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A NEW NON-SINGULAR INTEGRAL EQUATION FOR TWO-PARTICLE SCATTERING*

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Most recent numerical calculations of scattering and bound states due to the strong interactions make use of driving terms given by the Born approximation for single particle exchanges which are unitarized by solving the N/D equations[1] for the partial waves. This method has two disadvantages. In the first place, unitarization by N/D does not correctly represent the iterated exchanges of single particles (ladder diagrams) in the sense that the result does not quantitatively correspond in the non-relativistic limit to solving the Schroedinger equation for a potential with the same Born approximation.[2] A related difficulty is that, although the N/D equation is explicitly unitary, the phase shifts for higher partial waves do not approach zero as $k^{2\ell+1}$ at threshold, unless subtractions are made in the dispersion relation (which introduces arbitrary parameters and causes difficulties with the asymptotic behavior at infinity) or some other arbitrary prescription[3] is used. We present below a new representation for the two-particle scattering amplitude which avoids both difficulties simultaneously, while preserving the unitarity of the two-particle scattering matrix. A further advantage of the approach given below is that it leads to a non-singular integral equation for the behavior of the scattering matrix off the energy shell, whose kernel is explicitly determined by the (off-shell) Born approximation for the interaction, and which is equivalent to the Schroedinger equation in the non-relativistic case. This off-shell behavior is explicitly separated from the two-body scattering matrix on the energy shell, giving a simple representation for the T matrix needed to construct the kernels for the Faddeev equations[4] for the three-body problem. Hence this representation will allow separate study of those features of the three-body problem coming from the (experimentally determinable) two-body T matrix, and those due to the (model dependent) off-shell behavior.

If we denote the usual non-relativistic wave function in the center-of-mass system by $\tilde{\Psi}_{\underline{k}}(\underline{r}) = \exp(i\underline{k}\cdot\underline{r}) + \tilde{\varphi}_{\underline{k}}(\underline{r})$, or in momentum space by $\Psi_{\underline{k}}(\underline{p}) = \delta(\underline{p}-\underline{k}) + \varphi_{\underline{k}}(\underline{p})$, and make the partial wave expansion $\varphi_{\underline{k}}(\underline{p}) = (1/2\pi^2)\Sigma_\ell(2\ell+1)P_\ell(\hat{\underline{k}}\cdot\hat{\underline{p}})a_k^\ell(p)$, it is easy to show that

$$a_k^\ell(p) = kt_\ell(k)\left[i\int_0^\infty j_\ell(kr)h_\ell(kr)r^2dr + \int_0^\infty [\eta_\ell(kr) - \text{ctn}\,\delta_\ell j_\ell(kr) - w_k(r)]j_\ell(pr)r^2dr\right] \quad (1)$$

where $t_\ell(k) = \exp\left(i\delta_\ell(k)\right)\sin\delta_\ell(k)/k$ and $w_k(r)$ is the real wave function which approaches $\eta_\ell(kr) - \text{ctn}\,\delta_\ell j_\ell(kr)$ outside the range of forces. Hence we can represent the scattered wave function in momentum space by

$$a_k^\ell(p) = \frac{t_\ell(k)f_\ell(k,p)}{p^2 - i\epsilon - k^2} \quad (2)$$

with

$$f_\ell(k,p) = (p/k)^\ell + (p^2 - k^2)\int_0^\infty drkr^2 j_\ell(pr)[\eta_\ell(kr) - \text{ctn}\,\delta j_\ell(kr) - w_k(r)] \quad (3)$$

We see immediately from Eq. (3) that $f_\ell(k,k) = 1$, and that f is real for all real values of p and k, since the integral is real and finite. If we substitute this representation into the Schroedinger equation in momentum space, i.e., $(k^2-p^2)a_k^{\ell}(p) = V_{\ell}(p,k) + (2/\pi)\int_0^{\infty} dq\ q^2 V_{\ell}(p,q)a_k^{\ell}(q)$, we find immediately that

$$t_{\ell}(k)f_{\ell}(k,p) = -V_{\ell}(p,k) - t_{\ell}(k)(2/\pi)\int_0^{\infty} dq \frac{q^2}{q^2 - i\epsilon - k^2} V_{\ell}(p,q)f_{\ell}(k,q) \qquad (4)$$

Hence, from the property $f_{\ell}(k,k) = 1$, the on-shell t matrix is given by

$$t_{\ell}(k) = \frac{-V_{\ell}(k,k)}{1 + (2/\pi)\int_0^{\infty} dq \frac{q^2}{q^2 - i\epsilon - k^2} V_{\ell}(k,q)f_{\ell}(k,q)} \qquad (5)$$

Note a) that $t_{\ell}(k)^{-1}$ = (a principal value integral) - ik making it explicitly unitary, and b) that the numerator is just the Born approximation, thus insuring the correct $k^{2\ell+1}$ threshold behavior for $\delta_{\ell}(k)$ as $k \to 0$ provided only that the denominator remains finite in this limit, as we now demonstrate.

Substituting Eq. (5) back into Eq. (4), we find that f satisfies the integral equation

$$f_{\ell}(k,p) = \frac{V_{\ell}(p,k)}{V_{\ell}(k,k)} + \frac{2}{\pi}\int_0^{\infty} dq \frac{q^2}{q^2 - k^2}\left[\frac{V_{\ell}(k,q)V_{\ell}(p,k)}{V_{\ell}(k,k)} - V_{\ell}(p,q)\right] f_{\ell}(k,q) \qquad (6)$$

Note that the kernel of the equation is finite at $q = k$, so that the equation is of the Fredholm type. For a Yukawa potential, the off-shell Born approximation is proportional to $(1/2pq)Q_{\ell}[(p^2+q^2+\mu^2)/2pq]$, which gives a square-integrable kernel, showing that the equation should be readily soluble by numerical methods. Note further that k occurs only as a parameter in the equation, not as a variable, so that any singularities due to the vanishing of $V_\ell(k,k)$ can cause no essential difficulty. In particular, although $V_\ell(p,k)$ vanishes at threshold, for any reasonable potential such as that considered above, this behavior can be factored as $p^\ell k^\ell$ times a function which is finite for either (or both) p and k equal to zero. Referring back to Eq. (6), we see that this leaves the kernel finite at $k = 0$, the only singularity coming from the inhomogeneous term, which goes as $k^{-\ell}$. Consequently $f_\ell(k,p)$ will also behave as $k^{-\ell}$ as k goes to zero, but if this behavior is inserted in the denominator of Eq. (5), we see that this precisely cancels the k^ℓ behavior of V in the integrand. The denominator is therefore finite at $k = 0$, and the threshold behavior coming from the Born term in the numerator is preserved.

It is clear that if the potential vanishes at some point k_0 other than threshold, and this zero can be factored out as $(p-k_0)^a(k-k_0)^a$, the argument just given shows that $t_\ell(k_0)$ will also vanish at this point. In general we would not expect this zero to be factorable, and also would not expect $t_\ell(k_0)$ to vanish where the Born approximation does. However, if $V_\ell(k_0,k_0) = 0$ and $t_\ell(k_0) \neq 0$, we see that Eq. (4) requires that $1 + (2/\pi)\int_0^\infty dq\, q^2 V_\ell(k_0,q) f_\ell(k_0,q)/(q^2-i\epsilon-k^2) = 0$, thus insuring consistency with Eq. (5).

In order to construct the kernels of the Faddeev equations for the three-body problem (assuming only two-body interactions), we require the general off-shell T matrix $\langle \underline{q}|T(z)|\underline{p}\rangle$ defined by the Lippman-Schwinger[5] equations $T = V + VG_0T$ and $G = G_0 + G_0VG$, which are equivalent to

$G_o T G_o = G - G_o$. For the case at hand the full Green's function G is given by $\int_0^\infty d^3k \Psi_{\underline{k}}^*(\underline{q})\Psi_{\underline{k}}(\underline{p})/(z+i\epsilon-k^2/2m)$ and the free-particle Green's function by $G_o^{-1} = (z+i\epsilon-p^2/2m)\delta(\underline{p}-\underline{q})$, so, if we make the partial wave expansion

$$\langle \underline{q}|T(z)|\underline{p}\rangle = \frac{1}{4\pi^2 m}\Sigma_\ell(2\ell+1)P_\ell(\hat{\underline{p}}\cdot\hat{\underline{q}})T^\ell_{qp}(\zeta) \qquad (7)$$
$$\text{with } \zeta^2 = 2mz$$

the partial-wave T matrices are given in terms of the notation defined above by

$$T^\ell_{qp}(\zeta) = (\zeta^2-p^2)(\zeta^2-q^2)\int_0^\infty dk\,\frac{k^2 a_k^{\ell *}(q)a_k^\ell(p)}{\zeta^2+i\epsilon-k^2} - (\zeta^2-q^2)a_p^{\ell *}(q) - (\zeta^2-p^2)a_q^\ell(p) \qquad (8)$$

Using our representation for $a_k^\ell(p)$ and the completeness relation, the definition of T in terms of G and G_o reduces to

$$T^\ell_{qp}(\zeta) = \int_0^\infty wdk/(\zeta^2+i\epsilon k^2) - \tfrac{1}{2}\left[t_\ell^*(p)f_\ell(p,q) + \int_0^\infty wdk/p^2-i\epsilon-k^2)\right]$$
$$- \tfrac{1}{2}\left[t_\ell(q)f_\ell(q,p) + \int_0^\infty wdk/(q^2+i\epsilon-k^2)\right] \qquad (9)$$

where

$$w(k^2;q,p) = (2/\pi)\sin^2\delta_\ell(k)f_\ell(k,p)f_\ell(k,q) \qquad (10)$$

The symmetry of the T matrix in p and q is guaranteed by the completeness relation, which can be shown to require that

$$t_{\ell}^{*}(p)f_{\ell}(,q) + \int_{0}^{\infty} w dk/(p^2-i\epsilon-k^2) = t_{\ell}(q)f_{\ell}(q,p) + \int_{0}^{\infty} w dk/(q^2+i\epsilon-k^2) \qquad (11)$$

Note that if we put one of the particles on the energy shell, we also obtain a simple result for the half-off-shell T matrix recently discussed by Sobel[6] in connection with p-p bremsstrahlung, namely

$$T_{\zeta p}(\zeta) = -t_{\ell}(\zeta)f_{\ell}(\zeta,p) \qquad (12)$$

Note also that if we make use of the unitarity relation $\sin^2\delta_{\ell}(k) = k^2 t_{\ell}^{*}(k)t_{\ell}(k) = (ik/2)[t_{\ell}^{*}(k) - t_{\ell}(k)] = (ik/2)[t_{\ell}(-k)-t_{\ell}(k)]$, we can[6] convert the dk integration to a contour integral along the real axis with a half-circle at infinity. This allows us to separate out the term coming from the singularity of the real axis and obtain

$$T_{qp}^{\ell}(\zeta) = -t_{\ell}(\zeta)f_{\ell}(\zeta,q)f_{\ell}(\zeta,p) + \text{non-separable terms} \qquad (13)$$

To see that this can be expected to be a dominant contribution to T, we evaluate $f_0(k,p)$ for a wave function which differs from its asymptotic value by $e^{-\mu r}$ using Eq. (3) and find that $f_0(k,p) = 1 + (k^2-p^2)/[(p^2-k^2)^2 + 2\mu^2(p^2+k^2) + \mu^4]$, showing that other contributions from the singularities of w are separated from the dominant singularity on the real axis by at least the range of forces. Since this leading term is separable in p and q, it will reduce the Faddeev equations to (coupled) integral equations in a single variable, which are readily soluble on a computer. It is therefore to be hoped that this representation will provide a good first approximation for at least some aspects of the three-body problem. Finally, it should be emphasized that $t_\ell(k)$ can be determined without ambiguity directly from two-particle scattering experiments, independent of any assumptions (other than that they are of short range) about the nature of the strong interactions. We can therefore construct different local, non-local, or velocity-dependent models for the strong-interactions leading to identical results for $t_\ell(k)$, and study separately the influence of these differing physical assumptions about the strong-interactions on the three-body problem, independently from the (experimentally determinable) two-particle amplitude $t_\ell(k)$ which we adopt.

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5. B. A. Lippman and J. Schwinger, Phys. Rev. 79, 469 (1950).
6. M. I. Sobel, Phys. Rev. 138, B1517 (1965).
7. We see from Eq. (3) that f_ℓ will be odd rather than even in p and k for odd ℓ, but since only the symmetric product $f_\ell(k,q)f_\ell(k,p)$ occurs in the weight function w, this will be a function only of k^2, which is all we need to extend the k integration down to $-\infty$.