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# PARTICLE MASSES AND SU(6) SYMMETRY BREAKING*

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ABSTRACT

The transformation properties of the mass-splitting strong interaction are discussed in the framework of the SU(6)-scheme. Using the experimental mass values of the 56 baryons and the 35 mesons, the reduced matrix elements of all the possible mass tensor operators are calculated. It is found that the SU(3) symmetry breaking terms transform mainly as the 35 representation of SU(6) whereas the SU(3)-conserving, SU(6)-violating terms do not have simple transformation properties.

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# I. INTRODUCTION

It is well known that an exact determination of the transformation properties of the symmetry breaking interactions is crucial for understanding the relevance of any given approximate symmetry to physical phenomena. An "almost exact symmetry" like isotopic spin is usually capable of describing various properties of particles within a reasonable accuracy, and we do not have to worry about its symmetry breaking terms while dealing with ordinary strong interaction processes. However, when we discuss, for example, the SU(3) symmetry, we must always consider symmetry breaking contributions. These contributions may turn out to be of the same order of magnitude as the SU(3)-symmetric terms of the interaction. This is the case for the $\pi$-$\eta$ mass difference or for certain decays and scattering processes. The predictive power of the symmetry is, of course, reduced when non-symmetric terms are allowed but some results can still be obtained, using the well-known SU(3) transformation properties of the medium strong symmetry breaking interaction.[1] The knowledge of these transformation properties is mainly based on the great success of the Gell-Mann-Okulo mass formula[2] which is derived by neglecting all symmetry breaking contributions transforming like SU(3)-representations which are higher than the octet. Similar assumptions concerning the SU(3) behaviour of the electromagnetic[3] and weak interactions[4] led to additional results which were found to be in good agreement with experiment.

The introduction of the SU(6) symmetry[5] scheme led immediately to various suggestions of mass formulae for the known multiplets, and to other applications in which the SU(6) transformation properties of the strong electromagnetic and weak symmetry breaking interactions were involved.

It is our purpose in this paper to explore further the problem of mass formulae in the framework of SU(6), trying to arrive at definite conclusions concerning the SU(6) properties of the various symmetry breaking terms of the strong interactions.

It should be immediately noted that the significance of a given mass relation is closely related to the number of possible contributions to the symmetry breaking mechanism that are omitted from the final formula. It is generally true that all the masses of all states within a multiplet of a given group can be correctly expressed by a linear combination of all the allowed symmetry breaking terms for the given multiplet. A trivial example might illustrate this: Consider an SU(3) triplet of quarks and assume that the mass operator transforms like a linear combination of all $I_z = Y = 0$ (but not necessarily $I = 0$) components of all SU(3) representations. For the triplet, we find that the allowed contributions come from the SU(3) singlet and from the $I = 0$ and $I = 1$ members of the octet. We have three terms in the mass formula and three masses in our multiplet. Without additional assumptions the formula is trivially satisfied for any three mass values. Furthermore, if we introduce an exact isospin symmetry, we reduce the number of allowed terms in the formula (no $I = 1$ term) but also reduce the number of different masses in the SU(3) triplet. This property is a general property of all mass formulae in which the mass is expressed as the most general linear combinations of tensor operators with a given set of quantum numbers. What we do in such cases is simply to express one complete set of states in terms of another and the equality between the number of independent terms in the general mass formula and the number of independent mass values is a trivial consequence of the equal number of linearly independent states in any two equivalent set of states.

This leads us to the main idea of this work. The simplest way (in principle) of finding which terms in the most general mass formula can be omitted and which are needed, is to calculate the reduced matrix elements of all possible tensor operators for a given multiplet of particles, and then - to neglect those contributions which are consistent with zero or are extremely small with respect to the other terms. This simple method had been applied to many other physical problems involving complete sets of tensor operators. It was also applied to the problem of SU(3) mass formulae.[6] We feel that applying it to the analogous SU(6) problem could clarify the somewhat obscured situation with respect to mass formulae of SU(6) which is a consequence of incomplete "guesses" concerning the transformation properties of the mass operator according to this algebra. In Section II, we discuss some previous suggestions and describe the details of the method that we apply. The $\underline{56}$ baryons and the $\underline{35}$ mesons are discussed in detail in Sections III and IV. We analyze the results in Section V.

## II. METHOD OF CALCULATION

The simplest possible assumption concerning the transformation properties of the mass-splitting interaction is, naturally, to assume that it transforms like the $J = I = Y = 0$ member of a $\underline{35}$.[7] This is an operator which transforms like a component of an SU(3) - octet but as a singlet of $SU(4)_I$, the group generated by the matrices $\sigma_i \tau_j$, where $i,j = 0,1,2,3$; $\sigma_o = \tau_o = 1$; $\sigma_i$ are the Pauli spin matrices and $\tau_j$ are the isospin (Pauli) matrices. Consequently, all states within the same $SU(4)_I$ representation remain degenerate even when the symmetry breaking mechanism included contributions of this $\underline{35}$ - operator to all orders. The following mass equalities remain when only this kind of a symmetry breaking interaction is introduced:

$$K = K^*,\ \pi = \rho,\ \Lambda = \Sigma = Y^*,\ N = N^*,\ \Xi = \Xi^* \qquad (1)$$

It is clear that the symmetry breaking interaction must contain terms which transform like the components of higher representations of SU(6). For the mesons these terms may transform like the J = I = Y = 0 members of the $\underline{189}$ and $\underline{405}$ representations whereas for the baryons - the higher contributions are included in the $\underline{405}$ and $\underline{2695}$. Mass expressions for some of these terms were written down by Beg and Singh[8] who noticed that one of the meson terms (the SU(3) octet of the $\underline{189}$) can be omitted without getting into contradictions with experiment.

If we allow all SU(3) representations to contribute we have, for the baryons, eight terms which describe the eight masses of the N, Λ, Σ, Ξ, $N^*$, $Y^*$, $\Xi^*$, Ω. These terms can be denoted by $\underline{1}^1$, $\underline{35}^8$, $\underline{405}^1$, $\underline{405}^8$, $\underline{405}^{27}$, $\underline{2695}^8$, $\underline{2695}^{27}$, $\underline{2695}^{64}$.

If we assume that the $\underline{64}$ of SU(3) does not contribute, we remain with seven terms but only seven masses, as the Ω-mass is already fixed by:

$$\Omega = N^* + 3\,\Xi^* - 3Y^* \qquad (2)$$

If we further assume that the SU(3) $\underline{27}$ is negligible we remain with five SU(6) terms and five independent masses, say - N, Λ, Σ, $N^*$, $Y^*$ while Ξ and $\Xi^*$ are given by:

$$\Xi = \frac{1}{2}\,(3\Lambda + \Sigma - 2N) \qquad (3)$$

$$\Xi^* = 2Y^* - N^* \qquad (4)$$

Finally, if we assume that even the SU(3) - octet contribution is small (in the approximation where every SU(3) multiplet is degenerate) we remain with two SU(6) terms ($\underline{1}^1$ and $\underline{405}^1$) and two independent masses - a common octet mass and a common decuplet mass.

In every stage of this chain of approximations we are able to calculate the allowed reduced matrix elements of the mass-splitting. We will be able to say that the SU(6) transformation properties of the symmetry breaking interaction are simple and that a useful SU(6) mass formula is obtainable only if some of the SU(6) reduced matrix elements will turn out to be negligible with respect to the others. The simplest way of deciding this is to calculate explicitly all eight reduced matrix elements, and consider their relative magnitudes. We expect, of course, that the terms which transform like the $\underline{27}$ and $\underline{64}$ of SU(3) will be always small and we shall see whether some of the SU(3) - octet terms are also small. We include the $\underline{27}$ and $\underline{64}$ terms in the calculation in order to obtain some information on the relative magnitude of these terms (which are known to be very small from SU(3) considerations) and the SU(6) terms that we will try to neglect.

Note that in the case of the 56 baryons there are no two states with the same spin, isospin and hypercharge. Consequently, no mixing problems arise. This is not the case for the mesons, as the $\omega$ and $\varphi$ are mixed and a nonvanishing off-diagonal matrix element exists in the mass operator. The possible contributions for the meson masses may come from the following SU(6) tensor operators:

$$\underline{1}^1,\ \underline{35}^8,\ \underline{189}^1,\ \underline{189}^8,\ \underline{189}^{27},\ \underline{405}^1,\ \underline{405}^8,\ \underline{405}^{27}$$

Note that the product $\underset{\sim}{35} \times \underset{\sim}{35}$ contains three additional representations: A second $\underset{\sim}{35}$, a $\underline{280}$ and a $\overline{\underline{280}}$. However, these are antisymmetric with respect to the exchange of the two multiplied $\underset{\sim}{35}$'s and hence cannot contribute to the <u>meson</u> masses. The physical masses are those of the $\pi$, K, $\eta$, $\rho$, $K^*$, $\omega$, $\varphi$ and an $\omega \leftrightarrow \varphi$ "transition mass". These can be expressed in terms of our eight tensor operators. We cannot calculate all the reduced matrix elements without additional assumptions, as we do not know the experimental value of the $\omega \leftrightarrow \varphi$ term. We shall discuss this difficulty later and consider the different possibilities of defining the physical $\omega$ and $\varphi$ in a way which enables us to calculate the required reduced matrix elements.

The explicit calculation is carried out by applying the Wigner-Eckart theorem and using the SU(6) Clebsch-Gordan coefficients.[10] The reduced matrix elements are defined by:

$$(\lambda\mu_1 IY\sigma | M_{\lambda'}{}^{\mu'} | \lambda\mu_2 IY\sigma) = (\lambda || M_{\lambda'}{}^{\mu'} || \lambda) \begin{pmatrix} \lambda & \lambda' & \lambda \\ \mu_1\sigma & \mu' 0 & \mu_2\sigma \end{pmatrix} \qquad (5)$$

$\lambda'$, $\mu'$ are the dimensionalities of the SU(6) and SU(3) representations of the mass term, $\lambda$, I, Y and $\sigma$ are respectively, the SU(6) dimensionality, the isospin, the hypercharge and the spin of the considered states, $\mu_1$ and $\mu_2$ are their SU(3) representations (which are not necessarily identical) and $\begin{pmatrix} \lambda & \lambda' & \lambda \\ \mu_1\sigma & \mu' 0 & \mu_2\sigma \end{pmatrix}$ is the appropriate Clebsch-Gordan coefficient. In the two cases that we consider, no contribution appear twice and only one reduced matrix element is obtained for a given set of $(\lambda', \mu')$. We normalize all reduced matrix elements in the same way, guaranteeing that the SU(6) - singlet contribution will be equal to the average mass of the multiplet.

## III. THE BARYONS

Using the Wigner-Eckart theorem (Eq. (5)) we now express the masses of the eight baryons within the $\underline{56}$ in terms of the reduced matrix elements $B_1$, $B_2$, ..., $B_8$ which stand for the $\underline{1}^1$, $\underline{35}^8$, $\underline{405}^1$, $\underline{405}^8$, $\underline{405}^{27}$, $\underline{2695}^8$, $\underline{2695}^{27}$ and $\underline{2695}^{64}$ representations, respectively. The transformation matrix is given by:

$$\begin{pmatrix} N \\ \Lambda \\ \Sigma \\ \Xi \\ N^* \\ Y^* \\ \Xi^* \\ \Omega \end{pmatrix} = \begin{pmatrix} 1 & 1 & -5 & -7 & 3 & -2 & -21 & 0 \\ 1 & 0 & -5 & -4 & -9 & 6 & 63 & 0 \\ 1 & 0 & -5 & 4 & -1 & -6 & 7 & 0 \\ 1 & -1 & -5 & 3 & 3 & 8 & -21 & 0 \\ 1 & 1 & 2 & 1 & 6 & 1 & 3 & 1 \\ 1 & 0 & 2 & 0 & -10 & 0 & -5 & -4 \\ 1 & -1 & 2 & -1 & -6 & -1 & -3 & 6 \\ 1 & -2 & 2 & -2 & 18 & -2 & 9 & -4 \end{pmatrix} \begin{pmatrix} R_1 \\ \sqrt{\frac{7}{6}}\, B_2 \\ \frac{1}{\sqrt{10}}\, B_3 \\ \frac{\sqrt{14}}{10}\, B_4 \\ \frac{\sqrt{14}}{30}\, B_5 \\ \frac{\sqrt{21}}{15}\, B_6 \\ \frac{1}{15}\, B_7 \\ \frac{1}{\sqrt{10}}\, B_8 \end{pmatrix} \qquad (6)$$

We use the experimental masses given by Rosenfeld et al. and include the electromagnetic mass differences (where they are known) in the quoted experimental "errors." This is, of course, necessary in any procedure which assumes equal masses within any given isospin multiplet. The values which we use are:

| | |
|---|---|
| $N = 939 \pm 1$ | $N^* = 1236 \pm 2$ |
| $\Lambda = 1115$ | $Y^* = 1382 \pm 1$ |
| $\Sigma = 1193 \pm 4$ | $\Xi^* = 1529 \pm 1$ |
| $\Xi = 1318 \pm 3$ | $\Omega = 1675 \pm 3$ |

The information concerning electromagnetic mass differences between the different components of the $N^*$, $Y^*$ or $\Xi^*$ is extremely poor and we did not include it in the quoted "errors."

The results of the calculation are the following (in MeV):

$$\begin{aligned} B_1(\underline{1}^1) &= 1316 \pm 2 \\ B_2(\underline{35}^8) &= -142 \pm 1 \\ B_3(\underline{405}') &= 104 \pm 2 \\ B_4(\underline{405}^8) &= 23 \pm 1 \\ B_5(\underline{405}^{27}) &= 0 \pm 1 \\ B_6(\underline{2695}^8) &= -4 \pm 2 \\ B_7(\underline{2695}^{27}) &= 1 \pm 1 \\ B_8(\underline{2695}^{64}) &= 0 \pm 1 \end{aligned} \tag{8}$$

As expected, all 27 and 64 SU(3) contributions are consistent with zero. The SU(3) octet terms transform mainly like a 35 of SU(6), a smaller contribution being obtained from the 405 and a still smaller contribution from the 2695. Neglecting the 2695 we predict

$$\Xi^* - Y^* = \Xi - \Sigma \tag{9}$$

which is not too well satisfied (125 MeV versus 147 MeV). Neglecting the $\underline{405}^8$ we obtain the following formula:

$$m = a + bY + cJ(J + 1) \tag{10}$$

which implies:

$$\Sigma = \Lambda \tag{11}$$

$$N^* - N = Y^* - \Lambda = \Xi^* - \Xi \tag{12}$$

This can be used at most as a crude approximation which tells us that the masses are changing in both the octet and the decuplet with no strong isospin dependence and approximately with the same dependence on the hypercharge.

## IV. THE MESONS

We denote the $\underline{1}^1$, $\underline{35}^8$, $\underline{189}^1$, $\underline{189}^8$, $\underline{189}^{27}$, $\underline{405}^1$, $\underline{405}^8$ and $\underline{405}^{27}$ reduced matrix elements by $M_1, M_2, \ldots, M_8$, respectively and use them for expressing the eight "experimental" quantities $\pi$, K, $\eta$, $\rho$, $K^*$, $\omega_8$, $\omega_1$, X where $\omega_8$ and $\omega_1$ are defined as members of the SU(3)-octet and SU(3) singlet and X is the $\omega_1$-$\omega_8$ "transition" mass. Note that $\omega_8$ does not have to satisfy a Gell-Mann - Okubo relation with $K^*$ and $\rho$ as we allow mass terms which transform like a $\underline{27}$ of SU(3). We know that:

$$\omega_1 + \omega_8 = \omega + \varphi \tag{13}$$

$$\omega_1 \omega_8 - X^2 = \omega\varphi \tag{14}$$

This means that two of the three quantities $\omega_1$, $\omega_8$, X can be expressed in terms of the experimental values of $\omega$ and $\varphi$ and the value of the third quantity. The transformation matrix for the mesons is

$$\left|\begin{matrix}\pi\\ K\\ \eta\\ \rho\\ K^*\\ \omega_1\\ \omega_8\\ X\end{matrix}\right| \left|\begin{matrix} 1 & 2 & -9 & 18 & 1 & -9 & -18 & -3\\ 1 & -1 & -9 & -9 & -3 & -9 & 9 & 9\\ 1 & -2 & -9 & -18 & 9 & -9 & 18 & -27\\ 1 & 2 & 1 & 2 & -1 & 5 & 14 & -1\\ 1 & -1 & 1 & -1 & 3 & 5 & -7 & 3\\ 1 & 0 & 16 & 0 & 0 & -16 & 0 & 0\\ 1 & -2 & 1 & -2 & -9 & 5 & -14 & -9\\ 0 & 2\sqrt{2} & 0 & -10\sqrt{2} & 0 & 0 & -10\sqrt{2} & 0\end{matrix}\right| \left|\begin{matrix} M_1\\ \frac{\sqrt{70}}{16} M_2\\ \frac{\sqrt{14}}{24} M_3\\ \frac{\sqrt{7}}{24} M_4\\ \frac{\sqrt{42}}{24} M_5\\ \frac{\sqrt{10}}{24} M_6\\ \frac{\sqrt{14}}{48} M_7\\ \frac{\sqrt{14}}{24} M_8\end{matrix}\right| \qquad (15)$$

We use the following experimental $(\text{mass})^2$ values (in $(\text{BeV})^2$):

$$\begin{matrix} \pi = 0.019 \pm 0.0005 & \rho = 0.582 \pm 0.005 \\ K = 0.244 \pm 0.002 & K^* = 0.794 \pm 0.002 \\ \eta = 0.301 \pm 0.001 & \omega = 0.613 \pm 0.001 \\ & \varphi = 1.040 \pm 0.001 \end{matrix} \qquad (16)$$

There are various possibilities of defining $\omega_8$, $\omega_1$, and X. We can assume that $\omega_8$ is defined by a Gell-Mann - Okubo mass relation for the $\rho$, $K^*$, $\omega_8$ octet. This gives us two different solutions for X. If we analyze these two solutions we find that in one of them the physical $\varphi$ is almost a pure singlet of $SU(4)_I$ and the $\omega$ is almost purely in a $\underline{15}$ of $SU(4)_I$ while in the other solution the $\varphi$ is very close to a pure $\underline{15}$ and the $\omega$ is almost purely an $SU(4)_I$ singlet. In the first case the decay $\varphi \to \rho + \pi$ will be very small (and forbidden if $\varphi$ is a pure $SU(4)_I$ singlet[11]); in the second case $\omega \to \rho + \pi$ should be small. We naturally prefer the first possibility which is consistent with the experimental data. Experimentally $\varphi \to \rho\pi$ is of the order of 15% of the total width of the $\varphi$, but the appropriate matrix element is much smaller as the phase space factor tends to enhance the $\rho\pi$ and $3\pi$ modes over the dominant $K\bar{K}$ decay mode of the $\varphi$.

Alternatively we could use the small $\varphi \to \rho + \pi$ decay matrix element in order to say that the $\varphi$ is a pure $SU(4)_I$ singlet to a very good approximation and to use this as a definition of $\omega_1$, $\omega_8$ and X. In this case $\omega_8$ will not satisfy the GMO mass formula exactly but will be within 4% from the GMO prediction. This discrepancy is of the same order of magnitude as the discrepancy of the pseudoscalar meson masses, and both are very easily accounted for by a small contribution of an SU(3) $\underline{27}$ operator. We, hence, choose to define the physical $\varphi$ as a pure $SU(4)_I$ singlet. The physical $\omega$ is consequently a pure $\underline{15}$ of $SU(4)_I$ and the $\omega_1$ and $\omega_8$ masses satisfy:

$$\begin{aligned} \omega_1 &= \frac{2}{3}\omega + \frac{1}{3}\varphi = \phantom{-}0.755 \pm 0.001 \\ \omega_8 &= \frac{1}{3}\omega + \frac{2}{3}\varphi = \phantom{-}0.899 \pm 0.001 \qquad (17) \\ X &= \frac{2}{3}(\omega - \varphi) = -0.200 \pm 0.001 \end{aligned}$$

The reduced matrix elements which are obtained from (15) are:

$$\begin{aligned} M_1(\underline{1}^1) &= \phantom{-}0.602 \pm 0.002 & M_5(\underline{189}^{27}) &= -0.008 \pm 0.001 \\ M_2(\underline{35}^8) &= -0.139 \pm 0.002 & M_6(\underline{405}^1) &= \phantom{-}0.147 \pm 0.002 \\ M_3(\underline{189}^1) &= \phantom{-}0.186 \pm 0.001 & M_7(\underline{405}^8) &= -0.002 \pm 0.002 \\ M_4(\underline{189}^8) &= -0.001 \pm 0.001 & M_8(\underline{405}^{27}) &= -0.002 \pm 0.001 \end{aligned} \qquad (18)$$

The SU(3) 27 contributions are both small but are not consistent with zero, as expected. The octet symmetry breaking transforms like a pure 35 of SU(6), both the $189^8$ and $405^8$ being consistent with zero[12] (and smaller than the $189^{27}$!). The complete meson spectrum can be explained in terms of three symmetry breaking terms: $35^8$, $189^1$, and $405^1$. The general formula is:

$$m^2 = a + bJ(J + 1) + cC_2^{(3)} + d[2S(S + 1) - C_2^{(4)} + \frac{1}{4} Y^2] \tag{19}$$

and the obtained relations are:[11]

$$K^* - \rho = K - \pi \tag{20}$$

$$(\omega - \rho)(\varphi - \rho) = \frac{4}{3} (K^* - \rho)(\omega + \varphi - 2K^*) \tag{21}$$

## V. DISCUSSION

It is clear that no simple mass formula for SU(6) may be obtained, which describes the mass spectra of both the 56 and the 35 representations. However, apart from the expected small values of all SU(3) 27 and 64 contributions, we find some interesting regularities in the calculated reduced matrix elements. The SU(6) symmetry is broken in two steps. In the first stage SU(6) is broken without breaking SU(3). This is usually done by an interaction which has large contributions from all possible SU(3) singlets within the various SU(6) representations. It is found that the transformation properties of the symmetry breaking in this stage are not simple in any sense. However, when we investigate the SU(6) transformation properties of the SU(3)-symmetry breaking term we find in both cases that the 35 representation of SU(6) is dominant. For the mesons this dominance is very clear and all other contributions vanish. For the baryons the situation is not that clear and the contribution of the $\underline{405}^8$ is about 18% of the $\underline{35}^8$ term. In fact, the $\underline{35}$-dominance in the SU(3)-octet term is a good approximation to the extent that the $\Sigma$-$\Lambda$ mass difference can be neglected. Unfortunately, the simplest case in which we would like to apply this[13] is the well-known discrepancy between the exact SU(3) prediction and the experimental branching ratios of the $Y^*$ decays. It is obvious that the $\Sigma$-$\Lambda$ mass difference is playing an important role in this problem and our approximation is not valid. The application of this $\underline{35}$ dominance to other problems could be very interesting, provided that all other symmetry breaking effects [i.e., breaking of W-spin symmetry or the SU(6) symmetry breaking while reducing SU(6) to SU(3)] are proved to be irrelevant to the considered problem.

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