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W SPIN FOR ANY SPIN*

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ABSTRACT

The collinear group $SU(6)_W$ enables us to discuss vertices, form factors and other collinear processes in an SU(6)-theory which is consistent with relativistic invariance. It leads, however, to a classification of the particles which is different from that of the "static" $SU(6)_S$. The general "W-spin" properties of an arbitrary spin state constructed from any number of basic spin $\frac{1}{2}$ objects are discussed in detail. Explicit formulae for expressing the eigenstates of $\vec{W}^2$ as linear combinations of ordinary spin states are given and some properties of the transformation matrices are discussed. The relation between W-spin and ordinary S-spin in the framework of the $SU(2) \otimes SU(2)$ algebra is generalized to an arbitrary Lie algebra of the form $G \otimes G$. Some examples of such generalized W-type algebras are considered and the special case of $SU(6)_W$ and $SU(6)_S$ is discussed in detail. An explicit formula for calculating the $SU(6)_W$ properties of an arbitrary component of a representation of the non-chiral $U(6) \otimes U(6)$ is given. Some explicit transformation matrices between the eigenstates of W-spin and S-spin are given in the appendix.

I. INTRODUCTION

It has been pointed out that the difficulties encountered in formulating a relativistic version of the static[1] SU(6) theory can be avoided, for various sets of processes, by applying the approximate $U(6) \otimes U(6)$ symmetry to particles at rest[2] and its appropriate subgroups to collinear and coplanar processes. In particular, the collinear group $SU(6)_W$[3,4] which commutes with the Lorentz transformations in the z direction (and with the Dirac Hamiltonian for a free particle moving in that direction) enables us to discuss three-particle vertices, form factors[3] and processes such as two-body decays and forward and backward scattering.[4,5]

The difference between the "old" SU(6) classification of mesons and baryons[1] and their $SU(6)_W$ classification[4] stems from the different relative phases of the "ordinary S-spin" and the "W-spin" raising and lowering operators for quarks and antiquarks.[4] In order to classify particles according to $SU(6)_W$ or $SU(2)_W$, it is sufficient to define the W-spin operators for states having zero momentum, as the classification remains unchanged when the particles are moving with an arbitrary momentum in the z-direction. The W-spin operators for quarks and antiquarks at rest are defined as follows:[4]

$$\text{For quarks:} \qquad W_q^+ = S_q^+; \; W_q^- = S_q^-; \; W_q^z = S_q^z \quad . \tag{1}$$

$$\text{For antiquarks:} \qquad W_{\bar{q}}^+ = -S_{\bar{q}}^+; \; W_{\bar{q}}^- = -S_{\bar{q}}^-; \; W_{\bar{q}}^z = S_{\bar{q}}^z \quad . \tag{2}$$

Clearly, for any system which includes only quarks or only antiquarks we find: $\vec{W}^2 = \vec{S}^2$. This simple relation does not hold, however, for systems containing both quarks and antiquarks. Moreover, $\vec{S}^2$ does not necessarily commute with $\vec{W}^2$ and an eigenstate of $\vec{W}^2$ may be described, in general, as a linear combination of eigenstates of $\vec{S}^2$ having different total ordinary spins. It is our purpose in this paper to discuss the general problem of calculating the W-spin properties of an arbitrary spin state, and to present explicit formulae for the general $W \leftrightarrow S$ transformation matrices.

Our discussion is based on the assumption that all the involved spin states are constructed from basic spin $\frac{1}{2}$ objects which may or may not be identified as physical particles (quarks and antiquarks). All we really need is the assumption that the spins S_q and $S_{\bar{q}}$, satisfying Eqs. (1) and (2), respectively, are well defined[6] for all our states. In a simple quark model S_q is the total spin of the quarks and $S_{\bar{q}}$ is the total spin of the antiquarks. However, our arguments do not depend on the existence of quarks. They can be applied to any one of the following situations:

a) All states are classified <u>at rest</u> into the representations of the non-chiral $U(6) \otimes U(6)$. (For spin purposes, $U(2) \otimes U(2)$ is, of course, sufficient.) It is then automatically guaranteed that the particles are described by known linear combinations of eigenstates of $\vec{S}_q^2$ and $\vec{S}_{\bar{q}}^2$ where S_q is the spin associated with the first $U(6)$ [or $U(2)$] group and $S_{\bar{q}}$ is associated with the second. Notice that in this case we do not assume anything about the existence of quarks (although we may still use them as a convenient mathematical tool).

b) Particle states are classified according to a symmetry group which includes the non-chiral $U(6) \otimes U(6)$ as a subgroup. In this case S_q and $S_{\bar{q}}$ are, again, defined as the spins associated with the two $U(6)$ groups. However, these spins are not necessarily identical to the total spins of the quarks and antiquarks, respectively. S_q and $S_{\bar{q}}$ are now redefined as the total spins of the positive parity and negative parity basic spinors, respectively.[7] In a (compact) $U(12)$

or a (non-compact) $U(6,6)$ scheme[8] in which both positive parity quarks and negative parity pseudoquarks are proposed, (at least as mathematical entities) S_q will be the total spin of all quarks and antipseudoquarks whereas $S_{\bar{q}}$ is the total spin of all antiquarks and pseudoquarks.

c) The W-spin can be defined for any system of physical quarks even without a $U(6) \otimes U(6)$ classification. This can be done for any particle which is constructed from S-wave quarks and antiquarks (and, possibly, pseudoquarks and antipseudoquarks), provided that the total number of basic particles in the system is well defined (e.g., a given particle is a three-quark object with no additional quark-antiquark pairs).

d) Electromagnetic and weak currents, defined on the basis of a quark model can be classified according to $SU(2)_W$ and $SU(6)_W$[9]. In this case, positive parity currents will have $S_{\bar{q}} = 0$ and, consequently, $S = W$. Negative parity currents do not necessarily satisfy this, as for them $S_{\bar{q}} \neq 0$, and their W-spin values will be determined by their S_q and $S_{\bar{q}}$ according to our formulae.

e) Particles off the mass shell may have "virtual components" with spins different from their ordinary spin (e.g., the $S = 0$ fourth component of a virtual vector meson). The W-spins of such components can be easily determined by using the following rule: The so-called "large components" (those with $\gamma_o = +1$) of a Dirac four-spinor describing a $J^P = \frac{1}{2}^+$ object behave like quarks (and contribute to S_q); the "small components" (with $\gamma_o = -1$) behave like pseudoquarks (and contribute to $S_{\bar{q}}$). The large components ($\gamma_o = +1$) of the four spinors of the antiparticle behave like antiquarks (with $S_{\bar{q}}$) and the small components ($\gamma_o = -1$) like antipseudoquarks (with S_q). Every spin state is then constructed from basic $J^P = \frac{1}{2}^+$ objects with their antiparticles, using the Bargman-Wigner formalism (or any other free field equation for arbitrary spin). The S_q and $S_{\bar{q}}$ values are then well defined, both for the "real" and the "virtual" components.

Notice that in this case, the results may depend on the number of basic Dirac spinors which are used. For example. The virtual components of an $S = 1$ meson which is described by one pair of basic spinors (e.g., according to the Duffin-Kemmer equation) will differ from those of a vector meson obtained from three pairs of basic spinors. This ambiguity is present in any field theoretical description of a particle with an arbitrary spin.

The material of the paper is organized as follows. In Section II we discuss the $SU(2) \otimes SU(2)$ algebra which includes both the W-spin and the S-spin operators, and we present explicit formulae for the $W \longleftrightarrow S$ transformations. Some properties of the transformation coefficients are discussed in Section III, while in Section IV we generalize our procedure to some bigger groups. Finally, in Section V, the transition coefficients between $SU(6)_W$ and $SU(6)_S$ are calculated.

II. W SPIN FOR ANY SPIN

Consider a particle state having well-defined values of S_q and $S_{\bar{q}}$ and a total spin S, and denote the z-components of these spins by M_q, $M_{\bar{q}}$ and M, respectively. We know that:

$$\vec{S} = \vec{S}_q + \vec{S}_{\bar{q}} \tag{3}$$

$$W^z = S^z = M = M_q + M_{\bar{q}} \quad . \tag{4}$$

From Eqs. (1) and (2) we obtain:

$$W^+ = S_q^+ - S_{\bar{q}}^+ \tag{5}$$

$$W^- = S_q^- - S_{\bar{q}}^- \quad . \tag{6}$$

The six operators S_q^+, S_q^-, S_q^z, $S_{\bar{q}}^+$, $S_{\bar{q}}^-$, $S_{\bar{q}}^z$ form an $SU(2) \otimes SU(2)$ algebra,[10] the generators of which may be also chosen as S^+, S^-, W^+, W^-, $R = M_q - M_{\bar{q}}$ and $M = M_q + M_{\bar{q}} = W^z = S^z$. Using the well-known properties of $SU(2)$ and $SU(2) \otimes SU(2)$[11] [which are locally isomorphic to $SO(3)$ and $SO(4)$, respectively] we can calculate the explicit formula for applying the lowering operator W^- to a spin state $|S, M>$. Defining

$$S_q + S_{\bar{q}} = r \tag{7}$$

$$S_q - S_{\bar{q}} = t \tag{8}$$

we obtain the following expression:[12]

$$\begin{aligned} W^- |S, M> = {} & \frac{t(r+1)}{S(S+1)} \sqrt{(S-M+1)\,(S+M)}\, |S, M-1> - \\ & - \frac{1}{S} \sqrt{\frac{(S+M)\,(S+M-1)\,[(r+1)^2 - S^2]\,(S^2-t^2)}{(2S+1)\,(2S-1)}}\, |S-1, M-1\rangle + \\ & + \frac{1}{S+1} \sqrt{\frac{(S-M+2)\,(S-M+1)\,[(r+1)^2-(S+1)^2]\,[(S+1)^2-t^2]}{(2S+3)\,(2S+1)}}\, |S+1, M-1>. \end{aligned} \tag{9}$$

In order to express the eigenstates of $\vec{W}^2$ in terms of the S-spin states, the following procedure may be used. For any set of values of S_q and $S_{\bar{q}}$ we start by considering the state $|r, r>$ satisfying:

$$r = S = W = M = S_q + S_{\bar{q}} \quad .$$

Using Eq. (9), we apply W_- to this state and obtain all $2W + 1$ components of the $W = r$ multiplet. The maximal W^z component of the next W-spin multiplet $(W = r - 1)$ is then found from its orthogonality to the state $|r, r-1>$, and all other components of the $W = r - 1$ multiplet can be, again, obtained by applying W^-. By continuing this procedure we can express all the W-spin multiplets in terms of the appropriate ordinary spin states, and we can calculate the transformation matrices between the eigenstates of $\vec{S}^2$ and $\vec{W}^2$ for any value of S_q, $S_{\bar{q}}$ and M.

It turns out, however, that this procedure, though very simple in principle, is not very convenient for calculating a given $W \leftrightarrow S$ coefficent, when other coefficients are not known, because it requires a successive application of Eq. (9) to higher states. We therefore present here an alternative procedure of calculating the $W \leftrightarrow S$ transformations, expressing the coefficients themselves in terms of the ordinary SU(2) Clebsch-Gordan coefficients. Consider a system with a given set of values for S_q, $S_{\bar{q}}$ and $M = S^z = W^z$. Its total S and W may be in the range

$$\max\ (|t|, M) \leq W \leq r$$

$$\max\ (|t|, M) \leq S \leq r$$

where r and t are defined by Eqs. (7) and (8). All states having the given values of S_q, $S_{\bar{q}}$, M may be labeled according to any one of the following sets of quantum numbers:

I. S_q, M_q, $S_{\bar{q}}$, $M_{\bar{q}}$.

II. S, M, S_q, $S_{\bar{q}}$.

III. W, M, S_q, $S_{\bar{q}}$.

The transformation between the eigenstates of the operators of set I and those of set II is given by the usual SU(2) Clebsch-Gordan coefficients. For every set of values of S_q, $S_{\bar{q}}$ and M we define the usual matrix:[13]

$$A^{S}_{M_q}(S_q, S_{\bar{q}}, M) = (S_q M_q S_{\bar{q}} M_{\bar{q}} | SM) \quad . \tag{10}$$

A similar matrix gives the transformation between sets I and III. However, all the SU(2) Clebsch-Gordan coefficients in this matrix are multiplied by the phase factor $(-1)^{S_{\bar{q}}-M_{\bar{q}}}$ which reflects the minus signs in the definitions of $W^{\pm}$ for antiquarks [Eq. (2)]. The transformation matrix B is then defined by

$$B^{W}_{M_q}(S_q, S_{\bar{q}}, M) = (-1)^{S_{\bar{q}}-M_{\bar{q}}} (S_q M_q S_{\bar{q}} M_{\bar{q}} | WM) \quad . \tag{11}$$

The transformation between the sets of operators II and III can now be easily calculated.[14]

$$\begin{aligned} C^{W}_{S}(S_q, S_{\bar{q}}, M) &= \sum_{M_q} \left(A^{S}_{M_q}\right)^{\dagger} \cdot B^{W}_{M_q} = \\ &= \sum_{M_q} (-1)^{S_{\bar{q}}-M_{\bar{q}}} (SM| S_q M_q S_{\bar{q}} M_{\bar{q}}) (S_q M_q S_{\bar{q}} M_{\bar{q}} | WM) \end{aligned} \tag{12}$$

where $A^{\dagger}$ is the transposed of A.

Using the standard phases of the ordinary SU(2) Clebsch-Gordan coefficients and the exact form of Eq. (12), we automatically obtain standard phases for the C^{W}_{S} coefficients. Some of these coefficients are given in the appendix.

III. SOME PROPERTIES OF THE COEFFICIENTS $C_S^W(S_q, S_{\bar{q}}, M)$

1. For $S_q = 0$ or $S_{\bar{q}} = 0$, Eq. (12) degenerates into $C_S^W = (-1)^{S_{\bar{q}}-M_{\bar{q}}} \delta_S^W$. This is consistent with our previous observation that $\vec{W}^2 = \vec{S}^2$ for any "pure" system of quarks or antiquarks. This last equality is now extended to systems in which all quark spins or all antiquark spins are coupled to zero.
2. For $S_q = S_{\bar{q}} = \frac{1}{2}$, we obtain the already well-known W-S spin flip for the vector and pseudoscalar mesons. The two M=0 states of the $q\bar{q}$ system have $S = 0$, $W = 1$ and $S = 1$, $W = 0$ respectively.[4]
3. For $S_q = S_{\bar{q}}$, we obtain from Eq. (9) an interesting "selection rule." Since $t = S_q - S_{\bar{q}} = 0$, no transition between two states with the same S-spin can be induced by applying W^-. Consequently, states with odd and even values of S cannot be mixed in the same $|W, M>$ states and vice versa. The $q\bar{q}$ system is, of course, a simple example to this rule.
4. The matrix C_W^S is symmetric. This reflects the symmetrical relation between the S-spin and the W-spin generators in $SU(2) \otimes SU(2)$.
5. It can be shown that:[15]

$$e^{i\pi W_x} = P_{int} e^{i\pi S_x} \tag{13}$$

where P_{int} is the intrinsic parity. We know that:

$$e^{i\pi W_x} | W, M > = (-1)^{W-M} | W, -M > \tag{14}$$

$$P_{int} e^{i\pi S_x} | S, M > = P_{int} (-1)^{S-M} | S, -M >. \tag{15}$$

Consequently:

$$C_S^W (S_q, S_{\bar{q}}, M) = (-1)^{S-W} \cdot P_{int} \cdot C_S^W (S_q, S_{\bar{q}}, -M) \qquad (16)$$

or:

$$C_S^W (S_q, S_{\bar{q}}, M) = (-1)^{S-W-2S_{\bar{q}}} C_S^W (S_q, S_{\bar{q}}, -M) \quad . \qquad (16')$$

For the case $M = 0$ we find:

$$(-1)^{S-W} = (-1)^{2S_{\bar{q}}} = P_{int} \qquad (17)$$

Namely, for an odd number of antiquarks the zero helicity state connects odd S to even W and even S to odd W. For an even number of antiquarks even S can be connected only to even W and odd S to odd W.

6. It is clear that states with high S-spins have, in general, components with small W-spins. This means that W-spin conservation does not lead to any general ΔS selection rule. This is extremely important, as otherwise the π-π or N-π decay modes of resonances with high spins $(2, \frac{5}{2}, \frac{7}{2}$, etc.) would be forbidden in an $SU(2)_W$ invariant theory.

IV. THE GENERALIZED W-SPIN

The $SU(2) \otimes SU(2)$ analysis of W-spin and S-spin can be generalized to any algebra of the form $G \otimes G$. We denote the generators of the two commuting G

algebras by g_i', g_i'' and assume that they satisfy the usual commutation relations:

$$\left[g_i', g_j'\right] = C_{ij}^k g_k' \tag{18}$$

$$\left[g_i'', g_j''\right] = C_{ij}^k g_k'' \quad . \tag{19}$$

The "diagonal" subalgebra G^D is then defined by:

$$g_i^D = g_i' + g_i'' \quad . \tag{20}$$

Clearly:

$$\left[g_i^D, g_j^D\right] = C_{ij}^k g_k^D \tag{21}$$

We can now divide the operators of G into two sets:

$$G = K + P \tag{22}$$

such that: (1) K is a subalgebra of G, (2) the operators in P transform like one or more irreducible representations of K and (3) the commutator of any two operators in P belongs to K. We may then define:

$$\text{For } g_i' \epsilon K \qquad g_i^W = g_i' + g_i'' \tag{23}$$

$$\text{For } g_i' \epsilon P \qquad g_i^W = g_i' - g_i'' \tag{24}$$

Again:

$$\left[g_i^W, g_j^W\right] = C_{ij}^k g_k^W \tag{25}$$

and the g_i^W operators form an algebra G^W isomorphic to G^D. Both G^W and G^D have the same subalgebra K, but they differ in all their other generators. The general transformation between the eigenstates of the G^W and G^D operators can be easily expressed in terms of the Clebsch-Gordan coefficients of G, using a straightforward generalization of Eq. (12).

The simplest method of finding all possible isomorphic G^W subalgebras of $G \otimes G$ is based on the analogous problem of finding all non-compact versions of a compact algebra G. In both cases all we have to do is to find all subalgebras K of G which satisfy the conditions (1) - (3). For every such K we can construct another G^W algebra (and, similarly, a non-compact version of the compact G). Since we know the solution to the second problem (at least for the classical algebras) we also know how to construct all G^W algebras.

We now consider a few examples:

a) For $G \equiv SU(2)$ the only possible subalgebra K is U(1). Consequently, we have only one G^W algebra: $SU(2)_W$ [similarly, SU(2) has only one non-compact form SU(1,1) whose maximal compact subalgebra is U(1)].

b) For $G \equiv SU(3)$ we may have two G^W algebras corresponding to the two possible K-subalgebras — SO(3) and U(2). Denoting the SU(3) generators by $I^{\pm}, U^{\pm}, V^{\pm}, I^Z, Y$, we may consider the SO(3) subalgebra:

$$\begin{aligned} K^+ &= U^+ + V^+ \\ K^- &= U^- + V^- \\ K^Z &= I^Z \end{aligned} \qquad (26)$$

The other five generators transform like a $K = 2$ tensor:

$$
\begin{aligned}
Q^{++} &= I^{+} \\
Q^{+} &= U^{+} - V^{+} \\
Q^{o} &= Y \\
Q^{-} &= U^{-} - V^{-} \\
Q^{--} &= I^{-} \quad .
\end{aligned}
\tag{27}
$$

We can now start from $SU(3) \otimes SU(3)$ and define:

$$K_i^W = K_i' + K_i'' \tag{28}$$

$$Q_j^W = Q_j' - Q_j'' \ . \tag{29}$$

The operators K_i^W, Q_j^W form an $SU(3)$ algebra.

Another W-type $SU(3)$ which may have physical application can be constructed by identifying the subalgebra K with the isospin-hypercharge $U(2)$ or with the U-spin-electric charge $U(2)$. In the framework of the coplanar $SU(3) \otimes SU(3)$[2,16] we can construct in this way an $SU(3)$ subalgebra whose generators are:

$$W_x I^+, W_x I^-, I^z, W_x V^+, W_x V^-, U^+, U^-, Y .$$

Under this subalgebra the $W_x = 0$ states of ρ^+, ω, ρ^- form an $SU(2)$ triplet[17] which can be used for calculating coplanar processes.

In the framework of the non-chiral $U(3) \otimes U(3)$ one might also consider a W-type $SU(3)$ algebra defined by

$$I^+, I^-, I^z, \gamma_o V^+, \gamma_o V^-, \gamma_o U^+, \gamma_o U^-, Y.$$

Such an algebra might be a good symmetry even in the presence of the usual $SU(3)$ symmetry breaking since only the isospin-hypercharge $U(2)$ subgroup is common to this $SU(3)$ and the ordinary $SU(3)$.[18]

c) In the case $G \equiv SU(6)$ we can construct G^W algebras in various different ways, corresponding to the following possible subalgebras K:

$$U(5),\ SU(4) \otimes SU(2) \otimes U(1),\ SU(3) \otimes SU(3) \otimes U(1),\ SO(6),\ Sp(6).$$

We are interested in the case $K \equiv SU(3) \otimes SU(3) \otimes U(1)$ since this is the algebra of all generators belonging to both $SU(6)_S$ and $SU(6)_W$. This $SU(3) \otimes SU(3)$ may be considered as the collinear subgroup of $SU(6)_S$[19] [to be distinguished from $SU(6)_W$ which is the collinear subgroups of $U(6) \otimes U(6)$]. The explicit expression for the transformation between $SU(6)_S$ and $SU(6)_W$ is obtained by using the method of Section II and is discussed in detail in the next section.

V. $SU(6)_W$ AND $SU(6)_S$

Both $SU(6)_W$ and $SU(6)_S$ are subgroups of the non-chiral $U(6) \otimes U(6)$. We denote the sets of $SU(3)$ quantum numbers $[(\lambda,\mu)II_zY]$ for the two $U(6)$ groups by Λ_q and $\Lambda_{\bar{q}}$, respectively, and the total $SU(3)$ quantum numbers [those of $SU(6)_S$ or $SU(6)_W$] by Λ. It is a priori clear that the transformation between $SU(6)_W$

and $SU(6)_S$ conserves, in addition to $S^z = W^z = M$, all the quantum numbers of SU(3). Furthermore, it can mix states of different $SU(6)_S$ representations only within the same $U(6) \otimes U(6)$ multiplet which we denote by $(\mu_q, \mu_{\bar{q}})$. We, therefore, start our analysis by considering all states belonging to the $(\mu_q, \mu_{\bar{q}})$ representation of the non-chiral $U(6) \otimes U(6)$, which have the same values of Λ and M [e. g., all octet states with M = 0 in the $(21, \overline{21})$]. We then follow the procedure of Section II. All the considered states can be labeled by the following sets of quantum numbers:

I. $\mu_q, \Lambda_q, S_q, M_q, \alpha_q, \mu_{\bar{q}}, \Lambda_{\bar{q}}, S_{\bar{q}}, M_{\bar{q}}, \alpha_{\bar{q}}$

II. $\mu_q, \mu_{\bar{q}}, \mu_S, \Lambda, S, M, \beta$

III. $\mu_q, \mu_{\bar{q}}, \mu_W, \Lambda, W, M, \gamma$

where μ_S and μ_W are the dimensionalities of the representations of $SU(6)_S$ and $SU(6)_W$, respectively, and $\alpha_q, \alpha_{\bar{q}}, \beta, \gamma$ are additional quantum numbers which may be needed for distinguishing between two identical $SU(3) \otimes SU(2)$ representations within the same SU(6) multiplet. It is convenient to denote the set of quantum numbers $(\Lambda_q, S_q, M_q, \alpha_q)$ by R_q and the set $(\Lambda_{\bar{q}}, S_{\bar{q}}, M_{\bar{q}}, \alpha_{\bar{q}})$ by $R_{\bar{q}}$.

The transition between the sets I and II is then given by the usual SU(6) Clebsch-Gordan coefficients:[20]

$$A^{\mu_S S\beta}_{R_q R_{\bar{q}}}(\mu_q, \mu_{\bar{q}}, \Lambda, M) = (\mu_q R_q; \mu_{\bar{q}} R_{\bar{q}} | \mu_S \Lambda S \beta) \quad . \tag{30}$$

The transition between the sets I and III is given by the same coefficients multiplied by $(-1)^{S_{\bar{q}}-M_{\bar{q}}}$:

$$B^{\mu_W W\gamma}_{R_q R_{\bar{q}}}(\mu_q, \mu_{\bar{q}}, \Lambda, M) = (-1)^{S_{\bar{q}} - M_{\bar{q}}} \cdot (\mu_q R_q; \mu_{\bar{q}} R_{\bar{q}} \mid \mu_W \Lambda W \gamma) \quad . \tag{31}$$

Finally:

$$C^{\mu_W W\gamma}_{\mu_S S\beta}(\mu_q, \mu_{\bar{q}}, \Lambda, M) = \sum_{R_q R_{\bar{q}}} (-1)^{S_{\bar{q}} - M_{\bar{q}}} (\mu_S \Lambda S\beta \mid \mu_q R_q; \mu_{\bar{q}} R_{\bar{q}}) \cdot \tag{32}$$

$$\cdot (\mu_q R_q; \mu_{\bar{q}} R_{\bar{q}} \mid \mu_W \Lambda W \gamma)$$

or, in matrix notation $C = A^\dagger B$.

The simplest non-trivial C-matrix is the 2×2 matrix obtained for the SU(3) singlet, $M = 0$, quark-antiquark system. In this case (μ^S, S) obtains the values (35, 1) and (1, 0) and the explicit form of C is:

$$C^{\mu_W W}_{\mu_S S}(6, \bar{6}, 1, 0) = 1 - \delta^W_S \quad . \tag{33}$$

Namely the $S = 0$, $SU(6)_S$-singlet is a $W = 1$ state in the 35 of $SU(6)_W$ and the $S = 1$ state of the 35 of $SU(6)_S$ is a $W = 0$ $SU(6)_W$ singlet.

FOOTNOTES AND REFERENCES

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3. K. J. Barnes, P. Carruthers and F. von Hippel, Phys. Rev. Letters 14, 81(1965); K. J. Barnes, Phys. Rev. Letters 14, 798 (1965).

4. H. J. Lipkin and S. Meshkov, Phys. Rev. Letters 14, 670 (1965).

5. J. C. Carter, J. J. Coyne, S. Meshkov, D. Horn, M. Kugler and H. J. Lipkin, Phys. Rev. Letters 15, 373 (1965).

6. "Well defined," for our purposes, means that a given particle with spin S can be uniquely described as a linear combination of eigenstates of $\vec{S}_q^2$ and $\vec{S}_{\bar{q}}^2$.

7. This generalized definition of W-spin was introduced by H. J. Lipkin and S. Meshkov, Phys. Rev., in print.

8. See e.g. Y. Dothan, M. Gell-Mann and Y. Ne'eman, Phys. Letters 17, 148 (1965). Pseudoquarks are negative parity SU(3) triplets transforming like the $(\underline{1}, \underline{6})$ representation of $U(6) \otimes U(6)$.

9. The W-spin classification of the scalar, pseudoscalar, vector, axial vector and tensor currents can be easily obtained from the classification of all Dirac γ-matrices which was given in Ref. 4.

10. Some properties of this $SU(2) \otimes SU(2)$ algebra were discussed by K. Ahmed, S. A. Dunne, M. Martinis and J. R. Poston, Phys. Rev., to be published.

11. See e.g. W. Pauli, Continuous groups in quantum mechanics, CERN 56-31.

12. Equation (9) can be obtained by using some of the relations given by W. Pauli, Ref. 11. For simplicity, we did not include the full normalization factor in Eq. (9). This can be easily calculated for any given specific case.

13. Tables of SU(2) Clebsch-Gordan coefficients organized in matrices of this form, can be found, for example, in A. H. Rosenfeld et al., UCRL-8030 (wallet cards).

14. Notice that the correct normalization is guaranteed in this case by the properties of the ordinary SU(2) C-G coefficients.

15. This relation was first noticed and discussed by Lipkin and Meshkov, Ref. 7.

16. The two commuting SU(3) algebras are defined by $(1 \pm W_x)\lambda_\alpha$, $\alpha = 1, \ldots, 8$ where λ_α are the usual SU(3) generators.

17. ω is the singlet-octet mixture which includes no strange quarks;

$$\omega = \sqrt{\frac{2}{3}}\,\omega_1 + \sqrt{\frac{1}{3}}\,\omega_8 \ .$$

18. This possibility is considered by S. Coleman (private communication).

19. D. V. Volkov, JETP Letters, 1, 129 (1965). This $SU(3) \otimes SU(3)$ is defined by the matrices $(1 \pm \sigma_z)\lambda_\alpha$.

20. J. C. Carter, J. J. Coyne and S. Meshkov, Phys. Rev. Letters 14, 523 (1965); 14, 850 (1965) C. L. Cook and G. Murtaza, Nuovo Cimento, 39, 531 (1965). These tables include the explicit coefficients for the products 35×56, $35 \otimes 35$ and $56 \otimes \overline{56}$.

APPENDIX: SOME C_S^W COEFFICIENTS

We present here a few $C_S^W(S_q, S_{\bar{q}}, M)$ coefficients. The tables include the coefficients for $M \geq 0$. The coefficients for $M < 0$ can be obtained from those by Eq. (16').

I. $S_q = \frac{1}{2}$; $S_{\bar{q}} = \frac{1}{2}$

$M = 1$

	1
1	1

$M = 0$

	1	0
1	0	-1
0	-1	0

II. $S_q = 1$; $S_{\bar{q}} = \frac{1}{2}$

$M = \frac{3}{2}$

	$\frac{3}{2}$
$\frac{3}{2}$	1

$M = \frac{1}{2}$

	$\frac{3}{2}$	$\frac{1}{2}$
$\frac{3}{2}$	$\frac{1}{3}$	$-\frac{2\sqrt{2}}{3}$
$\frac{1}{2}$	$-\frac{2\sqrt{2}}{3}$	$\frac{1}{3}$

III. $S_q = 1; \quad S_{\bar{q}} = 1$

M = 2

	2
2	1

M = 1

	2	1
2	0	-1
1	-1	0

M = 0

	2	1	0
2	$\frac{1}{3}$	0	$\frac{2\sqrt{2}}{3}$
1	0	1	0
0	$\frac{2\sqrt{2}}{3}$	0	$\frac{1}{3}$

IV. $S_q = \frac{3}{2}; \quad S_{\bar{q}} = \frac{1}{2}$

M = 2

	2
2	1

M = 1

	2	1
2	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
1	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

M = 0

	2	1
2	0	-1
1	-1	0

V. $S_q = 2$; $S_{\bar{q}} = \frac{1}{2}$

$M = \frac{5}{2}$

	$\frac{5}{2}$
$\frac{5}{2}$	1

$M = \frac{3}{2}$

	$\frac{5}{2}$	$\frac{3}{2}$
$\frac{5}{2}$	$\frac{3}{5}$	$-\frac{4}{5}$
$\frac{3}{2}$	$-\frac{4}{5}$	$-\frac{3}{5}$

$M = \frac{1}{2}$

	$\frac{5}{2}$	$\frac{3}{2}$
$\frac{5}{2}$	$\frac{1}{5}$	$-\frac{2\sqrt{6}}{5}$
$\frac{3}{2}$	$-\frac{2\sqrt{6}}{5}$	$-\frac{1}{5}$

VI. $S_q = \frac{3}{2}$; $S_{\bar{q}} = 1$

$M = \frac{5}{2}$

	$\frac{5}{2}$
$\frac{5}{2}$	1

$M = \frac{3}{2}$

	$\frac{5}{2}$	$\frac{3}{2}$
$\frac{5}{2}$	$\frac{1}{5}$	$-\frac{2\sqrt{6}}{5}$
$\frac{3}{2}$	$-\frac{2\sqrt{6}}{5}$	$-\frac{1}{5}$

$M = \frac{1}{2}$

	$\frac{5}{2}$	$\frac{3}{2}$	$\frac{1}{2}$
$\frac{5}{2}$	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{2\sqrt{5}}{5}$
$\frac{3}{2}$	$-\frac{2}{5}$	$\frac{13}{15}$	$\frac{2\sqrt{5}}{15}$
$\frac{1}{2}$	$\frac{2\sqrt{5}}{5}$	$\frac{2\sqrt{5}}{15}$	$\frac{1}{3}$

VII. $S_q = 2$; $S_{\bar{q}} = 1$

M = 3

$$\begin{array}{c|c|} & 3 \\ \hline 3 & 1 \\ \hline \end{array}$$

M = 2

$$\begin{array}{c|c|c|} & 3 & 2 \\ \hline 3 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ \hline 2 & -\frac{2\sqrt{2}}{3} & -\frac{1}{3} \\ \hline \end{array}$$

M = 1

$$\begin{array}{c|c|c|c|} & 3 & 2 & 1 \\ \hline 3 & -\frac{1}{15} & -\frac{4\sqrt{5}}{15} & \frac{4}{5} \\ \hline 2 & -\frac{4\sqrt{5}}{15} & \frac{2}{3} & \frac{\sqrt{5}}{5} \\ \hline 1 & \frac{4}{5} & \frac{\sqrt{5}}{5} & \frac{2}{5} \\ \hline \end{array}$$

M = 0

$$\begin{array}{c|c|c|c|} & 3 & 2 & 1 \\ \hline 3 & -\frac{1}{5} & 0 & \frac{2\sqrt{6}}{5} \\ \hline 2 & 0 & 1 & 0 \\ \hline 1 & \frac{2\sqrt{6}}{5} & 0 & \frac{1}{5} \\ \hline \end{array}$$