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UNSUBTRACTED DISPERSION RELATIONS AND THE RENORMALIZATION OF THE WEAK AXIAL-VECTOR COUPLING CONSTANTS*

by

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ABSTRACT

Assuming that the equal-time commutation rules for the vector and axial-vector current octets proposed by Gell-Mann are valid and that the divergence of the $\Delta S = 0$, $\Delta I = 1$ axial current is a strongly convergent operator obeying unsubtracted dispersion relations and dominated by low frequency contributions, we derive a sum rule for the renormalization of the neutron axial β-decay constant, G_A, by the strong interactions. The result agrees with that previously obtained from the assumption that the axial-current divergence is proportional to the pion field. The results are generalized to the strangeness-changing leptonic decays in the context of Cabibbo theory and generalized Goldberger-Treiman relations and used to compute the d/f ratio for the weak baryon-axial current coupling and an independent value of G_A.

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I. INTRODUCTION

Recent calculations of the effects of the strong interactions in renormalizing the axial-vector coupling constant in β-decay,[1,2] $g_A = G_A/G_V$, give good agreement with the experimental value. These results were derived from the following three assumptions.

1. The equal-time commutators of the spatial integrals of the time components of the hadron currents measured to first order in the weak and electromagnetic interactions, the "charges," obey the algebra of $SU(3) \times SU(3)$ as postulated by Gell-Mann et al.[3]

2. The effective Hamiltonian for leptonic decay of the hadrons is a current-current interaction which couples the appropriate members vector and axial-vector current octets of the strongly interacting particles to the usual $\gamma_\mu(1-\gamma_5)$ current of the leptons through the simple combination $V_\mu \pm A_\mu$.[4]

3. Partially Conserved Axial Current (PCAC) hypothesis. The divergence of the $\Delta S = 0$ axial-vector current is proportional to the pion field.[5-8]

$$\partial^\alpha A^i_\alpha(x) = -\, i\, \sqrt{2}\, \mu^2\, M g_A/g_{\pi n}\; \varphi^i_\pi(x), \quad i = 1,2,3. \tag{I.1}$$

where $\varphi^i_\pi(x)$ is the renormalized Heisenberg field of the π-mesons, μ is the pion mass, M is the nucleon mass, $g_{\pi n}$ is the rationalized renormalized π-nucleon coupling constant.

In this article, we derive the sum rule for g_A [Eq. (II.14)], from a more general form of PCAC analogous to that used by Bernstein et al.[9] to derive the Goldberger-Treiman relation. We assume that the divergence of the axial current is a highly convergent operator whose matrix elements satisfy unsubtracted dispersion relations in the four-momentum transfer squared, q^2. For small q^2 and certain values at the other variables in the problem, these matrix elements may be dominated by nearby poles.

These notions will be made more precise in the theoretical development of Section II where we treat the problem of formulating an unambiguous definition and region of validity for pole dominance of matrix elements of the axial current divergence when these matrix elements are functions of more than one invariant variable. In Section III the results are generalized to include the $\Delta S = 1$ leptonic decays in the context of Cabibbo theory[10] and generalized Goldberger-Treiman[11] relations. The numerical evaluation of the sum rules is discussed in Section IV. The results give $|g_A| \simeq 1.2$, and a d/f ratio similar to other estimates. While there are considerable numerical uncertainties in the evaluation of the sum rule for $\Delta S = 1$ decays, the general consistency with Cabibbo theory is good and is strong evidence against the explanation of the suppression of $\Delta S = 1$ decays relative to $\Delta S = 0$ decays as a strong interaction renormalization effect.

II. THEORETICAL DEVELOPMENT FOR $\Delta S = 0$ DECAYS

As a starting point we consider a matrix element of the time-ordered product of two components of the axial-vector current between one-proton states of equal momentum

$$R_{\alpha\beta} = \int d^4x \, e^{iq.x} < P \left| T \left(A^+_\alpha(x) \; A^-_\beta(0) \right) \right| P > \tag{II.1}$$

with $A^{\pm}_\alpha = A^1_\alpha \pm i \, A^2_\alpha$.

A^i, $i = 1,2,3$, are the isovector members of the octet of axial-vector currents. The tensor, $R_{\alpha\beta}$, is related to second-order forward scattering of a proton by an axial-vector field. From general invariance arguments, $R_{\alpha\beta}$ can be written as a sum of kinematic second-rank tensors formed from combinations of p, q, and the γ-matrices evaluated between Dirac spinors, each multiplied by appropriate normalization factors and a Lorentz-invariant scalar function. In the usual manner, the arguments of the scalar functions are chosen as the invariant variables in the problem, which in this case are

$$p^2 = M^2$$

$$q^2$$

$$p \cdot q = M\nu$$

or some linear combination of these three. ν can be considered as the energy of the particle incident on the proton in the rest system of the proton, the "laboratory system."

From Eq. (II.1) we obtain

$$q^\alpha R_{\alpha\beta}(q^2,\nu) = i\int d^4x\, e^{iq\cdot x}\left[< P \left| T\left(\partial^\alpha A^+_\alpha(x)\, A^-_\beta(0)\right)\right| P > + \delta(x_o) < P \left| \left[A^+_o(x),\, A^-_\beta(0)\right]\right| P >\right] \tag{II.2}$$

and

$$\begin{aligned} q^\alpha q^\beta R_{\alpha\beta} = \int d^4x\, e^{-iq\cdot x} \Big[& < P \left| T\left(\partial^\alpha A^+_\alpha(0)\, \partial^\beta A^-_\beta(x)\right)\right| P > \\ & - \delta(x_o) < P \left| \left[\partial^\alpha A^+_\alpha(0),\, A^-_o(x)\right]\right| P > \\ & + \delta(x_o)\, iq^\beta < P \left| \left[A^+_o(0),\, A^-_\beta(x)\right]\right| P > \Big] \end{aligned} \tag{II.3}$$

We have integrated by parts to cast Eq. (II.3) in the given form. Eq. (II.3) is the basic equation for deriving our results. The sum rule is obtained from Eq. (II.3) as a low energy theorem[12] in the limit $q^2 \to 0$, $\nu \to 0$. We proceed to evaluate the terms in Eq. (II.3) up to first order in ν. For fixed space-like or light-like q^2, the invariant functions in the decomposition of $R_{\alpha\beta}$ can be shown from the axioms of local field theory[13] to satisfy dispersion relations in ν. For $\nu \simeq 0$ the only singular term as $q^2 \to 0$ is the one-neutron pole at $q^2 + 2M\nu = 0$. That is, the contribution to $R_{\alpha\beta}(\nu \simeq 0,\ q^2 = 0)$ from the cuts is finite in this limit. Therefore, if we consider $q^\alpha q^\beta R_{\alpha\beta}$ and take the lim $q^\alpha \to 0$, the cut contributions are at least of second order, and the finite and first order terms on the left side of Eq. (II.3) come entirely from the one-neutron Born term.

This Born term will give a factor g_A^2. On the right side of Eq. (II.3), the term involving the time-ordered product of the axial current divergences will be related to the forward π-p scattering amplitude on the mass shell via analyticity in q^2. The assumed equal-time commutation rules determine the last term on the right. The combination of these various factors leads finally from Eq. (II.3) to a sum rule for g_A^2, Eq. (II.14).

In deriving Eq. (II.3) we have integrated by parts with respect to space and time variables and discarded surface terms. The spatial surface terms give no contribution if we use wave packets. The temporal surface terms at $t = \pm\infty$ vanish in the same manner if all the intermediate states inserted in our expressions lead to oscillating time behavior, that is, if all intermediate states have different energy from the one-proton state.[14] For $q^2 = 0$, the only dangerous term comes from the one-neutron intermediate state; in our calculation we shall explicitly assume the neutron mass, M_n, to be different from the proton mass, M_p. In final result we let $M_n = M_p$ and assume charge independence; the answer is insensitive to the order in which we let the various small quantities in the problem tend to zero. This procedure of keeping $M_n \neq M_p$ until the end of the calculation will have the additional advantage of allowing the derivation to be generalized immediately to renormalization of the strangeness changing decays, (Sec. III), where the Born terms involve nucleon-hyperon transitions and the masses are manifestly unequal.

For reference we note that the matrix element of the axial-vector current between proton and neutron is given by

$$\begin{aligned} < P(p_1)\left|A_\alpha^+(x)\right| N(p_2) > &= (2\pi)^{-3}\left[M_N M_p/(E_n E_p)\right]^{\frac{1}{2}} \\ & e^{iq.x} g_A \bar{u}_p(p_1)\left[\gamma_\alpha\gamma_5 F_1(q^2) - q_\alpha\gamma_5 F_2(q^2)\right] \tau^+ u_n(p_2). \end{aligned} \tag{II.4}$$

$q_1^\alpha = p_1^\alpha - p_2^\alpha$; $F_1(0) = 1$.

$\tau^+ = \frac{1}{2}(\tau_1 + i\tau_2)$ is a nucleon isotopic spin matrix.

If the effective Hamiltonian has V - A coupling, then g_A equals G_A/G_V, the ratio of axial-vector to vector coupling constants measured in ordinary β-decay. From Eq. (II.4)

$$< P\left|\partial^\alpha A_\alpha^+\right| N > = (2\pi)^{-3}\left[M_n M_p/(E_n E_p)\right]^{\frac{1}{2}} e^{iq.x} g_A D(q^2)\bar{u}_p(p_1)\gamma_5\tau^+ u_n(p_2). \tag{II.5}$$

$$D(q^2) = (M_n + M_p) F_1(q^2) - q^2 F_2(q^2) .$$

The assumption that $D(q^2)$ obeys an unsubtracted dispersion relation and that $D(0)$ is dominated by the one-pion pole at $q^2 = \mu^2$ leads to a derivation of the Goldberger-Treiman, (G-T), relation,

$$f_\pi = -\sqrt{2}\, g_A M/g_{\pi n} \; . \tag{II.6}$$

f_π is the decay constant of the charged pion defined by

$$< 0 \left| \partial^\alpha A_\alpha^+(0) \right| \pi^- > = -(2\pi)^{-3/2}(2E_\pi)^{-\frac{1}{2}} \mu^2 f_\pi \; . \tag{II.7}$$

With these definitions the Born contribution to $R_{\alpha\beta}$ can be evaluated as

$$\begin{aligned} q^\alpha q^\beta R_{\alpha\beta}^{\text{Born}} = N_p g_A^2 \Big[&(M_N + M_p + \nu) F_1^2(q^2) - 2F_1(q^2) \\ &+ D^2(q^2)(M_p - M_N + \nu)/(q^2 + M_p^2 - M_N^2 + 2M_p \nu) \Big] \; . \end{aligned} \tag{II.8}$$

$N_p = (2\pi)^{-3} M_p/E_p$.

The last term on the right-hand side of Eq. (II.3) is determined from the assumed equal-time commutation rules:

$$\begin{aligned} \delta(x_o)[A_o^+(0),\; A_\beta^-(x)] &= 2V_\beta^3(x)\delta^{(4)}(x) \\ &+ (\text{more singular terms}). \end{aligned} \tag{II.9}$$

V_β^3 is the third component of the total isotopic spin current. We generalize the SU(3) $\times$ SU(3) algebra to include commutators of time-components of currents with space-components.

The more singular terms of the equal-time commutator involve derivatives of delta functions.[15] In the integral of Eq. (II.3a) these terms give polynomials in q. Since the results of interest will be obtained in the $\lim q\alpha \to 0$, the derivatives of delta functions do not contribute in this calculation. From the delta function term in Eq. (II.9) one has

$$\int d^4x\, e^{iq\cdot x} \delta(x_o) i q^\beta < p \left| \left[A_o^+(0),\; A_\beta^-(x) \right] \right| p > = i N_p \nu \; . \tag{II.10}$$

Returning to Eq. (II.3) we have still to evaluate the first two terms on the right side. The equal-time commutator, $\left[\partial^{\alpha}A_{\alpha}^{+}(0),\ A_{\beta}^{-}(\vec{x},o)\right]$, is presumably proportional to $\delta^{(3)}(\vec{x})$. This leads to a finite q-independent term in Eq. (III.3a),

$$C = \int d^4x e^{-iq\cdot x}\delta(x_o) < P\left|\left[\partial^{\alpha}A_{\alpha}^{+}(o),\ A_o^{-}(x)\right]\right|P > . \qquad \text{(II.11a)}$$

Let the first term on the left side of (II.3) be denoted by

$$R(q^2,\nu) = \int d^4x e^{-iq\cdot x} < P\left|T\left(\partial^{\alpha}A_{\alpha}^{+}(o),\ \partial^{\beta}A_{\beta}^{-}(x)\right)\right|P > . \qquad \text{(II.11b)}$$

It is straightforward to show that $R(0,0) = C$. Thus, after evaluating $R(q,\nu)$, we need keep only terms proportional to ν. Since R involves matrix elements of the divergence of the axial current, we assume that for fixed ν, R satisfies an unsubtracted dispersion relation in q^2. For $\nu \simeq 0$, $q^2 = 0$, we assume that R is dominated by nearby singularities. These are the one-neutron Born pole at $q^2 + M_p^2 - M_N^2 + 2M_p\nu = 0$ and the one-pion poles at $q^2 = \mu^2$.

There is, however, a possible ambiguity in defining the residues of the poles.[16] In this problem, the independent variables may be taken as q^2 and $\sigma = \nu + aq^2$, and we can disperse in q^2 with $\sigma = 0$. As we vary the constant, "a", different parts of the total dispersion relation for $R(0,0)$ are associated with the residues of the poles and the integral over the continuum. The problem is to choose "a" to give the best pole approximation, to put as much as possible of the contribution to $R(0,0)$ into the nucleon and pion poles and make the corrections due to the integral over the branch cut, which will be neglected, as small as possible.

In the context of dominance by nearby singularities there is a natural, if somewhat arbitrary, criterion for a best pole approximation, namely, choose "a" to keep the threshold of the cut as far from the poles as possible. The locations of the singularities in the Real q^2 - Real ν plane which follow from perturbation theory are plotted in Fig. 1. For any fixed ν, R satisfies a dispersion relation in q^2. For q^2 fixed and not too timelike, R should obey a dispersion relation in ν with singularities on the Real ν-axis. The anomalous thresholds come from the dispersion-perturbative diagram shown in Fig. 2. From Fig. 1, it is seen that the criterion given above leads to the value $a = 0$, or $\sigma = \nu = 0$, as the best choice of the fixed second variable for writing a pole-dominated dispersion relation for $R(0,0)$. For $\nu = 0$, the cut has an anomalous threshold at $q^2 \simeq 8\mu^2$.

The choice of $\nu = 0$, $(a = 0)$, can be justified also by general symmetry arguments. The thresholds are determined by the masses of intermediate states in the s and u channels, where s, t, u are the usual Mandelstam variables. Here $t = 0$, so s and u are related to q^2 and ν by $\binom{s}{u} = M^2 + q^2 \pm 2M\nu$. For the purpose of specifying intermediate states in R, both s and u channels look like π-nucleon scattering and have the same intermediate states available. For a particular choice of "a", denote the residue of the pion pole by $\overline{R}(\nu = -a\mu^2/M, \mu^2)$. It follows from the statements above that $\overline{R}$ is an even function of a. To retain the symmetry between the s and u channels one should disperse in q^2 with $a = 0$.

For fixed $q^2 \simeq 0$, $R(q^2,\nu)$, which resembles a forward scattering amplitude, should satisfy a dispersion relation in ν, and we can separate R into contributions from the Born and continuum terms of the

ν-dispersion relation. Thus, for small q^2, ν

$$R(q^2,\nu) = iN_p \left[g_A^2 D^2(q^2)(M_p - M_n + \nu)/(q^2 + M_p^2 - M_n^2 + 2M_p\nu) + \tilde{R}(q^2,\nu)\right]. \tag{II.12}$$

This Born term cancels the singular term of $q^\alpha q^\beta R^{Born}_{\alpha\beta}$, Eq. (II.8), and clearly satisfies an unsubtracted dispersion relation in q^2. Therefore, $\tilde{R}$ has no one-neutron pole and must itself obey an unsubtracted dispersion relation in q^2. $\tilde{R}$ has double and single one-pion poles at $q^2 = \mu^2$ and a cut starting at $q^2 \simeq 8\mu^2$. In the spirit of our approach, the pole contributions dominate for $q^2 = 0$ and the integral over the branch cut is neglected. In the same manner it will be shown that the single pole contributions are small. The result from keeping only the double pion pole term is

$$\tilde{R}(0,\nu) = -f_\pi^2\, \tilde{T}_{\pi^- p}(\mu^2,\nu) \tag{II.13}$$

where $\tilde{T}_{\pi^- p}(\mu^2,\nu)$ is the invariant forward π-proton scattering amplitude on the mass shell and with the Born terms subtracted. From the usual dispersion relations[17] for the forward π-nucleon scattering amplitude,

$$\tilde{R}(0,\nu) = -f_\pi^2/\pi \int_\mu^\infty d\nu' \left[A_{\pi^- p}(\mu^2,\nu')/\nu' - \nu) + A_{\pi^- p}(\mu^2,-\nu')/\nu' + \nu)\right]. \tag{II.13a}$$

From unitarity and crossing symmetry,

$$A_{\pi^- p}(\nu) = \mathrm{Im}T_{\pi^- p}(\nu) = k\sigma_{\pi^- p}(\nu)$$

$$A_{\pi^- p}(-\nu) = k\sigma_{\pi^+ p}(\nu), \quad \nu > \mu\,,$$

where the σ's are total cross sections and k is the magnitude of the pion three-momentum in the laboratory system.

Recalling that we are interested in equating the terms $O(\nu)$ as $\nu \to 0$ in Eq. (II.3) we collect the results from Eqs. (II.8), (II.10) and (II.13a). Substituting we obtain the sum rule

$$g_A^2 = 1 - f_\pi^2/\pi \int_\mu^\infty \frac{k d\nu}{\nu^2} \left[\sigma_{\pi^- p}(\nu) - \sigma_{\pi^+ p}(\nu)\right] \qquad \text{(II.14)}$$

If we eliminate f_π by the G-T relation, we can determine g_A from strong interaction cross-sections only

$$1/g_A^2 = 1 + \frac{2M_n^2}{\pi g_{\pi n}^2} \int_\mu^\infty \frac{k d\nu}{\nu^2} \left[\sigma_{\pi^- p}(\nu) - \sigma_{\pi^+ p}(\nu)\right] . \qquad \text{(II.14a)}$$

We discuss the neglected single pion-pole terms. These can be written as

$$\tilde{R}(o,o)_{s.p.p.} = f_\pi r(\mu^2,0) + h.c. \qquad \text{(II.15)}$$

As an analytic function of q^2, r has no pion-pole

$$r(\mu^2,0) = \frac{1}{\pi} \int_{\sigma_o^2}^\infty \frac{\mathrm{Im}\, r(\sigma^2,o)}{\sigma^2 - \mu^2} d\sigma^2 .$$

$$\text{(II.16)}$$

$$\mathrm{Im}\, r \propto \sum_{m \neq \pi} < P\pi \left| j_P \right| m > < m \left| \partial^\alpha A_\alpha \right| 0 >$$

If we considered the matrix element for forward creation of a pion from a nucleon by scattering of the axial-current divergence, pole dominance at $q^2 = 0$ (off-mass-shell pions) would imply

$$\left| f_\pi T_{\pi N}(\mu^2,0) \right| \gg \left| r(0,0) \right| .$$

Since $\sigma_o^2 \sim 8\mu^2$, $r(\mu^2,0) \simeq r(0,0)$ and $r(\mu^2,0)$ is similarly unimportant compared to $f_\pi T_{\pi N}(\mu^2,0)$.

III. SUM RULES FOR $\Delta S = 1$ DECAYS

The results of the preceeding section can be applied to the $\Delta S = 1$ decays in the context of the Cabibbo theory of weak interactions if one accepts the generalization of the G-T relation to K-meson pole dominance for the divergence of the strangeness-changing axial currents. We briefly review the Cabibbo theory of leptonic decays.

The $SU(3) \times SU(3)$ commutation rules fix the relative scale of the vector and axial vector currents. The combinations

$$Q^i_{\pm} = \int \frac{1}{2} \left(V^i_o(x) \pm A^i_o(x)\right) d^3x, \quad i = 1,\ldots,8,$$

form two mutually commuting octets of chiral changes. The hadron current which couples to the leptons and is measured in decay processes, is a component of one of these chiral octets

$$J^{had}_{\mu} = \cos\theta \left(V^{1+i2}_{\mu} - A^{1+i2}_{\mu}\right) + \sin\theta \left(V^{4+i5}_{\mu} - A^{4+i5}_{\mu}\right) \tag{III.1}$$

The Cabibbo angle, θ , which determines the suppression of the $\Delta S = 1$ decays relative to the $\Delta S = 0$ decays is an input parameter to the structure of the effective weak Hamiltonian. The problem of whether the right-handed or left-handed current appears is determined from experiment. In the limit of exact $SU(3)$ symmetry the vector currents are unrenormalized, and their matrix elements between one baryon states have only f-type coupling. For the corresponding matrix elements of the axial current, we have in the $SU(3)$ limit

$$< B^i(p)\left|A^k_{\mu}(0)\right|B^j(p)> = g_A^{B_i B_j}\, \bar{u}(p)\gamma_{\mu}\gamma_5 u(p) = g_A\left[(1-\alpha)f_{ijk} + \alpha d_{ijk}\right]\bar{u}(p)\gamma_{\mu}\gamma_5 u(p).$$

$$i,j,k = 1,\ \ldots,\ 8. \tag{III.2}$$

where f and d are the usual Gell-Mann coupling coefficients, and we have neglected trivial normalization factors.

Empirically[11,18] this description gives a satisfactory fit to the presently available data on leptonic decays even though SU(3) is a badly broken symmetry. The origin of the lack of renormalization for the vector currents is suggested by the theorem of Ademollo and Gatto,[19] which shows that there is no renormalization of the vector currents to first order in the symmetry breaking because the space integrals of the time components of the vector currents are the generators of SU(3). There is no such theorem for the axial-vector current.

However, if we believe that the commutation rules of the vector and axial-vector currents are unchanged by the SU(3) breaking interactions, then the axial currents transform exactly as an irreducible octet tensor even in the presence of symmetry breaking. The experimental success of Eq. (III.2) in describing the axial matrix elements suggests then that the one-particle states may be nearly pure octet despite the large mass splitting due to SU(3) breaking.[20]

The divergence of the axial current, however, does not transform like a pure octet tensor if SU(3) is broken. Indeed, the Cabibbo theory gives explicit SU(3) violation of the matrix elements of $\partial^{\alpha} A_{\alpha}^{i}$ due to the mass splittings. The generalization of the G-T relations to the strangeness changing decays implies that the meson-baryon couplings have the same d/f ratio as the axial current-baryon vertex but the meson couplings show explicit dependence on the physical baryon masses. The results are the same as those of Freund and Nambu[21] who assumed that the currents are conserved but the states are not pure.

To check the consistency of this picture we obtain sum rules for the $\Delta S = 1$ decays. The procedure is exactly the same as in Section II. Start with matrix elements of $T(A_\alpha^{4+i5}(x)\, A_\beta^{4-i5}(0))$. Consider matrix elements of this time ordered product between both one-neutron states and one-proton states respectively. In the first case the Born terms are due to a Σ^- pole. In the second both Σ^0 and Λ^0 contribute. These give the sum rules[22]

$$1 = (g_A^{n\Sigma^-})^2 + \frac{f_K^2}{\pi} \int_{\nu_0}^{\infty} \frac{d\nu}{\nu^2} \left[A_{K^-n}(\nu) - A_{K^+n}(\nu)\right] \qquad \text{(III.3a)}$$

$$2 = (g_A^{p\Sigma^0})^2 + (g_A^{p\Lambda^0})^2 + \frac{f_K^2}{\pi} \int_{\nu_0}^{\infty} \frac{d\nu}{\nu^2} \left[A_{K^-n}(\nu) - A_{K^+p}(\nu)\right] \qquad \text{(III.3b)}$$

The A's are absorptive parts of forward scattering amplitudes. For $\nu > M_K$ they are proportional to total cross-sections, but the K-nucleon dispersion relations have cuts in unphysical region due to the hyperon-pion channels. f_K is a K-meson decay constant defined in analogy to Eq. (II.7). In Cabibbo theory, $f_K = f_\mu$, and the g_A's are given by Eq. (III.2). Making these substitutions in Eq. (III.3) and using G-T one obtains

$$1/g_A^2 = (1-2\alpha)^2 + \frac{2M_N^2}{\pi g_{\pi n}^{\ 2}} \int_{\nu_0}^{\infty} \frac{d\nu}{\nu^2} \left[A_{K^-n}(\nu) - A_{K^+n}(\nu)\right] \qquad \text{(III.4)}$$

$$1/g_A^2 = \left(1-2\alpha + (4/3)\alpha^2\right) + \frac{M_n^2}{\pi g_{\pi n}^{\ 2}} \int_{\nu_0}^{\infty} \frac{d\nu}{\nu^2} \left[A_{K^-p}(\nu) - A_{K^+p}(\nu)\right] \qquad \text{(III.5)}$$

Evaluating the dispersion integrals in Eq. (III.4), one determines c_A, and α or the d/f ratio. By considering also the commutator of the $\Delta S = 1$, $\Delta Q = 0$ axial currents and taking all possible diagonal matrix elements of the three canonical commutators between baryon states, one can derive six more sum rules. They involve unmeasurable scattering processes, but in the limit of exact SU(3), they can each be shown to be equivalent to Eqs. (II.13), (III.3a) or (III.3b).

IV. NUMBERICAL RESULTS AND CONCLUSIONS

A. $\Delta S = 0$ Axial Current

The sum rule for the $\Delta S = 0$ axial-vector [Eq. (II.14a)] is evaluated[23] using tabulated values[24] of the experimental pion-nucleon cross sections to $k = 5$ BeV/c. The very high-energy data is fitted with the exponential form[25]

$$\sigma_{\pi^+p}(\nu) - \sigma_{\pi^-p}(\nu) = b\nu^{-0.7} \qquad \text{(IV.1)}$$

The convergence of the integral in Eq. (II.14a) depends on the validity of the Pomeranchuk theorem but the numerical result is insensitive to the details of the high-energy behavior. The result is

$$1 - 1/g_A^2 = 0.246 \qquad \text{(IV.2a)}$$

or

$$|g_A| = |G_A/G_V| = 1.15 \qquad \text{(IV.2b)}$$

The best value calculated from experimental β-decay measurements is[25]

$$(G_A/G_V)_{\text{exp}} = -1.18 \pm 0.02 \,. \qquad \text{(IV.3)}$$

The theoretical uncertainties in Eq. (II.14) are due mainly to the continuum terms that have been discarded. This approximation is used for

$\tilde{R}$, Eq. (II.13), and in deriving the Goldberger-Treiman relation. From the comparison of G-T with experiments, the errors inherent in this type of approximation may be about 20% for the right-hand side of Eq. (III.2a) but only about 5% for g_A. For example, if we evaluate Eq. (II.14) using f_π as determined from experimental measurements of the lifetime of the charged pion,[27] the sum rule yields

$$|g_A| = 1.21 \qquad \text{(IV.2b')}$$

Errors in the calculated value of g_A from uncertainties in the exact high-energy behavior of π-proton cross sections and the best experimental value of $g_{\pi N}$ are about 1%. It is the effect of the (3,3) resonance in Eq. (II.13) that makes $|g_a| > 1$. In fact, the (3,3) resonance contribution alone gives $|G_A/G_V| \sim 1.35$, and the higher energy $I = 1/2$ resonances reduce this value. Thus, the (3,3) resonance does not saturate the sum rule.

From Fig. 1 some general conclusions can be drawn about the expected domain of validity of pion pole dominance in this problem. From the postulated equal-time commutation rules for the weak charges, one can obtain directly[1,14] a sum rule for g_a

$$g_a^2 = 1 + \int_{\mu+\mu^2/2M}^{\infty} \frac{d\nu}{\nu^2}\,[D^+(0,\nu) - D^-(0,\nu)] \qquad \text{(IV.4)}$$

with

$$D^{\pm}(0,\nu) = \operatorname{Im} R\,(0,\pm\nu), \;\; \nu > \mu + \mu^2/2M\,. \qquad \text{(IV.5)}$$

$D^{\pm}$ can be measured in high-energy neutrino reactions when the lepton is produced in the forward direction.[28]

This result is obtained by writing a disperion relation in ν for $R(0,0)$ in Eq. (II.3). The result, Eq. (II.14), is obtained by first taking the pole approximation in q^2 for $R(0,0)$ and then dispersing the residue of the pion pole in ν. For the integrand in Eq. (IV.4), however, one cannot justify directly replacing the matrix element of $D^{\pm}$ by their pion pole contributions.

This follows from Fig. 1 where it is seen that in the integration region for ν of Eqs. (II.14a) and (IV.4), the threshold of the cut in q^2 moves past the one-pion pole. That is, for physical ν there is no isolated pion pole, and multi-particle thresholds in q^2 are as close to $q^2 = 0$ as the one-pion state.[29]

Nevertheless, as $\nu \to \pm\infty$, a return to pion-pole dominance can be justified for $D^{\pm}(0,\nu)$. The position of the singularity from a state of invariant mass, M_j^2, in the s or u channel is given by

$$q^2 \pm 2M\nu = M_j^2 - M^2 .$$

Thus, as $|\nu| \to \infty$, the threshold for any state of finite mass moves off to $q^2 = \pm\infty$. The only singularities remaining near $q^2 = 0$ are the one-pion pole and very heavy inelastic states. The normal threshold at $q^2 = 9\mu^2$ remains fixed. In the spirit of PCAC, with $\partial^\mu A_\mu$ assumed to be a highly convergent operator satisfying unsubtracted dispersion relations one expects that very heavy states are unimportant in the spectral functions for $\partial^\mu A_\mu$. Therefore, as $|\nu| \to \infty$, the only important contribution near $q^2 = 0$ comes from the pion pole at $q^2 = \mu^2$. This leads to a derivation of Adler's proposed tests of PCAC in high-energy neutrino reactions.[23] The preceding argument explicitly uses the PCAC hypothesis in dispersion theory as a physical assumption about small numerators as well as a geometrical statement about large denominators or far-away singularities. Faith in such arguments in needed also to justify K-meson pole dominance of the divergence of $\Delta S = 1$ axial current.

B. Strangeness-Changing Currents

The numerical evaluation of the sum rules for the $\Delta S = 1$ currents, Eq. (III.4), is complicated by the presence of an unphysical region in the $\overline{K}$-nucleon channel extending below the elastic threshold. Above the threshold the integral can be expressed in terms of total cross sections as in the π-nucleon case. The sum rules can be written as

$$1/g_A^2 = (1-2\alpha)^2 + 2I(Kn) \tag{IV.6a}$$

$$1/g_A^2 = (1 - 2\alpha + (4/3)\alpha^2) + I(Kp) \tag{IV.6b}$$

The contributions of the different energy regions to the dispersion integrals for $I(Kn)$ and $I(Kp)$ are summarized in Table I and discussed below.

(a) Unphysical region. $\nu < M_K$

We assume that the only important contributions come from the $I = 1$ p-wave resonance $Y_1^*(1385)$, and the continuation of the $I = 0,1$ S-waves below threshold. The Y_1^* is a member of the decuplet of spin 3/2 resonances. To estimate the Y_1^* contribution to the K-nucleon integrals we assume a phenomenological B^*BM (resonance-baryon-meson), coupling

$$\mathcal{L}_{eff}(x) = - \lambda \overline{\psi}_\mu(x) \psi(x)\partial^\mu\varphi(x) \tag{IV.7}$$

where the $\psi_\mu(x)$, $\psi(x)$, $\varphi(x)$ are the field operators for the spin 3/2 resonance, spin 1/2 baryon and pseudoscalar mesons respectively. The resonance is described by the Rarita-Schwinger formalism.[30]

The decay width for a resonance is related to the effective coupling constant, λ, by

$$\Gamma = (k^3\lambda^2/24\pi)\ \left[(M_B^* + M_B)^2 - \mu_M^2\right] / M_{B^*}^2 \qquad \text{(IV.8)}$$

where k is the momenta of the baryon and meson in the center-of-mass system. M_B^*, M_B and μ_M are the masses of the resonance baryon and meson respectively. We assume that the coupling constants, λ, for all the B^*BM couplings are related by SU(3). The $Y_1^*N\bar{K}$ coupling is computed from the observed width[27] for $\Delta(1235) \to N + \pi$. Then the Y_1^* is inserted as a pole in the $\bar{K}N$ scattering amplitudes. Various estimates[31] of the effects of SU(3) breaking on the B^*BM couplings indicate that the Y_1^* contribution is uncertain within a factor of 2.

To evaluate the $I = 0$ and $I = 1$ the S-wave contribution below threshold we use the complex-scattering length, zero-effective range K-matrix formalism of Dalitz and Tuan.[32] Recent experiments on low energy K^- proton scattering by Kim and Sakitt et al.[33] give very similar solutions for the two complex scattering lengths. Each of their solutions shows resonance behavior below threshold in the $I = 0$ channel at the mass of the $Y_0^*(1405)$.

Both sets of scattering lengths were used to evaluate the S-wave contributions to the dispersion integrals below threshold. The integrals were truncated at the Σ-π threshold.

(b) Physical region $\nu > M_K$.

In the physical region for KN and $\overline{K}N$ scattering, the integrands in the dispersion integrals can be expressed in terms of total cross sections. For low energies, $P_K^{lab} < 0.3$ BeV/c, we use the $\overline{K}N$ cross sections given by the Kim and Sakitt solutions. The KN cross sections at low energy have been measured; they are small and smoothly varying. We have collected the available experimental data[34] for the integrals up to $P_K^{lab} = 6$ BeV/c. For $P_K^{lab} > 6$ BeV/c, the experimental cross-section differences[35] occurring in Eq. (III.4) can be reasonably well-fitted with an exponential form as in Eq. (IV.1). For KN cross sections we use[36]

$$\sigma_{K^-\binom{n}{p}} - \sigma_{K^+\binom{n}{p}} = b_{\binom{n}{p}}\, \nu^{-0.5} . \tag{IV.9}$$

The contributions from the asymptotic region, $P_K^{lab} > 20$ BeV/c, are about 10% of the total integral

The possible numerical errors for these integrals are estimated to be ~20%. This is in addition to errors introduced by the pole dominance approximation. Equations (IV.6a) and (IV.6b) can be solved simultaneously for α and g_A. There are two solutions to the resulting quadratic equation for α. One solution gives $\alpha \simeq 0$ and $g_A \simeq 0.85$ and is discarded. With the indicated errors the other solution is[37]

$$\begin{aligned} \alpha &= 0.75 \pm 0.10 \\ |g_A| &= 1.28 \pm 0.10 \end{aligned} \tag{IV.10}$$

The solutions for the two sets of results in Table I are closer than the statistical errors indicated above. Correcting the I's for the errors in G-T does not affect the solution for α but gives

$$|g_A| = 1.20 \pm 0.10 \tag{IV.10'}$$

The consistency with the value for g_A obtained from the $\Delta S = 0$ sum rule is quite good. The solution for α agrees within error limits with the best fits of Cabibbo theory to experimental data on semi-leptonic decays.[18] These give

$$\alpha = \begin{cases} 0.67 \pm 0.03 & \text{(Brene \underline{et} \underline{al}.)} \\ 0.63 & \text{(Willis \underline{et} \underline{al}.)} \end{cases} \qquad \text{(IV.11)}$$

These results yield a consistent theoretical picture of low-energy semi-leptonic processes with only two imput parameters for the weak interactions, G_V and the Cabibbo angle, θ . One should expect, however, that future precise measurements of semi-leptonic decays will show departures from complete SU(3) symmetry of the matrix elements of the currents. The evidence is quite strong, however, that the suppression of the $\Delta S = 1$ decays relative to $\Delta S = 0$ decays by $\tan\theta$ lies in the structure of the weak interactions and cannot be explained as a strong interaction renormalization effect.[38]

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LIST OF REFERENCES

1. S. L. Adler, Phys. Rev. Letters 14, 1051 (1965); Phys. Rev. (to be published).
2. W. I. Weisberger, Phys. Rev. Letters 14, 1047 (1965).
3. M. Gell-Mann, Phys. Rev. 125, 1067 (1962); Physics 1 (1964). R. P. Feynman, M. Gell-Mann, and G. Zweig, Phys. Rev. Letters 13, 678 (1964).
4. The relation of the algebraic relations to the specification of a universal weak coupling of leptons and hadrons has been discussed by M. Gell-Mann and Y. Ne'eman, Ann. Phys. (N.Y.) 30, 360 (1964).
5. M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).
6. J. Bernstein, M. Gell-Mann, and L. Michel, Nuovo Cimento 16, 560 (1960).
7. Y. Nambu, Phys. Rev. Letters 4, 380 (1960).
8. S. L. Adler, Phys. Rev. 137, B1022 (1965); 139 B1638 (1965).
9. J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo Cimento 17, 757 (1960). See also Ref. 3.
10. N. Cabibbo, Phys. Rev. Letters 10, 531 (1963).
11. M. Goldberger and S. Treiman, Phys. Rev. 110, 1178, 1478 (1958).
12. Similar methods have been used to deduce consequences of field theoretical versions of conserved and partially-conserved axial currents. Y. Nambu and D. Lurié, Phys. Rev. 125, 1429 (1962); Y. Nambu and E. Shrauner, Phys. Rev. 128, 862 (1962). See also Refs. 1 and 9.
13. N. N. Bogoliubov and V. D. Shirkov, Introduction to the Theory of Quantized Fields (Interscience Publishers, Inc., New York, 1959), Ch. IX.

14. This is the same problem discussed in the ingenious method for extracting sum rules from equal-time commutators developed by S. Fubini, G. Furlan and C. Rossetti (to be published).

15. J. Schwinger, Phys. Rev. Letters 3, 296 (1959).

16. This point has been emphasized to the author in correspondence with S. L. Adler.

17. G. F. Chew, M. L. Goldberger, F. E. Low, Y. Nambu, Phys. Rev. 106, 1337 (1957).

18. M. Brene, B. Hellesen and M. Roos, Phys. Letters 11, 344 (1964); W. Willis, et al., Phys. Rev. Letters 13, 291 (1964).

19. M. Ademollo and R. Gatto, Phys. Rev. Letters 13, 264 (1964).

20. This implies that the off-diagonal matrix elements of the SU(3) breaking interaction between initially non-degenerate states are small though the diagonal matrix elements may be large. In a naive potential picture this means that the second-order effects on the mass splittings will be very small.

21. P.G.O. Freund and Y. Nambu, Phys. Rev. Letters 13, 221 (1964).

22. These sum rules have been discussed independently by D. Amati, C. Bouchiat, and J. Nuyts (to be published). Equation (III.3b) has also been evaluated by C. A. Levinson and I. J. Muzinich (to be published); L. K. Pandit and J. Schecter (to be published).

23. We use the value $g_{\pi n}^2/4\pi = 14.6 \pm 0.03$ given by J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. 35, 737 (1963).

24. C. Hohlen, C. Ebel, and J. Giesecke, Z. Physik 180, 430 (1964).

25. G. von Dardel, D. Dekkers, R. Mermod, M. Wivargent, G. Weber and K. Winter, Phys. Rev. Letters 8, 173 (1962).

26. C. S. Wu, unpublished.

27. A. H. Rosenfeld, A. Barbaro-Galtieri, W. H. Barkas, P. L. Bastien, J. Kirz and M. Roos, Rev. Mod. Phys. 36, 977 (1964).

28. S. L. Adler, Phys. Rev. 135, B963 (1964).

29. In a field theoretic treatment these departures from simple pole dominance might be identified with the low-energy threshold corrections. See Ref. 1.

30. W. Rarita and J. Schwinger, Phys. Rev. 60, 61 (1941).

31. E. Johnson and E. R. McCliment, Phys. Rev. 139, B951 (1965). See also references listed here.

32. R. H. Dalitz and S. F. Tuan, Ann. Phys. 10, 307 (1960).

33. J. K. Kim, Phys. Rev. Letters 14, 29 (1965); M. Sakitt, T. B. Day, R. G. Glasser, N. Seeman, J. Friedman, W. E. Humphrey and R. R. Ross, Phys. Rev. 139, B719 (1965).

34. See Refs. 4 and 5 of R. Good and N. Xuong, Phys. Rev. Letters 14, 191 (1965); also, G. von Dardel, D. H. Frisch, R. Mermod, R. H. Milburn, P. A. Piroué, M. Vivargent, G. Weber and K. Winter, Phys. Rev. Letters 5, 333 (1960); P. L. Bastien, J. P. Berge, O. I. Dahl, M. Ferro-Luzzi, J. Kirz, D. H. Millen, J. J. Murray, A. H. Rosenfeld, R. D. Tripp and B. Watson, Proceedings of the International Conference on High-Energy Physics, Geneva, 1962, edited by J. Prenkti (CERN Scientific Information Service, Geneva, Switzerland, 1962), p. 373; W. F. Baker, R. L. Cool, E. W. Jenkins, T. F. Kycia, R. H. Phillips, A. L. Read, K. F. Riley and H. Ruderman, Proceedings of the Sienna International Conference on Elementary Particles (Società Italiana di Fisica, Bologna, Italy, 1963), p. 634; V. J. Stenger, W. E. Slater, D. H. Stork, H. K. Ticho, G. Goldhaber and S. Goldhaber, Phys. Rev. 134, B1111 (1964).

35. E. W. Jenkins, T. F. Kycia, B. A. Leontic, R. H. Phillips, A. L. Read and R. Rubinstein, Phys. Rev. 138, 933 (1965).

36. R. J. N. Phillips and W. Rarita, Phys. Rev. <u>139</u>, 1336 (1965).

37. For comparison we note the independent results, see Ref. 22,

$\alpha = 0.73$ (D. Amati, C. Bouchiat and J. Nuyts);

$\alpha = 0.63$ (C. A. Levinson and I. J. Muzinich).

38. R. Oehme, Ann. Phys. <u>33</u>, 108 (1965).

FIGURE CAPTIONS

1. Singularities of $R(q^2,\nu)$ in the Real q^2-Real ν plane which follow from perturbation theory.

2. Dispersion-perturbative diagram producing the anomalous threshold shown in Fig. 1. Invariant masses of the external and internal lines are indicated for $\binom{s}{u}$ channel.

TABLE I

	Unphysical Region		Physical Region		Total
	$Y_1^*(1385)$	S-Waves	$0 < P_k^{lab} < 6$ BeV/c	$P_k^{lab} > 6$ BeV/c	
I(Kp)	-0.010 -0.010	0.106 0.126	0.205 0.198	0.048 0.048	0.349 (Kim) 0.362 (Sakitt)
I(Kn)	-0.021 -0.021	0.055 0.055	0.134 0.107	0.027 0.027	0.195 (Kim) 0.168 (Sakitt)

Numerical contributions to the dispersion integrals in the sum rules for the $\Delta S = 1$ axial current.

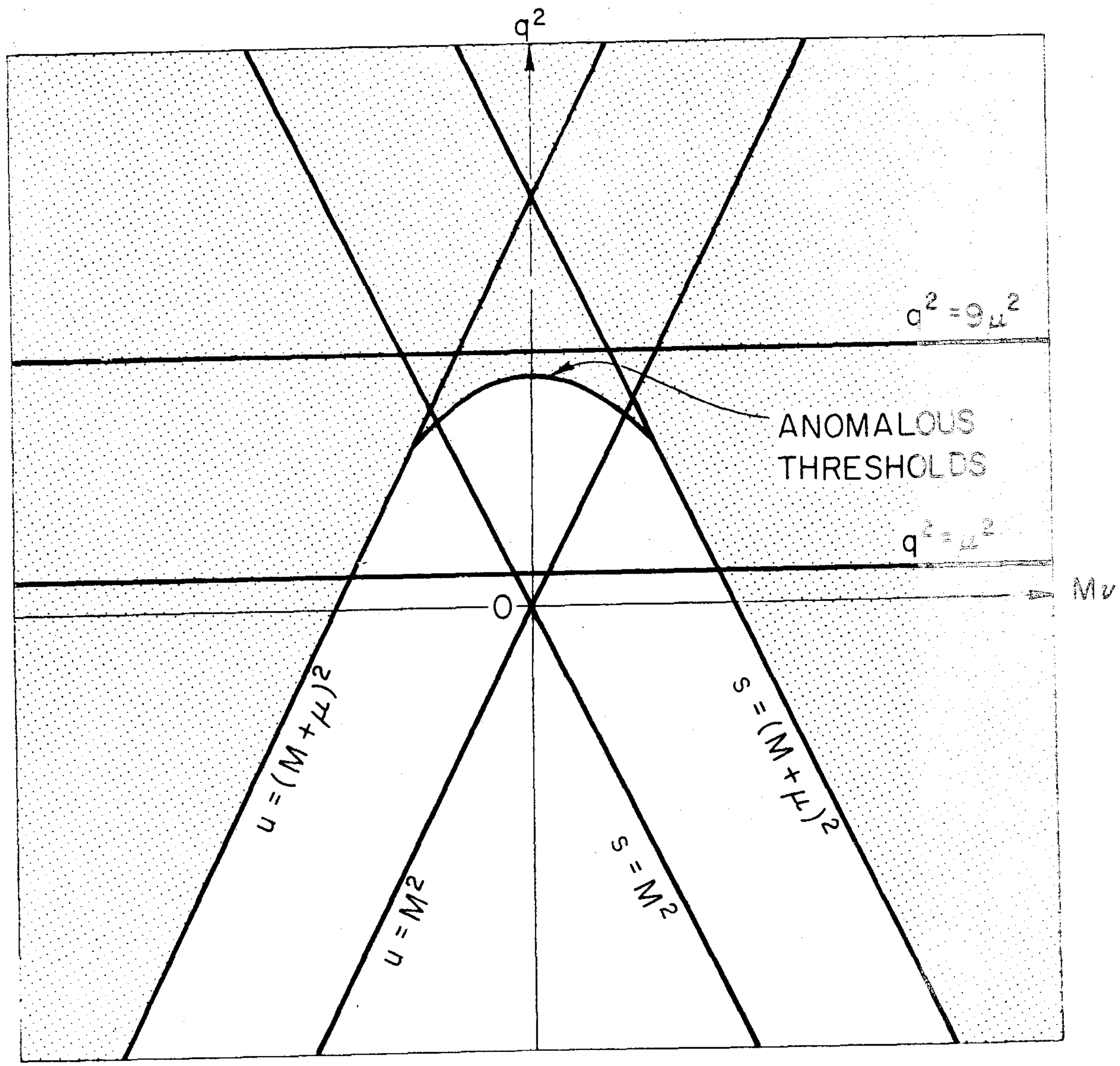

FIG. 1

418-1-A

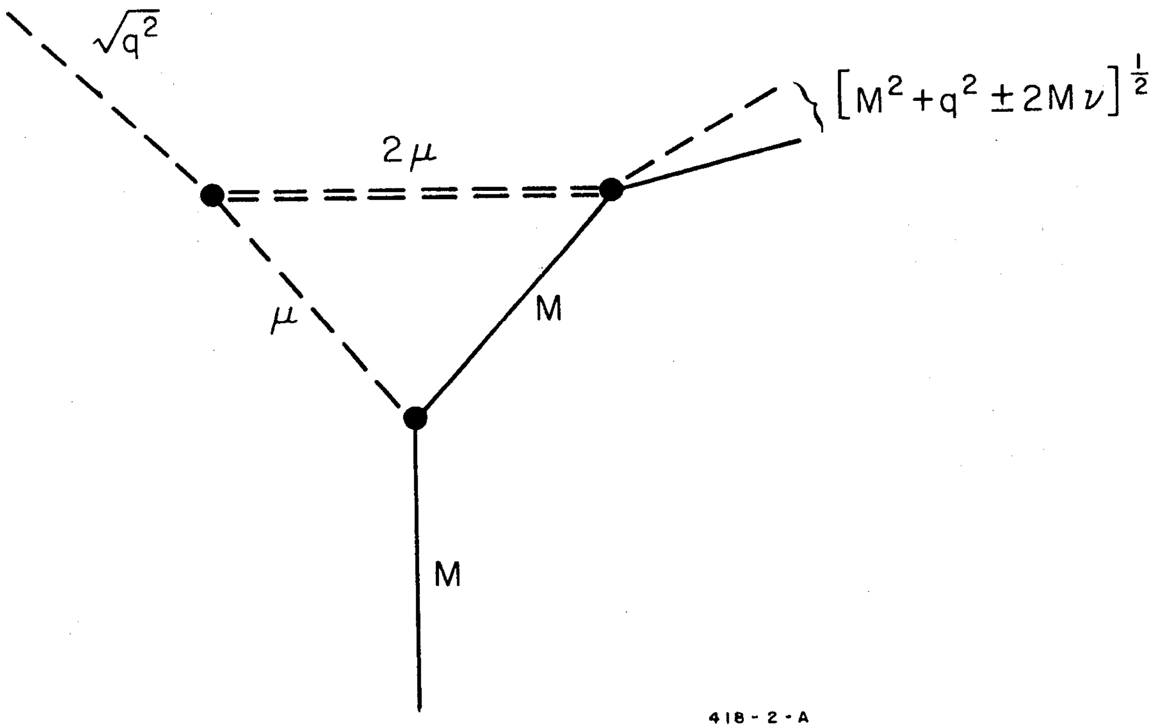

FIG. 2