SLAC PUB-132
August 1965

A GENERAL FIRST- AND SECOND-ORDER THEORY OF BEAM TRANSPORT OPTICS AND ITS APPLICATION TO THE DESIGN OF HIGH-ENERGY PARTICLE SPECTROMETERS *

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ABSTRACT

A general first- and second-order theory of beam transport optics has been developed. From this the first- and second-order matrix elements of bending magnets, quadrupoles, and sextupoles have been derived.

Utilizing this theory, very general first- and second-order theorems of beam transport optics have been formulated which have been extremely useful for designing single and multiple element magnetic optical systems. The theorems are expressed as functions of five characteristic first-order trajectories of a system. In fact, all of the first-and second-order optical properties of a system may be expressed in terms of these five trajectories.

A general discussion of the theory will be presented along with specific applications of the theory to the design of high-energy particle spectrometers.

(To be presented at the Fifth International Conference on High Energy Accelerators, Frascati, Italy, September 9-16, 1965.

* Work supported by the U. S. Atomic Energy Commission.

I. INTRODUCTION

For the past several years, we have been attempting to evolve at Stanford a more systematic procedure for solving beam transport problems. Two basic techniques have been utilized for this purpose. The first, which will be discussed in detail later, is a logical extension of the first-order matrix formalism to a matrix formalism which allows one to calculate systematically not only the first-order but also the second-order optical properties of beam transport systems. The second approach is the conventional one of computer ray tracing through a known field to the degree of precision demanded for the particular problem.

The advantage of the matrix formalism as we have evolved it, as compared to ray tracing, is that it provides us with a somewhat better physical insight into the physics of the problems and, as such, permits a more systematic procedure for solving problems. Having utilized the matrix method for finding a solution, we then use conventional ray-tracing techniques for verification and as a means for further refinement of the design if required.

The basic approach to formulating the matrix method has been as follows:

(1) The general differential equation describing the trajectory of a charged particle in a static magnetic field which possesses "midplane symmetry" is derived.

(2) A Taylor's series solution about a central trajectory is then assumed; this is substituted into the general differential equation and terms are retained to second-order.

(3) The first-order coefficients for monoenergetic rays satisfy the usual homogeneous differential equations characteristic of harmonic oscillator theory, and the first-order dispersion and the second-order coefficients of the Taylor's expansion satisfy second-order differential equations having "driving terms."

(4) The first-order dispersion and the second-order coefficients are then evaluated by a Green's function integral containing the characteristic driving function of the coefficient being evaluated.

In other words, the problem is nothing more or less than the old problem of the harmonic oscillator with driving terms; and as with the harmonic oscillator, we may readily draw general conclusions about a given second-order aberration by studying its characteristic driving function.

The task now is to transform this solution into a self-consistent second-order matrix formalism. I will demonstrate later how this has been accomplished.

By using the above procedure, we have derived the complete second-order matrix elements for a drift distance, quadrupoles, bending magnets, and sextupoles, including an impulse approximation for the input and output fringing field boundaries of bending magnets. This includes rotated input and output faces and curvatures on the input and output faces of the bending magnets. This entire formalism has then been programmed for a 7090 computer, which enables us to calculate within the above limitations the complete second-order properties of any combination of quadrupoles, sextupoles, bending magnets, and drift distances which one might choose to utilize.

Returning briefly now to the formulation of the general theory, all of the theory and the subsequent matrix elements have been derived and expressed in terms of five characteristic first-order trajectories of the system. Before identifying these trajectories, it should be mentioned that it is implicitly assumed from the beginning that a central trajectory is known and that the positions of other trajectories are always specified relative to this central trajectory. In other words, we have made the usual paraxial ray approximation.

The five characteristic trajectories are the following (identified by their initial conditions):

(1) The unit sine-like function s_x in the plane of bend where $s_x(0) = 0$

$s'_x(0) = 1$

(2) The unit cosine–like function c_x in the plane of bend where

$c_x(0) = 1 \qquad c'_x(0) = 0$

(3) The dispersion function d_x in the plane of bend where

$d_x(0) = 0 \qquad d'_x(0) = 0$

(4) The unit sine–like function s_y in the non–bend plane where

$s_y(0) = 0 \qquad s'_y(0) = 1$, and finally,

(5) The unit cosine–like function c_y in the non–bend plane where

$c_y(0) = 1 \qquad c'_y(0) = 0$

II. THE FORMULATION OF THE GENERAL THEORY

We begin with the usual relativistic equation of motion for a charged particle in a static magnetic field:

$$\dot{\vec{P}} = e\,(\vec{v} \times \vec{B}) \tag{1}$$

and immediately transform this expression to one in which time has been eliminated as a variable and we are left only with spatial coordinates. The curvilinear coordinate system utilized is shown in Fig. 1. With a little algebra, the equation of motion is readily expressed in the following vector forms:

$$\frac{d^2\vec{s}}{ds^2} = \frac{e}{P}\left(\frac{d\vec{s}}{ds} \times \vec{B}\right) \tag{2}$$

or

$$\vec{s}'' - \frac{1}{2}\,\frac{\vec{s}'}{(s')^2}\,\frac{d}{dt}\,(s')^2 = \frac{e}{P}\,s'\,(\vec{s}' \times \vec{B}) \tag{3}$$

where prime means the derivative with respect to t (the distance along the central trajectory).

By utilizing the expression for the differential line element in the chosen coordinate system, namely,

$$ds^2 = dx^2 + dy^2 + (1 + hx)^2 \, dt^2 \tag{4}$$

and expanding Eq. (3) into its component parts, retaining only terms through second–order, the x and y components of the equation of motion become:

$$\begin{aligned} x'' - h(1 + hx) - x'(hx' + h'x) &= \frac{e}{P} s' \left[y'B_t - (1 + hx) B_y \right] \\ y'' - y'(hx' + h'x) &= \frac{e}{P} s' \left[(1 + hx) B_x - x'B_t \right] \end{aligned} \tag{5}$$

The equation of motion for the central trajectory is found by taking the limit $x = x' = y = y' = 0$, from which $h = \frac{e}{p_o} B_y(o, o, t)$.

The field components B_x, B_y, and B_t in the curvilinear coordinate system may be derived from a scalar potential* ϕ, yielding the following result to second–order:

$$B_x(x, y, t) = \frac{\partial \phi}{\partial x} = A_{11} y + A_{12} xy + \ldots$$

$$B_y(x, y, t) = \frac{\partial \phi}{\partial y} = A_{10} + A_{11} x + \frac{A_{12}}{2!} x^2 + \frac{A_{30}}{2!} y^2 + \ldots \tag{6}$$

$$B_t(x, y, t) = \frac{1}{(1 + hx)} \frac{\partial \phi}{\partial t} = \frac{1}{(1 + hx)} \left[A'_{10} y + A'_{11} xy + \ldots \right]$$

*Midplane symmetry requires that ϕ be an odd function in y, i.e., $\phi(x, y, t) = - \phi(x, -y, t)$.

where the coefficients A_{1n} of the expansions are derivable from the midplane field $B_y(x, o, t)$.

$$A_{1n} = \left.\frac{\partial^n B_y}{\partial x^n}\right|_{\substack{x=o \\ y=o}} = \text{functions of } t \text{ only} .$$

and

$$A_{30} = -\left[A''_{10} + hA_{11} + A_{12}\right] \tag{7}$$

Studying the expansion for B_y for the midplane only,

$$B_y(x, o, t) = A_{10} + A_{11}\, x + \frac{1}{2!} A_{12}\, x^2 + \ldots$$

$$\begin{array}{cccc} \underline{\text{dipole}} & \underline{\text{quadrupole}} & \underline{\text{sextupole}} & \underline{\text{etc.}} \end{array} \tag{8}$$

$$= B_y\Big|_{\substack{x=o \\ y=o}} + \left.\frac{\partial B_y}{\partial x}\right|_{\substack{x=o \\ y=o}} x + \frac{1}{2!} \left.\frac{\partial^2 B_y}{\partial x^2}\right|_{\substack{x=o \\ y=o}} x^2 + \ldots$$

we can readily identify the various terms appearing in the equations as to whether they are of dipole, quadrupole, or sextupole origin and retain this identification throughout the remainder of the discussion. It is then convenient to define two dimensionless quantities $n(t)$ and $\beta(t)$ in terms of their quadrupole and sextupole origins, i.e.,

$$n(t) = -\left[\frac{1}{hB_y}\left(\frac{\partial B_y}{\partial x}\right)\right]_{\substack{x=o \\ y=o}} \qquad \beta(t) = \left[\frac{1}{2!\, h^2 B_y}\left(\frac{\partial^2 B_y}{\partial x^2}\right)\right]_{\substack{x=o \\ y=o}} \tag{9}$$

Making use of the equation of motion for the central trajectory, we may eliminate B_y in the expressions and rewrite them as follows:

$$-nh^2\left(\frac{P_o}{e}\right) = \left.\frac{\partial B_y}{\partial x}\right|_{\substack{x=o\\y=o}} \qquad \text{and} \qquad \beta h^3\left(\frac{P_o}{e}\right) = \frac{1}{2!}\left.\frac{\partial^2 B_y}{\partial x^2}\right|_{\substack{x=o\\y=o}} \tag{9a}$$

For a pure quadrupole field

$$B_y = \frac{B_o x}{a}$$

where B_o is the field at the pole and a is the radius of the aperture; hence, we obtain the identity

$$-nh^2 \equiv \left(\frac{B_o}{a}\right)\left(\frac{e}{P_o}\right) = k_q^2 \tag{9b}$$

and for a pure sextupole field

$$B_y = \frac{B_o}{a^2}(x^2 - y^2)$$

from which

$$\beta h^3 \equiv \left(\frac{B_o}{a^2}\right)\left(\frac{e}{P_o}\right) = k_s^2 \tag{9c}$$

Using these definitions, the equations of motion for x and y may, after a little algebra, be evolved into the following convenient forms:

$$\begin{aligned} x'' + (1-n)\,h^2 x &= h\delta + (2n - 1 - \beta)\,h^3 x^2 + h'xx' + \frac{1}{2}hx'^2 \\ &+ (2-n)\,h^2 x\delta + \frac{1}{2}(h'' - nh^3 + 2\beta h^3)\,y^2 \\ &+ h'yy' - \frac{1}{2}hy'^2 - h\delta^2 + \text{higher-order terms} \end{aligned} \tag{10}$$

$$y'' + nh^2 y = 2(\beta - n)\, h^3 xy + h'xy' - h'x'y + hx'y' + nh^2 y\delta \qquad (11)$$

$$+ \text{ higher-order terms}$$

where $\delta \equiv \dfrac{P - P_o}{P_o}$ and the constant e has been eliminated by incorporating the equation of motion for the central trajectory.

If now we assume a Taylor's expansion about the central orbit for x and y at the exit of a system, describing the position of an arbitrary trajectory with respect to the central trajectory as a function of the initial coordinates of the arbitrary trajectory, we have

$$\begin{aligned} x = \; & \overbrace{(x|x_o)x_o}^{c_x} + \overbrace{(x|x'_o)x'_o}^{s_x} + \overbrace{(x|\delta)\delta}^{d_x} \\ & + (x|x_o^2)x_o^2 + (x|x_o x'_o)x_o x'_o + (x|x_o\delta)x_o\delta \\ & + (x|x_o'^2)x_o'^2 + (x|x'_o\delta)x'_o\delta + (x|\delta^2)\delta^2 \\ & + (x|y_o^2)y_o^2 + (x|y_o y'_o)y_o y'_o + (x|y_o'^2)y_o'^2 \end{aligned} \qquad (12)$$

and

$$\begin{aligned} y = \; & \overbrace{(y|y_o)y_o}^{c_y} + \overbrace{(y|y'_o)y'_o}^{s_y} \\ & + (y|x_o y_o)x_o y_o + (y|x_o y'_o)x_o y'_o + (y|x'_o y_o)x'_o y_o \\ & + (y|x'_o y'_o)x'_o y'_o + (y|y_o\delta)y_o\delta + (y|y'_o\delta)y'_o\delta \end{aligned} \qquad (13)$$

Substituting these expansions into Eqs. (10) and (11), we derive a differential equation for each of the first– and second–order coefficients contained in the Taylor's expansions. When this is done, a systematic pattern evolves in the following way:

$$\begin{aligned} c'' + k^2 c &= o \\ s'' + k^2 s &= o \\ q'' + k^2 q &= f \end{aligned} \qquad (14)$$

where $k_x^2 = (1-n)h^2$ and $k_y^2 = nh^2$ for the x and y motions, respectively. The first two of these equations represent the equations of motion for the mono-energetic solution to the first-order part of the problem. The fact that there are two solutions, one for c and one for s, is a manifestation of the fact that the differential equation is second-order; hence, the two solutions differ only by the initial conditions of the characteristic s and c functions. The third differential equation is a type form which represents the solution for the first-order dispersion d_x and for the coefficients of the second-order aberrations in the system where the driving term f has a characteristic form for each of these coefficients. The third differential equation is solved by the Green's function integral

$$q = \int_0^t f(\tau)\ G(t-\tau)\ d\tau \tag{15}$$

It can be readily verified by substitution into the third equation that the correct Green's function is

$$G(t-\tau) = s(t)\ c(\tau) - s(\tau)\ c(t) \tag{16}$$

Thus, Eq. (15) becomes

$$q = s(t) \int_0^t f(\tau)\ c(\tau)\ d\tau - c(t) \int_0^t f(\tau)\ s(\tau)\ d\tau \tag{17}$$

The problem is then, in principle, solved if we know the driving term f and if we are able to evaluate the integrals contained in Eq. (17). The driving function f is readily obtained from substitution of the Taylor's expansions into the general differential Eqs. (10) and (11). The results of this substitution are expressed in Table I for the first-order dispersion and for all of the second-order coefficients which will occur for a system having midplane symmetry. All of the driving terms have been expressed in terms of the five characteristic first-order functions s_x, c_x, d_x, s_y, and c_y mentioned in the introduction. Also contained in the expressions are the parameters which characterize the expansion of the magnetic field to second-order, i.e., h, n, and β.

Going back to the definitions for n and β, it is possible to identify immediately the origin of the various terms contained in these driving terms. For example, any term containing the quantity nh^2 as a coefficient is of quadrupole origin, and any term containing the quantity βh^3 is of sextupole origin. The other terms are either of dipole origin or they result from cross product terms between the dipole and quadrupole elements of the system. The driving term expressions are completely rigorous to second-order for any magnetic field configuration possessing midplane symmetry; no assumptions have been made regarding the details of the fringing field or boundary shapes.

III. EVALUATION OF THE MATRIX ELEMENTS FOR HIGH-ENERGY PARTICLES

A considerable simplification results for the high-energy limit where the dipole, quadrupole and sextupole functions are physically separated, such that the cross product terms do not appear and such that the fringing field effects are small compared to the other dominant effects generated by the dipole, quadrupole, and sextupole elements of the system.

For the purpose of this discussion, the x plane is defined as the bend plane in which the particles are dispersed in momentum.

The focusing conditions imposed upon the system at the image planes are: At the x (bend-plane) image $s_x(i) = 0$, i.e., we assume point-to-point imaging; at the y (non-bend) image plane, we consider two cases:

(a) Point-to-point imaging, i.e., $s_y(i) = 0$, and

(b) Parallel-(line)-to-point imaging, i.e., $c_y(i) = 0$.

In the high-energy limit, the bending radius $\rho_o = \frac{1}{h} \gg 1$; the first-order focusing is accomplished predominately by quadrupole elements; and only $n = 0$ uniform-field bending magnets are considered.

In this limit, the following definitions are used for convenience:

$$-nh^2 = k_q^2 = \frac{B_q}{a_q(H\rho_o)} \quad \text{or} \quad k_q^2 \ell_q = \frac{1}{f_q} = \text{the quadrupole strength in the x(bend) plane} \tag{18}$$

and

$$\beta h^3 = k_s^2 = \frac{B_s}{a_s^2(H\rho_o)} \quad \text{or} \quad k_s^2 \ell_s = S = \text{the sextupole strength in the x(bend) plane} \tag{19}$$

where B_q and B_s are the field strengths at the poles of the quadrupole and sextupole, respectively, a_q and a_s are the radii of the apertures of the quadrupole and sextupole, and ℓ_q and ℓ_s are the equivalent magnetic lengths of the quadrupole and sextupole elements.

Using the Green's function solution, the equations for the first-order dispersion d_x and momentum resolution R_x reduce to the simple forms:

$$d_x = -c_x(i) \int_o^i s_x h d\tau = -c_x(i) \int_o^i s_x d\alpha \tag{20}$$

and

$$R_x x_o = -\frac{d_x}{c_x(i)} = \int_o^i s_x d\alpha \tag{21}$$

where $d\alpha$ is the differential angle of bend of the central trajectory of the system and x_o is the source size.

It follows from the general theory of Section II and the above focusing conditions that we obtain for the second-order x (bend) plane aberrations

$$q_x = - c_x(i) \int_0^i f s_x d\tau \tag{22}$$

for point-to-point imaging; for the second-order y (non-bend) plane aberrations

$$q_y = - c_y(i) \int_0^i f s_y d\tau \tag{23}$$

for point-to-point imaging (case a), and equal

$$q_y = s_y(i) \int_0^i f c_y d\tau \tag{24}$$

for parallel-(line)-to-point imaging (case b).

IV. APPLICATIONS OF THE GENERAL THEORY TO HIGH-ENERGY SPECTROMETER DESIGN

In high-energy spectrometers or beam transport systems where quadrupoles essentially control the first-order optics of the system, the second-order chromatic aberrations introduced by the quadrupoles are usually the dominant aberrations limiting the performance of the system. As an example of the use of the theory developed here, we shall calculate for some representative examples the angle ψ that the momentum focal plane makes with respect to the central trajectory.

For point-to-point imaging, it may be readily verified that

$$\tan\psi = - \left(\frac{d_x(i)}{c_x(i)}\right) \frac{1}{(x_i|x_0'\delta)} = \frac{\int_0^i s_x d\alpha}{(x_i|x_0'\delta)} \tag{25}$$

Let us now consider some representative quadrupole configurations and assume that the bending magnets are placed in a region having a large amplitude of the unit sine-like function s_x (to optimize the first-order momentum resolution).

Case I

Consider the simple quadrupole configuration shown in Fig. 2 with the bending magnets located in the region between the quadrupoles and $s_x' \cong 0$ in this region. For these conditions, $f_1 = \ell_1$, $s_x = \ell_1$ at the quadrupoles, and $f_2 = \ell_3$. From Table II, we have:

$$(x_i | x_0' \delta) \cong - c_x(i) \sum_q \frac{s_x^2}{f_q} = - c_x(i) \ell_1 \left(1 + \frac{\ell_1}{\ell_3}\right) = \ell_1 (1 + M_x) \qquad (26)$$

where we make use of the fact that $(\ell_3/\ell_1) = M_x = - c_x(i)$. M_x is the first-order magnification of the system.

Hence,

$$\tan \psi = \frac{\int_0^i s_x d\alpha}{(x_i | x_0' \delta)} \cong \frac{\alpha}{(1 + M_x)}$$

Case II

For a single quadrupole, Fig. 3, the result is similar

$$\tan \psi = \frac{K\alpha}{(1 + M_x)}$$

except for the factor $K < 1$ resulting from the fact that s_x cannot have the same amplitude in the bending magnets as it does in the quadrupole. Therefore $\int_0^i s_x d\alpha = K \ell_1 \alpha$.

Case III

Now let us consider a symmetric four-quadrupole array, Fig. 4, such that we have an intermediate image. Then

$$(x|x'_0\delta) = -\,2c_x(i)\,\ell_1\left[1 + (\ell_1/\ell_3)\right] = \text{twice that for Case I .}$$

Because of symmetry, $c_x(i) = M_x = 1$. Thus, we conclude

$$\tan\psi = -\,\alpha/2\left[1 + (\ell_1/\ell_3)\right]$$

In other words, the intermediate image has introduced a factor of two in the denominator and has changed the sign of ψ .

Conclusions

It is clear from these three examples that for high-energy systems where the total angle of bend α is a small quantity, ψ will be even smaller. It is for this reason that we have added sextupoles to the SLAC 20-GeV Spectrometer.

V. SECOND-ORDER MATRIX FORMALISM

The method for formulating the individual second-order matrix for a given element in a system is illustrated in Table V for the x plane case. The technique is similar for the y plane. The first three rows are derived directly from the general theory using the driving functions in Table I. However, in order to facilitate matching boundary conditions, the matrix is expressed in terms of a rectangular coordinate system x, y and z (see Fig. 1). The distinction is the introduction of θ and ϕ defined as follows:

$$\theta = \frac{dx}{dz} = \frac{x'}{z'} = \frac{x'}{1+hx}$$

$$\phi = \frac{dy}{dz} = \frac{y'}{z'} = \frac{y'}{1+hx}$$

Having formulated the second-order matrices for each element of a system, the total system optics is solved in the usual way by multiplying the individual matrices in the same manner as for a first-order problem. For further details, see Ref. 4.

Second-order matrix elements for drift distances, quadrupoles, sextupoles, bending magnets, and for fringing fields of bending magnets (using an impulse approximation) including rotated and curved entrance and exit boundaries of the bending magnets, have been derived.* These matrix elements have been incorporated into an IBM 7090 program called "TRANSPORT"[8] by S. K. Howry, C. H. Moore and H. S. Butler at the Stanford Linear Accelerator Center. We have used this program to finalize the design of all of the beam transport systems and high-energy spectrometers to be utilized at SLAC.

*See the list of references.

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TABLE I

The Driving Terms for the Coefficients

Listed in the first column are the coefficients in the expansions for the coordinates x and y ; they are indicated by means of the notation introduced in Eqs. (12) and (13). For general considerations, q has been used to represent any one of these coefficients. Listed in the second column are the corresponding driving functions f , which are related to the coefficients as shown by Eq. (17). This list includes all those functions f for the linear and quadratic coefficients which do not vanish identically.

q	f			
$d = (x\|\delta)$	$+h$			
$(x\|x_o^2)$		$+\ (2n-1-\beta)h^3c_x^2$	$+\ h'c_xc'_x$	$+\ \frac{1}{2}\ hc_x'^2$
$(x\|x_ox'_o)$		$+\ 2(2n-1-\beta)h^3c_xs_x$	$+\ h'(c_xs'_x+c'_xs_x)$	$+\ hc'_xs'_x$
$(x\|\delta x_o)$	$-\ (n-2)h^2c_x$	$+\ 2(2n-1-\beta)h^3c_xd$	$+\ h'(c_xd'+c'_xd)$	$+\ hc'_xd'$
$(x\|x_o'^2)$		$+\ (2n-1-\beta)h^3s_x^2$	$+\ h's_xs'_x$	$+\ \frac{1}{2}\ hs_x'^2$
$(x\|\delta x'_o)$	$-\ (n-2)h^2s_x$	$+\ 2(2n-1-\beta)h^3s_xd$	$+\ h'(s_xd'+s'_xd)$	$+\ hs'_xd'$
$(x\|\delta^2)$	$-\ h-(n-2)h^2d$	$+\ (2n-1-\beta)h^3d^2$	$+\ h'dd'$	$+\ \frac{1}{2}\ hd'^2$
$(x\|y_o^2)$		$+\ \frac{1}{2}(h''-nh^3+2\beta h^3)c_y^2$	$+\ h'c_yc'_y$	$-\ \frac{1}{2}\ hc_y'^2$
$(x\|y_oy'_o)$		$+\ (h''-nh^3+2\beta h^3)c_ys_y$	$+\ h'(c_ys'_y+c'_ys_y)$	$-\ hc'_ys'_y$
$(x\|y_o'^2)$		$+\ \frac{1}{2}(h''-nh^3+2\beta h^3)s_y^2$	$+\ h's_ys'_y$	$-\ \frac{1}{2}\ hs_y'^2$
$(y\|x_oy_o)$		$-\ 2(n-\beta)h^3c_xc_y$	$+\ h'(c_xc'_y-c'_xc_y)$	$+\ hc'_xc'_y$
$(y\|x_oy'_o)$		$-\ 2(n-\beta)h^3c_xs_y$	$+\ h'(c_xs'_y-c'_xs_y)$	$+\ hc'_xs'_y$
$(y\|x'_oy_o)$		$-\ 2(n-\beta)h^3s_xc_y$	$+\ h'(s_xc'_y-s'_xc_y)$	$+\ hs'_xc'_y$
$(y\|x'_oy'_o)$		$-\ 2(n-\beta)h^3s_xs_y$	$+\ h'(s_xs'_y-s'_xs_y)$	$+\ hs'_xs'_y$
$(y\|\delta y_o)$	$+\ nh^2c_y$	$-\ 2(n-\beta)h^3c_yd$	$-\ h'(c_yd'-c'_yd)$	$+\ hc'_yd'$
$(y\|\delta y'_o)$	$+\ nh^2s_y$	$-\ 2(n-\beta)h^3s_yd$	$-\ h'(s_yd'-s'_yd)$	$+\ hs'_yd'$

TABLE II

Applying the Greens' function solution, Eq. (22), in the high-energy limit as defined above for point-to-point imaging in the x(bend) plane, the second-order matrix elements reduce to:

$$(x|x_0^2) \cong -\frac{1}{2} c_x(i) \int_0^i c_x'^2 s_x \, d\alpha \quad + \quad c_x(i) \sum_j S_j c_x^2 s_x$$

$$(x|x_0 x_0') \cong -c_x(i) \int_0^i c_x' s_x' s_x \, d\alpha \quad + \quad 2c_x(i) \sum_j S_j c_x s_x^2$$

$$(x|x_0 \delta) \cong -c_x(i) \int_0^i c_x' d_x' s_x \, d\alpha \quad + \quad 2c_x(i) \sum_j S_j c_x s_x d_x - c_x(i) \sum_q \frac{c_x s_x}{f_q}$$

$$(x|x_0'^2) \cong -\frac{1}{2} c_x(i) \int_0^i s_x'^2 s_x \, d\alpha \quad + \quad c_x(i) \sum_j S_j s_x^3$$

$$(x|x_0' \delta) \cong -c_x(i) \int_0^i s_x' d_x' s_x \, d\alpha \quad + \quad 2c_x(i) \sum_j S_j s_x^2 d_x - c_x(i) \sum_q \frac{s_x^2}{f_q}$$

$$(x|\delta^2) \cong -\frac{c_x(i)}{2} \int_0^i (d_x')^2 s_x \, d\alpha \quad + \quad c_x(i) \sum_j S_j s_x d_x^2 - c_x(i) \sum_q \frac{s_x d_x}{f_q}$$

$$(x|y_0^2) \cong \frac{1}{2} c_x(i) \int_0^i c_y'^2 s_x \, d\alpha \quad - \quad c_x(i) \sum_j S_j c_y^2 s_x$$

$$(x|y_0 y_0') \cong c_x(i) \int_0^i c_y' s_y' s_x \, d\alpha \quad - \quad 2c_x(i) \sum_j S_j c_y s_y s_x$$

$$(x|y_0'^2) \cong \frac{1}{2} c_x(i) \int_0^i s_y'^2 s_x \, d\alpha \quad - \quad c_x(i) \sum_j S_j s_y^2 s_x$$

TABLE III

For point-to-point imaging in the y (non-bend) plane, Eq. (23), the high-energy limit yields:

$$(y|x_0y_0) \cong -\, c_y(i) \int_0^i c'_x c'_y s_y \, d\alpha \; - \; 2c_y(i) \sum_j S_j c_x c_y s_y$$

$$(y|x_0y'_0) \cong -\, c_y(i) \int_0^i c'_x s'_y s_y \, d\alpha \; - \; 2c_y(i) \sum_j S_j c_x s_y^2$$

$$(y|x'_0y_0) \cong -\, c_y(i) \int_0^i s'_x c_y s_y \, d\alpha \; - \; 2c_y(i) \sum_j S_j s_x c_y s_y$$

$$(y|x'_0y'_0) \cong -\, c_y(i) \int_0^i s'_x s'_y s_y \, d\alpha \; - \; 2c_y(i) \sum_j S_j s_x s_y^2$$

$$(y|y_0\delta) \cong -\, c_y(i) \int_0^i c_y d_x s_y \, d\alpha \; - \; 2c_y(i) \sum_j S_j c_y d_x s_y \; + \; c_y(i) \sum_q \frac{c_y s_y}{f_q}$$

$$(y|y'_0\delta) \cong -\, c_y(i) \int_0^i s_y d_x s_y \, d\alpha \; - \; 2c_y(i) \sum_j S_j d_x s_y^2 \; + \; c_y(i) \sum_q \frac{s_y^2}{f_q}$$

<u>TABLE IV</u>

For parallel-(line)-to-point imaging in the y (non-bend) plane, Eq. (24), the high energy limit yields:

$$(y|x_0 y_0) \cong s_y(i) \int_0^i c'_x c'_y c_y \, d\alpha + 2 s_y(i) \sum_j S_j c_x c_y^2$$

$$(y|x_0 y'_0) \cong s_y(i) \int_0^i c'_x s'_y c_y \, d\alpha + 2 s_y(i) \sum_j S_j c_x s_y c_y$$

$$(y|x'_0 y_0) \cong s_y(i) \int_0^i s'_x c'_y c_y \, d\alpha + 2 s_y(i) \sum_j S_j s_x c_y^2$$

$$(y|x'_0 y'_0) \cong s_y(i) \int_0^i s'_x s'_y c_y \, d\alpha + 2 s_y(i) \sum_j S_j s_x s_y c_y$$

$$(y|y_0 \delta) \cong + \; s_y(i) \int c'_y d' c_y \, d\alpha + 2 s_y(i) \sum_j S_j c_y^2 d_x - s_y(i) \sum_q \frac{c_y^2}{f_q}$$

$$(y|y'_0 \delta) \cong + \; s_y(i) \int s'_y d' c_y \, d\alpha + 2 s_y(i) \sum_j S_j s_y c_y d_x - s_y(i) \sum_q \frac{s_y c_y}{f_q}$$

TABLE V

$$
\begin{bmatrix} x \\ \theta \\ \delta \\ \hline x^2 \\ x\theta \\ x\delta \\ \theta^2 \\ \theta\delta \\ \delta^2 \\ y^2 \\ y\varphi \\ \varphi^2 \end{bmatrix}
=
\left[\begin{array}{ccc|ccccccccc}
c_x & s_x & d_x & (x|x_0^2) & (x|x_0\theta_0) & (x|x_0\delta) & (x|\theta_0^2) & (x|\theta_0\delta) & (x|\delta^2) & (x|y_0^2) & (x|y_0\varphi_0) & (x|\theta_0^2) \\
c'_x & s'_x & d'_x & (\theta|x_0^2) & (\theta|x_0\theta_0) & \text{etc.} & & & & & & \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & c_x^2 & 2s_x c_x & 2c_x d_x & s_x^2 & 2s_x d_x & d_x^2 & 0 & 0 & 0 \\
0 & 0 & 0 & c_x c'_x & c_x s'_x + c'_x s_x & \text{etc.} & & & & & & \\
0 & 0 & 0 & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & &
\end{array}\right]
\begin{bmatrix} x_0 \\ \theta_0 \\ \delta \\ \hline x_0^2 \\ x_0\theta_0 \\ x_0\delta \\ \theta_0^2 \\ \theta_0\delta \\ \delta^2 \\ y_0^2 \\ y_0\varphi_0 \\ \varphi_0^2 \end{bmatrix}
$$

An illustration of how to formulate the second-order matrices discussed in this article.

373-1-A

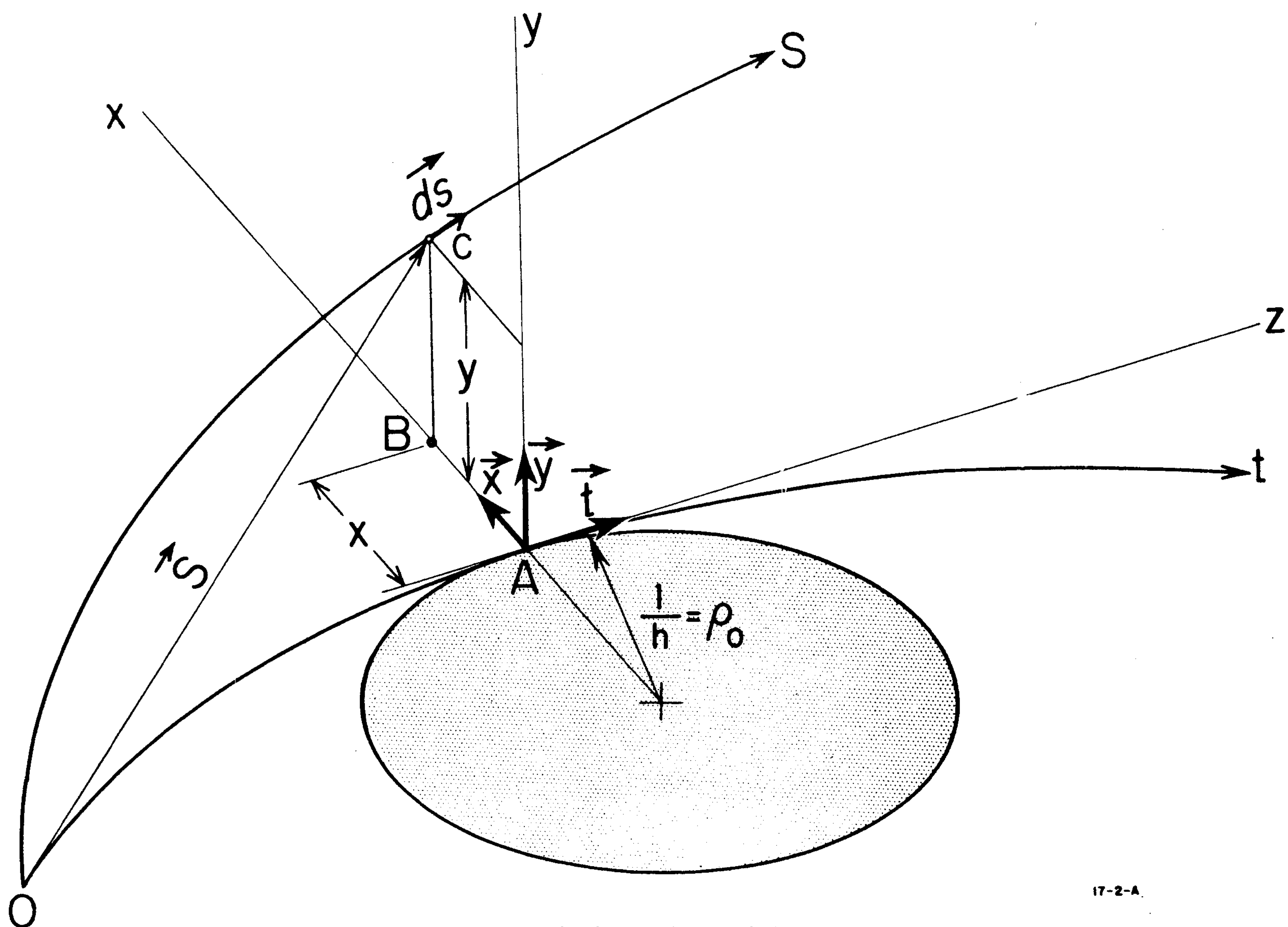


FIG. 1--The curvilinear coordinate system utilized in this system.

$$\tan \psi = \frac{\int_0^i s_x \, d\alpha}{(x_i | x_0' \delta)} \cong \frac{\alpha}{(1 + M_x)}$$

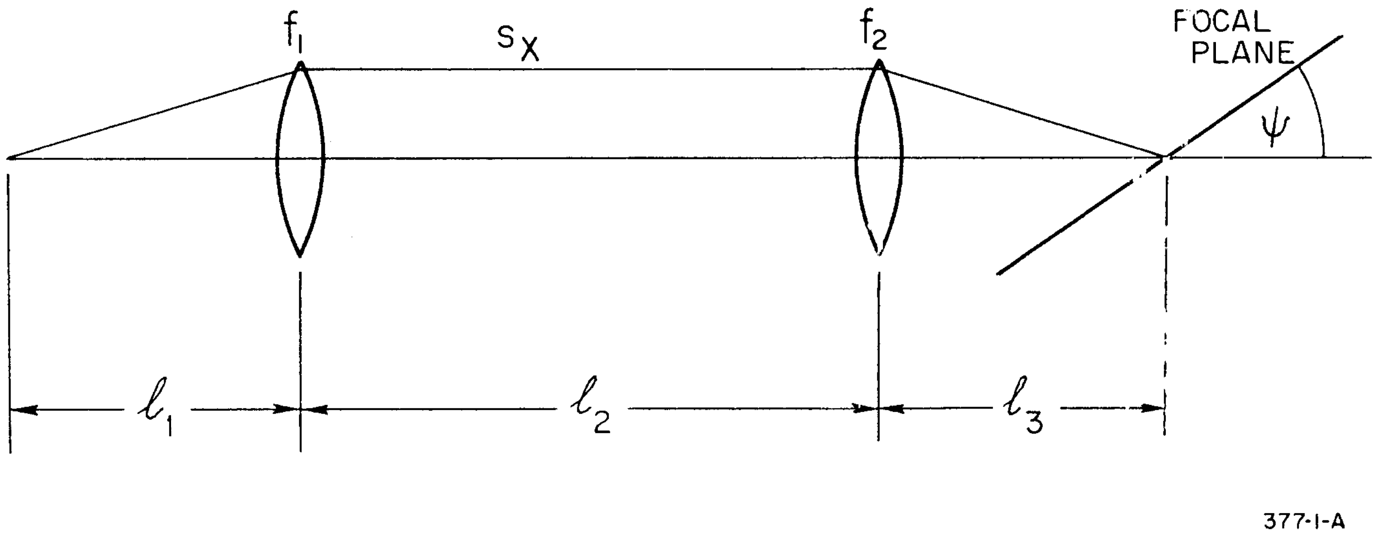
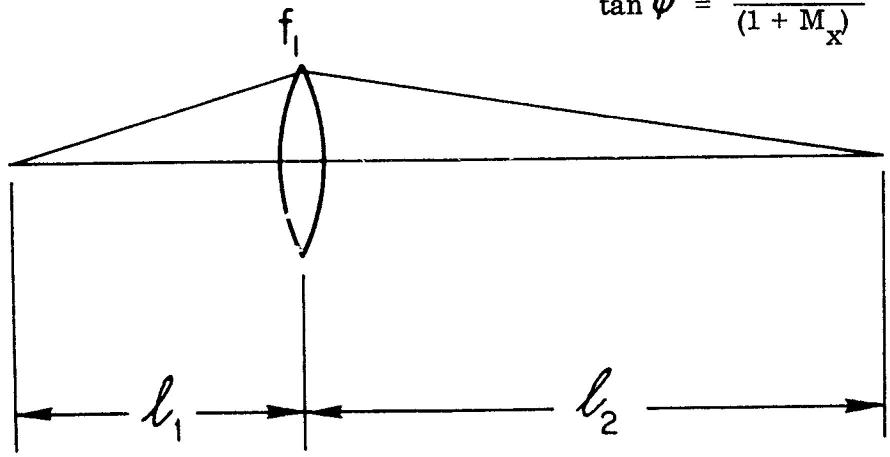


FIG. 2

$$\tan \psi = \frac{K\alpha}{(1 + M_x)}$$



377-2-A

FIG. 3

$$\tan \psi = -\alpha/2 \left[1 + (\ell_1/\ell_3) \right]$$

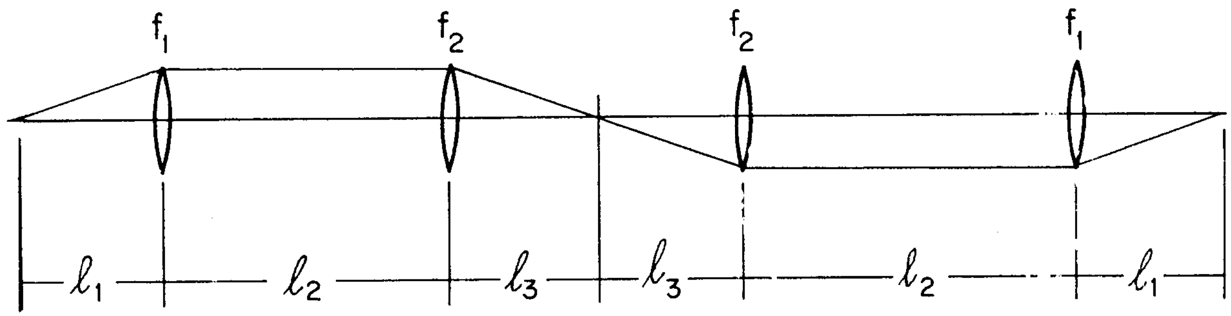


FIG. 4

377-3-A

LIST OF FIGURES