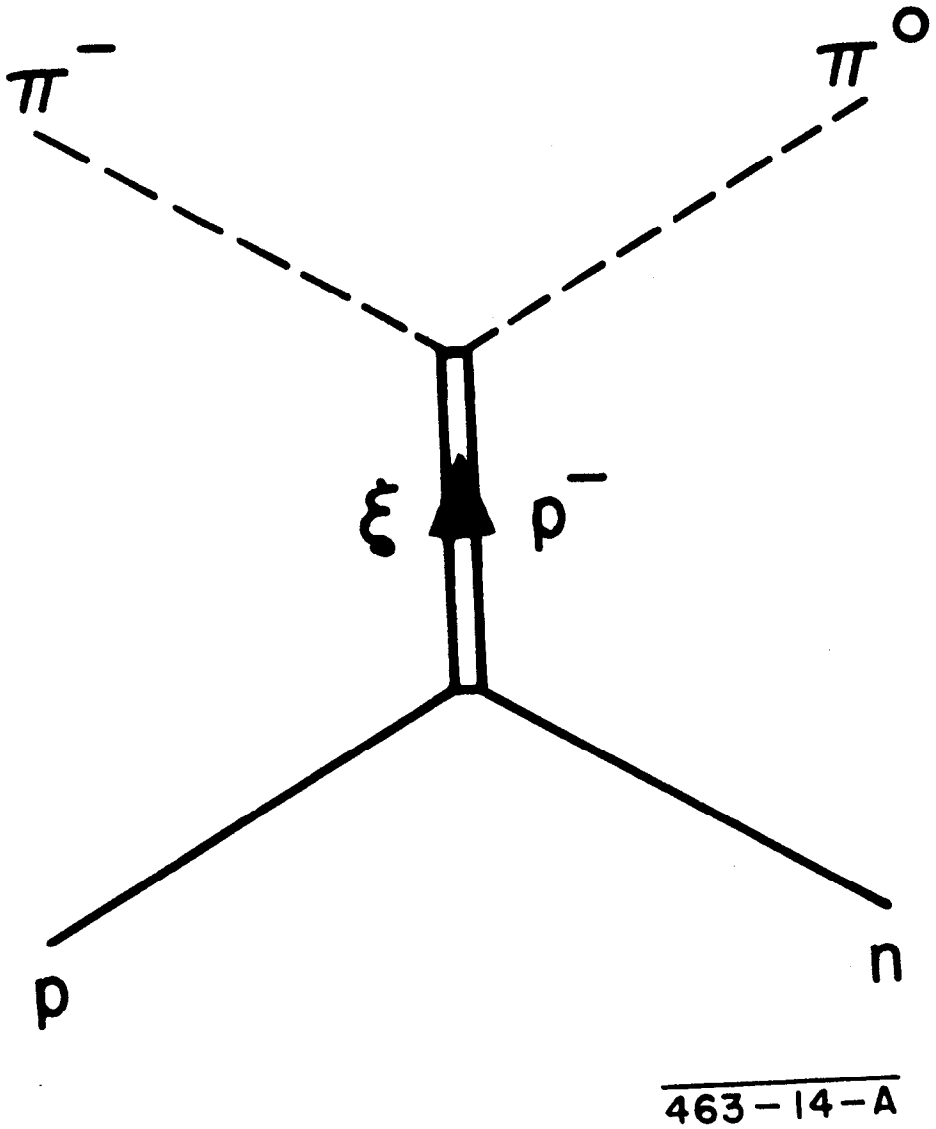


FIG. 14.

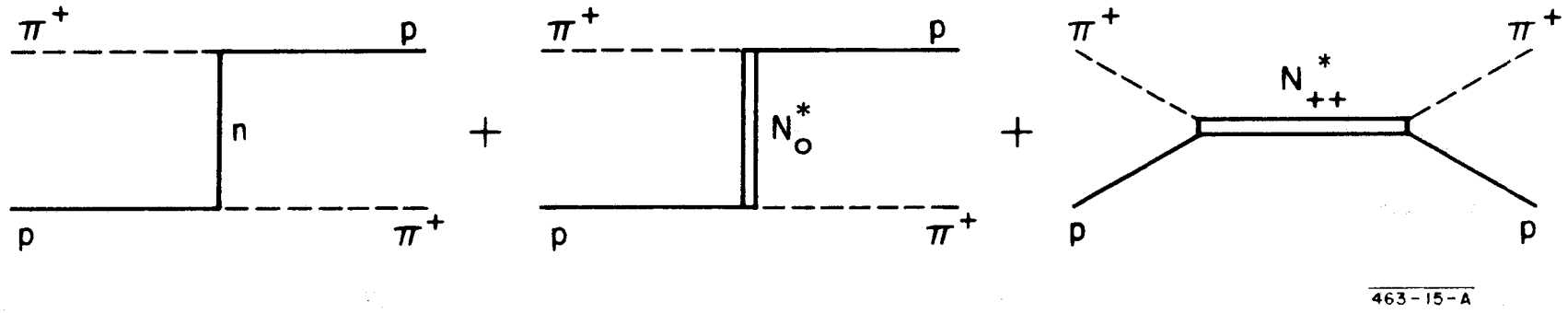


FIG. 15.

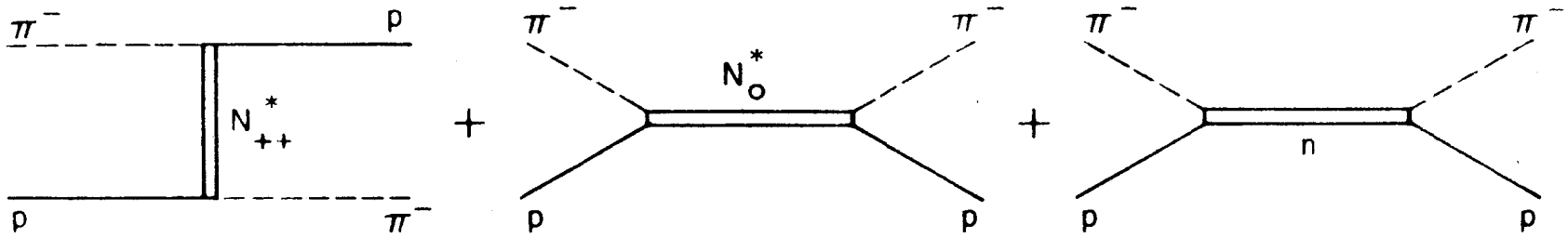


FIG. 16.