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CONSEQUENCES OF C-VIOLATING INTERACTIONS IN η^0 and X^0 DECAYS[*]

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ABSTRACT

In this article several decay modes of the η and the X^{o} (heavy η) meson are discussed under the assumption that there exist C-violating interactions which conserve parity and strangeness. Various speculations about the strength, symmetry, and electromagnetic properties of this interaction are considered to find out how these properties might be determined experimentally from these decays. From available experimental data, several limits for the strength of C-violating interactions are obtained.

INTRODUCTION

Since the discovery of the $\pi^+\pi^-$ decay mode of the long-lived neutral K-meson,[1 2,3] there has been a great deal of speculation concerning the nature of the interaction which is responsible for this CP violating transition. It has been pointed out recently[4,5,6] that a possible explanation is the existence of an interaction H which violates charge conjugation C, but conserves parity P and strangeness. The 2π decay of the K_2^0 occurs then as a second-order process in $H \times H_W$, where H_W is the CP conserving $\Delta S = 1$ weak interaction. The strength of H which accounts for the observed $K_2^0 \rightarrow 2\pi$ branching ratio is estimated to be of the order of electromagnetic interactions, but there is some uncertainty in this result because of the difficulties in calculating higher order processes.

The transformation properties under isospin and unitary spin of the C-violating interaction H are open to conjecture. It has been suggested that a) H might correspond to the same interaction that is responsible for the breakdown of SU_3 symmetry which transforms like the $T = 0$, $Y = 0$ member of the octet;[4] b) H may be part of the electromagnetic interaction of strongly interacting particles.[7,8] A number of experiments have been proposed to test the existence of H and examine its properties.[4-15]

We study here some decay modes of the η and the X^0 mesons under several assumptions about the strength and isospin transformation properties of H. Our object is to find out how these properties might be experimentally determined from these decays.

The existence of an interaction violating C-invariance may lead to an asymmetry between a particle and its anti-particle in a system which is produced from an eigenstate of C (more precisely, CPT). As explained below, if the strength of this interaction lies somewhere between that of electromagnetic and medium-strong interaction, this asymmetry would be expected to be particularly strong for the π^+ and the π^- in the decay $\eta \rightarrow \pi^+\pi^-\pi^0$ which is forbidden by the C-conserving strong interactions. It has been generally assumed that this transition is due to virtual electromagnetic processes which conserve C. The corresponding amplitude is symmetric under the exchange of π^+ and π^-, that is $C = +1$ for the final state. If there exists, in addition, a C-violating transition amplitude which is odd under $\pi^+ - \pi^-$ exchange, i.e., $C = -1$, the interference between these two amplitudes would give rise to a $\pi^+\pi^-$ asymmetry in η decay. The magnitude of this asymmetry depends, therefore, on the relative phase as well as magnitude of the C-conserving and C-violating amplitudes. If we assume CPT invariance, the relative phase is determined by the low energy $\pi - \pi$ interactions; it is 90° (no interference) in the absence of effective strong interactions.[15] Similarly, we consider also the decay modes $X^0 \rightarrow \pi^0 + \pi^- + \pi^+$ and $X^0 \rightarrow \eta\pi^+\pi^-$; the latter is allowed by strong interactions and no large $\pi^+\pi^-$ asymmetry is expected. In addition, C-violation allows the decay mode $\eta, X_0 \rightarrow \pi^0 e^+ e^-$ and $X^0 \rightarrow \eta e^+ e^-$ in second order in electromagnetic interactions. We relate here the branching ratio for these decays to the $\pi^+\pi^-$ asymmetry for several different models of C-violation. Other interesting decays like $X_0, \eta \rightarrow \pi^+\pi^-\gamma$ might also exhibit $\pi^+\pi^-$ asymmetry, but will not be discussed here.[7,9,10]

The topics which we cover have been arranged in the following manner. In Sec. I we consider the matrix element for $\eta \to \pi^+\pi^-\pi^0$ with C violation, and give expressions for the $\pi^+\pi^-$ asymmetry and the rate $\eta \to \pi^0 + e^+ + e^-$ for different assumptions about the properties of H. Most of this discussion is also applicable to the corresponding decay of the X^0 (heavy η), but important differences which arise because it has a heavier mass and different SU_3 properties will be pointed out explicitly. For simplicity we assume that the effect of final state π-π interactions is to give a constant phase between the C-conserving and C-violating amplitudes. In Sec. II we consider in more detail the effects of π-π interaction, in the approximation of including only interactions between pion pairs in the three-pion final state, and restrict the relevant π-π scattering to s and p wave. We give numerical calculations for the $\pi^+\pi^-$ asymmetry using both a scattering length and a resonance approximation for the S-wave, and assume dominance of the ρ meson for the p-wave. Finally, in Sec. III we summarize the main results and conclusions of our work.

I. η-Decay

We will discuss in this section the $\pi^+\pi^-$ asymmetry in the spectrum for the decay $\eta \to \pi^0\pi^+\pi^-$, the branching ratio $\Gamma(\eta \to \pi^0 e^+e^-)/\Gamma(\eta \to \pi^0\pi^+\pi^-)$ and similar transitions for the X_0 under the following two assumptions about the C-violating, but parity and strangeness-conserving interaction H:

a) H conserves isospin.[4]

b) H is part of the electromagnetic interaction.[7]

A few remarks are also made concerning the possibility that H violates isospin but does not correspond to case b).

a) If H conserves isospin I, the final 3π state from η decay obtained through this interaction has $I = 0$, which is totally antisymmetric under the exchange of any pair of pions. The corresponding decay amplitude is therefore a scalar function which is antisymmetric under the exchange of two pion momenta. The simplest form which satisfies this condition is

$$\frac{ic}{m^6}(s-t)(t-u)(u-s) \tag{1}$$

where $s = (p_+ + p_-)^2$, $t = (p_- + p_o)^2$ and $u = (p_+ + p_o)^2$ are the total center of mass energy squares of each pair of pions labeled by their four momenta p_+, p_- and p_o. The parameter c is a dimensionless coupling constant and m is a mass which corresponds to an inverse range due to the interaction H. More generally, c is a symmetric function of s, t, and u. For example, in a model based on a C-violating coupling[9,10,14]

$$g_\eta \vec{\rho}^{\mu}(\eta \partial_\mu \vec{\pi} - \partial_\mu \eta \vec{\pi})$$

the corresponding transition amplitude for $\eta \to \pi^+\pi^o\pi^-$ through ρ exchange (see Fig. 1) is given by Eq. 1 with

$$\frac{c}{m^6} = \frac{g_\eta g_{\rho\pi\pi}}{(s-m_\rho^2)(u-m_\rho^2)(t-m_\rho^2)} \tag{2}$$

where m_ρ is the complex ρ mass, and $g_{\rho\pi\pi}$ the $\rho\pi\pi$ coupling constant.[13] For the η decay, we can neglect in first approximation the kinetic energy of the pions in the denominator of Eq. (2); then $c = g_\eta g_{\rho\pi\pi}$ and $m = \sqrt{m_\rho^2 - (m_\eta - m_\pi)^2}$ in Eq. (1). On the other hand, for the X^0 decay this approximation is not valid, because in this case one of the π-pairs can form a real ρ resonance, i.e., the corresponding center of mass energy can equal the ρ mass.

Let us introduce polar coordinates r, θ in the Dalitz plot of the η to express the pion energies ω_0, ω_+ and ω_- in the rest frame of the η:

$$\begin{aligned}
\omega_0 &= \frac{m_\eta}{3} + r\cos\theta \\
\omega_+ &= \frac{m_\eta}{3} - \tfrac{1}{2} r(\cos\theta + \sqrt{3}\sin\theta) \\
\omega_- &= \frac{m_\eta}{3} - \tfrac{1}{2} r(\cos\theta - \sqrt{3}\sin\theta)
\end{aligned} \tag{3}$$

The boundary of the Dalitz plot is given by the condition $r(\theta) = R\,\chi(\theta)$, where $R = \frac{m_\eta}{3} - m_\pi$ is the kinetic energy of the π's and $\chi(\theta)$ is the solution of the cubic equation

$$\beta\chi^3(\theta)\cos 3\theta + (\beta+1)\chi^2(\theta) - 1 = 0 \quad , \quad \beta = \frac{2m_\eta(m_\eta - 3m_\pi)}{(m_\eta + 3m_\pi)^2} \tag{4}$$

for which $\chi(\pi/3) = 1$. In terms of these variables, the C-violating, $I = 0$ amplitude, Eq. (1), takes the alternative forms

$$ic\left(\frac{2m_\eta}{m^2}\right)^3 (\omega_+ - \omega_-)(\omega_0 - \omega_+)(\omega_- - \omega_0) = 3i\frac{\sqrt{3}}{4}\, c\left(\frac{2m_\eta}{m^2}\right)^3 r^3 \sin 3\theta \tag{5}$$

If we divide the Dalitz plot for the decay $\eta \to \pi^0\pi^+\pi^-$ into sextants $(j-1)\frac{\pi}{3} \leq \theta \leq j\frac{\pi}{3}$, $j = 1$ to 6, we note that the C-violating amplitude in Eq. (1) has the striking property that it changes sign between adjacent sextants. Since the $\pi^+\pi^-$ asymmetry is due to the interference of this amplitude with a C-conserving amplitude which is symmetric under $\pi^+ - \pi^-$ exchange, the sign of this asymmetry also changes between adjacent pairs of charge conjugate sextants.[14,15] This would provide strong evidence for the existence of an $I = 0$, C-violating transition.

We can readily estimate the magnitude of the $\pi^+ - \pi^-$ asymmetry Δ expected from Eq. (1). For orientation we assume that the C-conserving amplitude $\eta \to \pi^+\pi^0\pi^-$ is a constant a and let φ be the relative phase between a and c. If we define Δ to be the difference between $\eta \to \pi^0\pi^+\pi^-$ decays into charge conjugate sextants divided by the sum[14] we have

$$\Delta = 2 \sin\varphi \left(\frac{c}{a}\right) \frac{(2m_\eta)^3}{m^6} \frac{F(m_\eta)}{J(m_\eta)} \tag{6}$$

where

$$J(m_\eta) = \iint d\omega_+ d\omega_- = \frac{\sqrt{3}}{4}\int_0^{\pi/3} d\theta r^2(\theta) \tag{7}$$

and

$$F(m_\eta) = \int\int (\omega_+ - \omega_-)(\omega_+ - \omega_0)(\omega_0 - \omega_-)\, d\omega_+ d\omega_- \tag{8}$$

$$= \frac{3}{40} \int_0^{\pi/3} d\theta\, r^5(\theta) \sin 3\theta$$

The range of integration over the $j = 1$ sextant in the ω_+, ω_- variables in Eqs. (7) and (8) is given by the conditions $\omega_+ + \omega_0 + \omega_- = m_\eta$, $\sqrt{\omega_+^2 - m_\pi^2} + \sqrt{\omega_-^2 - m_\pi^2} \leqq \sqrt{\omega_0^2 - m_\pi^2}$ and $\omega_- \geq \omega_+$, $\omega_0 \geq \omega_-$. For the η meson a good approximation to the cubic Eq. (4) is

$$\chi(\theta) = 1 - \frac{\beta}{2}(\cos 3\theta + 1) \qquad \beta = 0.15 \tag{9}$$

(setting $\beta = 0$ in Eq. (9) corresponds to the non-relativistic limit), hence

$$\frac{F(m_\eta)}{J(m_\eta)} = \frac{3\sqrt{3}}{5\pi} R^3 \left(1 - \frac{3}{2}\beta\right) \tag{10}$$

The magnitude of the asymmetry is sensitive to the range parameter m. In the $\eta\rho\pi$ coupling model, $m \cong 4.6\, m_\pi$ (neglecting the ρ-width) and we obtain

$$\Delta \cong 5 \times 10^{-3} \sin\varphi \left(\frac{g_\eta}{a}\right) \tag{11}$$

If we ascribe the C-violating $I = 0$ amplitude to the $\eta\rho\pi$ coupling, we can also calculate the decay rate for the process[14] $\eta \to \pi^0 + e^+ + e^-$

by means of an effective gauge invariant coupling of the ρ meson and photons $f\partial_\mu \rho_\nu F^{\mu\nu}$. This decay proceeds through the sequence

$$\eta \to \pi^o + \rho^o$$
$$\quad\quad\quad\quad \lfloor\!\!\rightarrow \gamma$$
$$\quad\quad\quad\quad\quad\quad \lfloor\!\!\rightarrow e^+ + e^-$$

illustrated in Fig. 2 and the corresponding Feynman amplitude is given by

$$2(gfe)\,\frac{u(p_-)\not{k}v(p_+)}{(k^2 - m_\rho^2)} \tag{12}$$

where $u(p_-)$ and $v(p_+)$ are the electron and positron Dirac wave functions. Equation (12) can also be obtained from a subtracted dispersions relation assuming that the two-pion intermediate state in the region of the ρ resonance gives the dominant contribution, with $f = \frac{e}{g_{\rho\pi\pi}}$, where $g_{\rho\pi\pi}$ is the $\rho\pi\pi$ coupling constant.[16] Setting $g_{\rho\pi\pi}^2/4\pi \simeq 2$ to fit the ρ-width, we obtain for the corresponding rate $\Gamma(\eta \to \pi^o e^+ e^-) \simeq$

$$\Gamma(\eta \to \pi^o e^+ e^-) = \frac{2}{3\pi}\left(\frac{g_\eta^2}{4\pi}\right)\left(\frac{e^2}{4\pi}\right)^2 I(m_\eta, m_\rho, m_\pi) \tag{13}$$

where

$$I(m_\eta, m_\rho, m_\pi) = m_\eta \int_{m_\pi}^{(m_\eta^2+m_\pi^2)/2m_\eta} d\omega_o \frac{(\omega_o^2 - m_\pi^2)^{3/2}}{\left| m_\eta^2 + m_\pi^2 - 2m_\eta \omega_o - m_\rho^2 \right|^2} \tag{14}$$

In Eq. (14) we have set the electron and positron masses equal to zero. From Eq. (13) we obtain

$$\Gamma(\eta \to \pi^o e^+ e^-) = 20 \left(\frac{g_\eta^2}{4\pi}\right) \text{ev} \tag{15}$$

From SU_3 we have

$$\Gamma(\eta \to 2\gamma) = \frac{1}{3}\left(\frac{m_\eta}{m_\pi}\right)^3 \Gamma(\pi^o \to 2\gamma) .$$

Using the experimental value of the π^o lifetime $\Gamma(\pi^o \to 2\gamma) = 6.3$ ev we get $\Gamma(\eta \to 2\gamma) = 127$ ev and therefore

$$\frac{\Gamma(\eta \to \pi^o e^+ e^-)}{\Gamma(\eta \to 2\gamma)} \cong 0.16 \; \frac{g_\eta^2}{4\pi} \tag{16}$$

Experimentally[17] the upper limit to the branching ratio $\frac{\Gamma(\eta \to \pi^o e^+ e^-)}{\Gamma(\eta \to 2\gamma)}$ is $(0.7 \pm 0.7) \times 10^{-2}$, from which we deduce that

$$\frac{g_\eta^2}{4\pi} \lesssim 4.4 \times 10^{-2} \tag{17}$$

Since the decay rate $\eta \to \pi^0\pi^+\pi^-$ is mainly due to the C-conserving amplitude a we have for the corresponding rate

$$\Gamma(\eta \to \pi^0\pi^+\pi^-) = \frac{6\,a^2\,J(m_\eta)}{(4\pi)^2 m_\eta} \tag{18}$$

where $J(m_\eta)$ is defined in Eq. (7). Hence, the branching ratio

$$R_\eta \equiv \frac{\Gamma(\eta \to \pi^0 e^+ e^-)}{\Gamma(\eta \to \pi^0\pi^+\pi^-)} \tag{19}$$

is proportional to the unknown ratio of coupling constants squared $\left(\frac{g_\eta}{a}\right)^2$ which also appears in the asymmetry Δ in Eq. (6), and we obtain the inequality

$$R \geq K\Delta^2 \tag{20}$$

where

$$K = \frac{1}{18}\,\frac{e^2}{4\pi}\,m_\eta \left(\frac{m^2}{2m_\eta}\right)^6 \frac{J(m_\eta)}{F^2(m_\eta)}\,I(m_\eta, m_\rho, m_\pi) \tag{21}$$

which depends on known parameters. Numerically we find $K \cong 11$. The experimental upper limit for R_η is ~1%.[17] Hence we obtain in this model an upper limit for the asymmetry $\Delta \lesssim 3\%$.

In the next section we shall discuss in more detail the $\pi^+\pi^-$ asymmetry, taking into account more realistic π-π interactions. At this point we remark only that if $m \approx m_\rho$, the C-violating amplitude [Eq. (1)] is strongly suppressed relative to the C-conserving amplitude by the small Q value in η decay. Then the electromagnetic coupling of ρ mesons and photons [Eq. (6)] can give a comparable C-violating $\eta \to \pi^0\pi^+\pi^-$ amplitude through the process

$$\begin{array}{l} \eta \to \pi^0 + \rho^0 \\ \qquad\qquad\ \ \llcorner\!\!\to \gamma \\ \qquad\qquad\qquad\quad \llcorner\!\!\to \pi^+\pi^- \end{array}$$

in which the virtual ρ^0 disassociates into $\pi^+\pi^-$ through a photon (see Fig. 3). The corresponding $\pi^+\pi^-$ asymmetry will be discussed in connection with C-violation in electromagnetic interactions. On the other hand, for the heavier X_0 this suppression is not expected to occur. In particular, if we assume a C-violating $X\rho\pi$ coupling, $g_X \vec{\rho}^{\,\mu}(X\partial_\mu \pi - \partial_\mu X\vec{\pi})$ and neglect the ρ width, we obtain the following estimate for the rate $X^0 \to \pi^+\pi^-\pi^0$ (neglecting the contribution from the C-conserving electromagnetic contribution).

$$\begin{aligned} \Gamma(X^0 \to \pi^0\pi^+\pi^-) &\cong \Gamma(X^0 \to \pi^0\rho^0) + \Gamma(X^0 \to \pi^+\rho^-) + \Gamma(X^0 \to \pi^-\rho^+) \\ &= 3\Gamma(X^0 \to \pi^0 + \rho^0) \qquad\qquad (22) \\ &= \frac{3g_X^2}{2\pi}\,\frac{q^3}{m_\rho^2} \end{aligned}$$

where q is the ρ momentum in the rest frame of the X^0. Numerically

$$\Gamma(X^0 \to \pi^0\pi^+\pi^-) \cong 20 \left(\frac{g_X^2}{4\pi}\right) \text{MeV} \qquad (23)$$

Experimentally the width of the X^0 is found to be $\Gamma(X_0) \lesssim 4$ MeV,[18] while the branching ratio $\frac{\Gamma(X^0 \to \pi^+\pi^- \text{ neutrals})}{\Gamma(X^0)} \sim 5 \pm 4\%$.[19] Hence we obtain an upper limit for g_X ,

$$\frac{g_X^2}{4\pi} \lesssim 10^{-2} \tag{24}$$

which is comparable with Eq. (17) for $g_\eta^2/4\pi$.

Actually there are theoretical reasons to expect that $\Gamma(X_o)$ is much smaller than 4 MeV, because the electromagnetic decay $X^o \to \pi^+\pi^-\gamma$ is observed to occur in 20% of all X^o decays. Brown and Faier[20] and Dalitz and Sutherland[21] have estimated that $\Gamma(X_o) \sim 0.1$ MeV, which would imply

$$\frac{g_X^2}{4\pi} \approx 2.5 \times 10^{-4} \tag{25}$$

This would impose a severe limit on the possibility of an I conserving C-violating interaction. Note that if X is an SU_3 singlet, the $X\rho\pi$ coupling can be part of the SU_3 invariant coupling

$$g_X X_o \left\{ \vec{\rho}\vec{\pi} + \left(\frac{1}{\sqrt{3}}\,\omega + \sqrt{\frac{2}{3}}\,\varphi \right)\eta + K^*K \right\}$$

Hence the argument that C-conservation in strong interaction is due to internal symmetries[22] is not valid in this case.

Likewise we can calculate the decay rate for $X^o \to \pi^o e^+e^-$ via the ρ, and $X^o \to \eta e^+e^-$ via the ω and φ. We estimate the ω contribution using SU_3 symmetry to relate the ρ-γ and ω-γ coupling constants:

$f_{\omega\gamma} = \frac{1}{3} f_{\rho\gamma}$. (Similar results are obtained for the φ.)

$$\Gamma(X^0 \to \pi^0 e^+ e^-) = \frac{2}{3\pi}\left(\frac{g_X^2}{4\pi}\right)\left(\frac{e^2}{4\pi}\right)^2 I(m_X, m_\rho, m_\pi) \tag{26}$$

$$\Gamma_\omega(X^0 \to \eta e^+ e^-) = \frac{1}{27} \quad \frac{2}{3\pi}\left(\frac{g_X^2}{4\pi}\right)\left(\frac{e^2}{4\pi}\right)^2 I(m_X, m_\omega, m_\eta) \tag{27}$$

where I is the integral given in Eq. (14). Note that in the decay $X^0 \to \pi^0 e^+ e^-$ through a ρ, the $e^+ - e^-$ energy spectrum would exhibit an enhancement at the ρ-mass (see integrand of Eq. (14) for I). This enhancement would rule out a direct C-violating coupling of the photons (see case b).

Equations (26) and (27) give the following numerical estimates

$$\Gamma(X^0 \to \pi^0 e^+ e^-) \cong 2.2 \times 10^3 \left(\frac{g_X^2}{4\pi}\right) \text{ev} \tag{28}$$

and

$$\Gamma(X^0 \to \eta e^+ e^-) \cong 3 \times \left(\frac{g_X^2}{4\pi}\right) \text{ev} \tag{29}$$

Using Eq. 22 for $\Gamma(X^0 \to \pi^0 \pi^+ \pi^-)$ we arrive at

$$\frac{\Gamma(X^0 \to \pi^0 e^+ e^-)}{\Gamma(X^0 \to \pi^0 \pi^+ \pi^-)} \sim 1.1 \times 10^{-4} \tag{30}$$

and from the experimental upper limit[19] for the branching ratio $\Gamma(X \to 3\pi)/\Gamma(X \to \text{total}) \leq 5 \pm 4\%$.

$$\frac{\Gamma(X^o \to \pi^o e^+ e^-)}{\Gamma(X^o \to \eta\pi\pi)} \leq 5.5 \times 10^{-6} \tag{31}$$

which is independent of the uncertainty in the X^o width. If this branching ratio is much larger than the limit given here it would give a strong indication that C is violated by electromagnetic interactions.

(b) Corresponding to the assumption that the C-violating interaction H is an electromagnetic interaction,[7,8] we consider effective $\eta\pi^o\gamma$, $X^o\pi^o\gamma$ and $X^o\eta\gamma$ couplings as in the previous case, but now the intermediate ρ, ω and φ states are forbidden, if C is conserved by the strong interaction. If the assumed $C = +1$ component of the electromagnetic current transforms under SU_3 like the ordinary $C = -1$ current, the $\eta\pi^o\gamma$ vertex is suppressed, as pointed out by Cabbibo;[22] however, the corresponding argument is not applicable to the $X_o\pi^o\gamma$ coupling. We take the gauge invariant form

$$\frac{e}{m^2}\, \partial_\mu \eta \partial_\nu \pi^o F^{\mu\nu}. \tag{32}$$

where m is an undetermined mass, $F^{\mu\nu}$ is the electromagnetic field and e is the electric charge. Then in addition to the process $\eta \to \pi^o e^+ e^-$ this interaction gives rise to an $\eta \to \pi^+\pi^o\pi^-$ decay amplitude B corresponding to the second order process $\eta \to \pi^o + \gamma$, $\gamma \to \pi^+ + \pi^-$ in which the virtual photon converts into a $\pi^+\pi^-$ pair. From Eq. (32) we obtain

$$B = \frac{1}{2} i \frac{e^2}{m^2} (t - u)$$

$$= ie^2 \frac{m_\eta}{m^2} (\omega_- - \omega_+) \qquad (33)$$

$$= \sqrt{3}\, ie^2 \frac{m_\eta}{m^2} r \sin \theta$$

which is antisymmetric under $\pi^+ - \pi^-$.

If we assume again that the C-conserving amplitude for $\eta \to \pi^0\pi^+\pi^-$ is a constant $ae^{i\varphi}$, we obtain the following simple expression for the $\pi^+\pi^-$ asymmetry $\Delta = \frac{R - L}{R + L}$ where $R(L)$ correspond to the decay rate into the region of the Dalitz plot for which $\omega_+ \geq \omega_- (\omega_+ \leq \omega_-)$,

$$\Delta = \sin \varphi \frac{2e^2 m_\eta}{3m^2 a} \frac{P(m_\eta)}{J(m_\eta)} \qquad (34)$$

where

$$P(m_\eta) = \int_{\omega_- > \omega_+} (\omega_- - \omega_+) d\omega_+ d\omega_-$$

$$= \frac{1}{2} \int_0^\pi d\theta r^3(\theta) \sin \theta \qquad (35)$$

and $J(m_\eta)$ is given by Eq. (7). In the approximation that the η-decay pions are treated non-relativistically we obtain

$$\Delta = \sin\varphi \; \frac{32}{\sqrt{3}} \left(\frac{e^2}{4\pi}\right) \frac{m_\eta}{am^2} R \tag{36}$$

The decay rate for the process $\eta \to \pi^0 e^+ e^-$ obtained from Eq. (32) is

$$\Gamma(\eta \to \pi^0 e^+ e^-) = \frac{1}{3\pi} \left(\frac{e^2}{4\pi}\right)^2 \frac{m_\eta}{m^4} H(m_\eta) \tag{37}$$

where

$$H(m_\eta) = \int_{m_\pi}^{(m_\eta^2+m_\pi^2)/2m_\eta} d\omega_0 (\omega_0^2 - m_\pi^2)^{3/2} \tag{38}$$

Again, the branching ratio

$$R_\eta = \frac{\Gamma(\eta \to \pi^0 e^+ e^-)}{\Gamma(\eta \to \pi^0 \pi^+ \pi^-)}$$

is related to the asymmetry Δ by Eq. (20), where now the constant K is

$$K = \frac{1}{2} \frac{H(m_\eta) J(m_\eta)}{P^2(m_\eta)} \sim 70 \tag{39}$$

independent of the unknown range parameter m. The experimental upper limit[17] of 1% for R_η implies $\Delta \lesssim 1.3\%$. The corresponding results for the decay $X^0 \to \pi^0 \pi^+ \pi^-$ and $X^0 \to \pi^0 e^+ e^-$ are obtained by replacing

m_X for m_η in Eqs. (34) and (37). If we estimate the rate $X^0 \to \pi^0\pi^+\pi^-$ entirely from the C-violating transition due to the $X_0\pi^0\gamma$ coupling we obtain

$$\Gamma(X^0 \to \pi^0\pi^+\pi^-) = \frac{1}{\pi}\left(\frac{e^2}{4\pi}\right)^2 \frac{m_X}{m} Q(m_X) \tag{40}$$

where

$$Q(m_X) = \iint d\omega_+ d\omega_- (\omega_+ - \omega_-)^2 |F_\pi(s)|^2 \tag{41}$$

$F_\pi(S)$ is the pion form factor, approximated by

$$\frac{m_\rho^2}{m_\rho^2 - s - i\Gamma_\rho}$$

and s is the total $\pi^-\pi^+$ energy squared in their own center-of-mass. From this we obtain

$$\frac{\Gamma(X^0 \to \pi^0 e^+ e^-)}{\Gamma(X^0 \to \pi^0\pi^+\pi^-)} \sim \frac{1}{5} \tag{42}$$

Using the experimental upper limit of the branching ratio of $X^0 \to \pi^0\pi^+\pi^-$ and Eq. (42) we obtain the condition

$$\frac{\Gamma(X^0 \to \pi^0 e^+ e^-)}{\Gamma(X^0 \to \eta\pi^+\pi^-)} \lesssim 1\% \tag{43}$$

We note finally that many results of this section are also applicable to the case that H violates isospin, but does not arise from couplings to photons. For example, we could have an $\eta\rho\pi$ coupling in which only the neutral ρ enters, leading to $\Delta I = 2$, C-violating transitions. Such a possibility could be distinguished experimentally since in this case the asymmetry Δ, Eq. (36), is $\left(\frac{g_{\eta\rho\pi}}{e}\right)^2 \sim 270$ larger than for C-violation in electromagnetic interactions.

II. Effect of the Final State π-π Interaction

In this section we consider a more realistic approximation to the $\eta, X^0 \to \pi^+\pi^-\pi^0$ transition amplitude $\mathcal{M}$ which takes into account the effect of S- and P-wave final state π-π interactions. In the approximation that only a pair of pions interact at a time, we write

$$\mathcal{M}(s,tu) = \frac{a}{D_0(s)} + b\left[\frac{(s-u)}{D_1(t)} + \frac{(s-t)}{D_1(u)}\right]$$
$$+ ic\left[\frac{(t-u)}{D_1(s)} + \frac{(u-s)}{D_1(t)} + \frac{(s-t)}{D_1(u)}\right] \tag{2.1}$$
$$+ id\,\frac{(t-u)}{D_1(s)}$$

where

$$D_\ell(s) = \exp - \frac{s}{\pi}\int_{4m_\pi^2}^{\alpha} \frac{\delta_\ell(s')}{s'(s'-s-ie)}\,ds' \tag{2.2}$$

is the familiar function for two-particle final state interactions; the subscript ℓ denotes the ℓ-th partial wave and δ_ℓ is the corresponding phase shift. The first and second terms on the right-hand side of Eq. (2.1) are the C-conserving total isospin $I = 1$ amplitudes, corresponding to S-wave, $I = 0$, π-π scattering, and P-wave π-π scattering respectively; the third and fourth terms are the C-violating amplitudes in the total $I = 0$ and $I = 2$ states including the effects of P-wave π-π scattering. We have assumed that the S-wave π-π scattering in the $I = 2$ state can be neglected, since there is no evidence that it is large at low energy.[23]

CPT invariance requires that the coefficients a, b, c and d be real.

We note that application of the D-function, Eq. (2.2), to include final state interactions can be justified only in the case that the π-π scattering is purely elastic, or provided there is a resonance. However, we shall also use it here in the scattering length approximation for the $I = 0$, S-wave π-π scattering, which includes charge exchange scattering; it can be shown that it gives similar results to a more detailed treatment.

For our numerical calculations we will make two approximations for the S-wave, $I = 0$, π-π amplitude: 1) a constant scattering length α and 2) a Breit-Wigner resonance.[24] For the P-wave π-π amplitude we use a Breit-Wigner resonance corresponding to the ρ-meson. Thus we set

$$\frac{1}{D_0^{(1)}(s)} \propto \frac{1}{1 - i\alpha q(s)} \tag{2.3}$$

$$\frac{1}{D_0^{(2)}(s)} \propto \frac{1}{m_\sigma^2 - s - i\gamma_\sigma \frac{q(s)}{\sqrt{s}}} \tag{2.4}$$

and

$$\frac{1}{D_1(s)} \propto \frac{1}{m_\rho^2 - s - i\gamma_\rho \frac{q^3(s)}{\sqrt{s}}} \tag{2.5}$$

where $q(s) = \frac{1}{2}\left(s - 4m_\pi^2\right)^{\frac{1}{2}}$, and m_σ and γ_σ correspond to the position and reduced width of an assumed S-wave $I = 0$ resonance. We have chosen values of these parameters to fit the experimental η-decay spectrum,[25] see Fig. 4. For the mass and width of the ρ-meson we take $m_\rho = 750$ and

$\Gamma_\rho = 110$ MeV, corresponding to a reduced width $\gamma_\rho = 0.15$.

Note that for the decay of the η, the kinematics restrict the π-π systems well below the ρ-peak, so that Eq. (2.1) can be approximated by

$$\mathcal{M}_0(s,t,u) = \frac{a}{D_0(s)} + 3b(s-s_0) + \frac{ic}{m^6}(s-u)(u-t)(t-s) + id(u-t) \qquad (2.6)$$

where $3s_0 = m_\eta^2 + 3m_\pi^2$ and $m = \sqrt{m_\rho^2 - (m_\eta - m_\pi)^2}$, neglecting the ρ-width.

The $\pi^+ - \pi^-$ asymmetry in η-decay is determined by the interference between the S-wave π-π scattering amplitude and the C-violating amplitudes, i.e., from Eq. (2.6) we see that this asymmetry is proportional to

$$\mathrm{Im}\left(\frac{1}{D_0(s)}\right)\left[\frac{c}{m^6}(s-u)(u-t)(t-s) + d(u-t)\right] \qquad (2.7)$$

In Tables I and II we present some numerical results for the transition probabilities, P_j, $j = 1$ to 6, into the sextants of the Dalitz plot,

$$P_j = \iint_{j\text{-th sextant}} |\mathcal{M}|^2 d\omega_+ d\omega_- \Big/ \iint |\mathcal{M}|^2 d\omega_+ d\omega_-$$

for several values of the C-violating parameter in the scattering length and in the resonance approximation for the S-wave. These values have been chosen to give a $\pi^+\pi^-$ asymmetry of 2 to 5%. We note that the π^+-π^- asymmetry for all three pairs of conjugate sextants is roughly comparable unless there is a cancellation between the $I = 0$ and the $I = 2$ C-violating amplitudes. If the $I = 0$ amplitude dominates, the

π^+-π^- asymmetry for the middle pair of sextants, $j = 2$ and 5 changes sign.

In our approximation, we can also calculate the branching ratio $B = \Gamma(\eta \rightarrow 3\pi^0)/\Gamma(\eta \rightarrow \pi^0\pi^+\pi^-)$. Neglecting the π^0 - π^+ mass difference, and assuming an S-wave scattering length $a = 0.5$ and 1.0, we obtain $B = 1.4$, while for an S-wave π-π resonance at $m_\sigma = 420$ MeV with $\Gamma_\sigma = 100$ MeV, we obtain[26] $B = 1.3$. Experimentally[25] $B = 0.90 \pm 0.24$.

III. Summary and Conclusion

Let us summarize the main results obtained in this paper.

In Sec. I, Part a, we discuss some consequences of the possible existence of a C-violating interaction H which conserves isospin. Its effect of the η^o and X^o decay was studied by introducing respectively the C-violating coupling $g_\eta \eta \vec{\rho} \cdot \vec{\pi}$, which also violates SU_3 symmetry, and $g_{X_o} X_o \left[\vec{\rho}\vec{\pi} + \left(\sqrt{\frac{1}{3}}\, \omega + \sqrt{\frac{2}{3}}\, \varphi \right) \eta + K^* K \right]$ which is SU_3 invariant. Using the relation between the width of $\pi^o \to 2\gamma$ and $\eta \to 2\gamma$ given by SU_3, and the present experimental upper limit[17] for the ratio $\Gamma(\eta \to \pi^o e^+ e^- / \Gamma\eta \to 2\gamma)$, we arrive at the condition $\frac{g_\eta^2}{4\pi} \lesssim 4 \times 10^{-2}$, Eq. (17). Correspondingly, this leads to an upper limit to the $\pi^+\pi^-$ asymmetry in the decay mode $\eta \to \pi^o \pi^+ \pi^-$ of the order of a few percent. From the upper limit of the experimental branching ratio[19] $\Gamma(X^o \to \pi^o \pi^+ \pi^- / \Gamma(X^o \to \text{all}) \sim 5 \pm 1\%$, and the width of X^o, we obtain the condition $\frac{g_X^2}{4\pi} \lesssim 10^{-2}$, Eq. (24). On the other hand, from a theoretical estimate[20,21] for $\Gamma(X^o \to \pi^o \pi^+ \pi^-) \sim 0.1$ MeV, we get a more severe restriction on the magnitude of the C-violating coupling constant $g_X, g_X^2/4\pi \lesssim 2.5 \times 10^{-4}$, Eq. (25). Furthermore, independent of the assumed width for the X^o, we arrive at the condition, Eq. (31), that $\Gamma(X^o \to \pi^o e^+ e^-)/\Gamma(X^o \to \eta\, \pi^+ \pi^-) \lesssim 5 \times 10^{-6}$ if the C-violating interaction H conserves isospin. This last relation is interesting because the detection of the mode $X^o \to \pi^o e^+ e^-$ with a much higher branching ratio would give a good indication that C is violated by electromagnetic interaction. From the present analysis of η and X^o decays, we conclude that there is no large C-violation in either SU_3-conserving or SU_3-violating interactions.

Corresponding to the assumption that the C-violating interaction H is an electromagnetic interaction, Part b, we introduce $\eta^0\pi^0\gamma$ and $X^0\pi^0\gamma$ coupling. The experimental upper limit[17] for $\eta^0 \to \pi^0 e^+ e^-$ gives an upper limit of ~1% to the $\pi^+\pi^-$ asymmetry in $\eta^0 \to \pi^0\pi^+\pi^-$ due to the intermediate photon converting into a $\pi^+\pi^-$ pair. From the upper limit[19] for $\Gamma(X^0 \to \pi^0\pi^+\pi^-)$ we similarly obtain $\Gamma(X^0 \to \pi^0 e^+ e^-)/\Gamma(X^0 \to \eta^0\pi^+\pi^-) \lesssim 1\%$, Eq. (43).

Finally, we discuss briefly the possibility that H violates isospin, but it is not connected with the electromagnetic interaction. In this case, it is possible to have a large $\pi^+\pi^-$ asymmetry in $\eta \to \pi^0\pi^+\pi^-$ decay without any appreciable branching ratio for the mode $\eta \to \pi^0 e^+ e^-$.

In Sec. II we consider a more realistic approximation for the final state π-π interactions in the decay modes $\eta, X^0 \to \pi^0\pi^+\pi^-$. In our model, CPT invariance implies that the $\pi^+\pi^-$ asymmetry is proportional to the S-wave π-π scattering cross section as well as to the magnitude of the C-violating $\Delta I = 0$ and 2 amplitudes, Eq. (2.7). In Tables I and II we give the relative population of the sextants of the Dalitz plots for a scattering length and for a resonance approximation to the S-wave amplitude respectively, which are fitted to the experimental η-decay spectrum. We note that if the C-violating amplitude conserves isospin, the $\pi^+\pi^-$ asymmetry changes sign for the middle pair of sextants, but even a small $\Delta I = 2$ admixture can alter this effect appreciably.

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TABLE I

α_o	b/a	c/a	d/a	P_1	P_2	P_3	P_4	P_5	P_6
0.5	0.085	0	0.03	0.085	0.163	0.267	0.258	0.147	0.080
		90	0.0	0.085	0.150	0.268	0.258	0.159	0.080
		90	0.03	0.089	0.158	0.272	0.253	0.150	0.078
		-90	0.03	0.083	0.168	0.261	0.262	0.144	0.081
1.0	0.106	0	0.03	0.085	0.167	0.279	0.263	0.158	0.071
		90	0.0	0.083	0.142	0.280	0.263	0.158	0.073
		90	0.03	0.091	0.158	0.287	0.255	0.142	0.067
		-90	0.03	0.079	0.176	0.269	0.270	0.129	0.076

Transition probabilities P_j $j = 1$ to 6 into the sextants of the Dalitz plot for several values of the $I = 0$ and $I = 2$ C-violating parameter c and d, defined in Eq. (2.7), using two scattering lengths α and a corresponding C-conserving amplitude b which fits the experimental η decay spectrum, Fig. 4. The units for these parameters are in pion masses.

TABLE II

c/a	d/a	P_1	P_2	P_3	P_4	P_5	P_6
0	0.03	0.094	0.156	0.265	0.252	0.141	0.091
90	0.0	0.094	0.145	0.267	0.250	0.152	0.092
90	0.03	0.096	0.152	0.274	0.243	0.144	0.090
-90	0.03	0.093	0.160	0.257	0.259	0.138	0.092

Same as Table I, but using a resonance approximation for the $\pi\pi$ $I = 0$, S-wave which fits the η-decay spectrum, Fig. 4, corresponding to $m_\sigma = 420$ MeV and $\Gamma_\sigma = 100$ MeV, see Eq. (2.4).

FIGURE CAPTIONS

1. Decay $\eta \to \pi^+\pi^-\pi^0$ through a virtual ρ^0 which violates C. If isospin is conserved, there are also corresponding processes via virtual ρ^+ and ρ^- .

2. Decay $\eta \to \pi^0 e^+ e^-$ through a virtual photon coupled to ρ^0 .

3. Decay $\eta \to \pi^0\pi^+\pi^-$ through a virtual photon coupled to ρ^0 .

4. Comparison of the experimental π^0 spectrum in η decay, reference 25, with a) phase space, b) the scattering length approximation, with $\alpha = 0.5\ m_\pi^{-1}$, and c) the resonance approximation, with $m_\sigma = 420$ MeV and $\Gamma_\sigma = 100$ MeV.

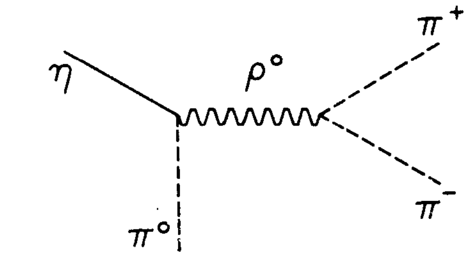

FIG. 1

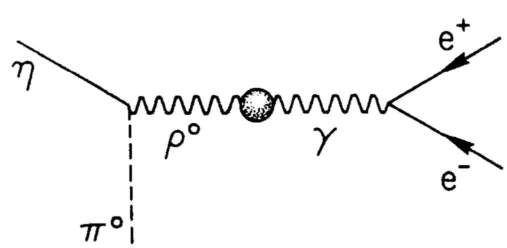

FIG. 2

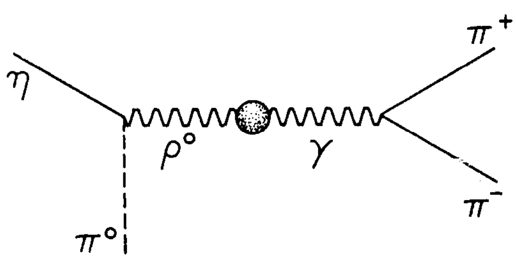

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FIG. 3

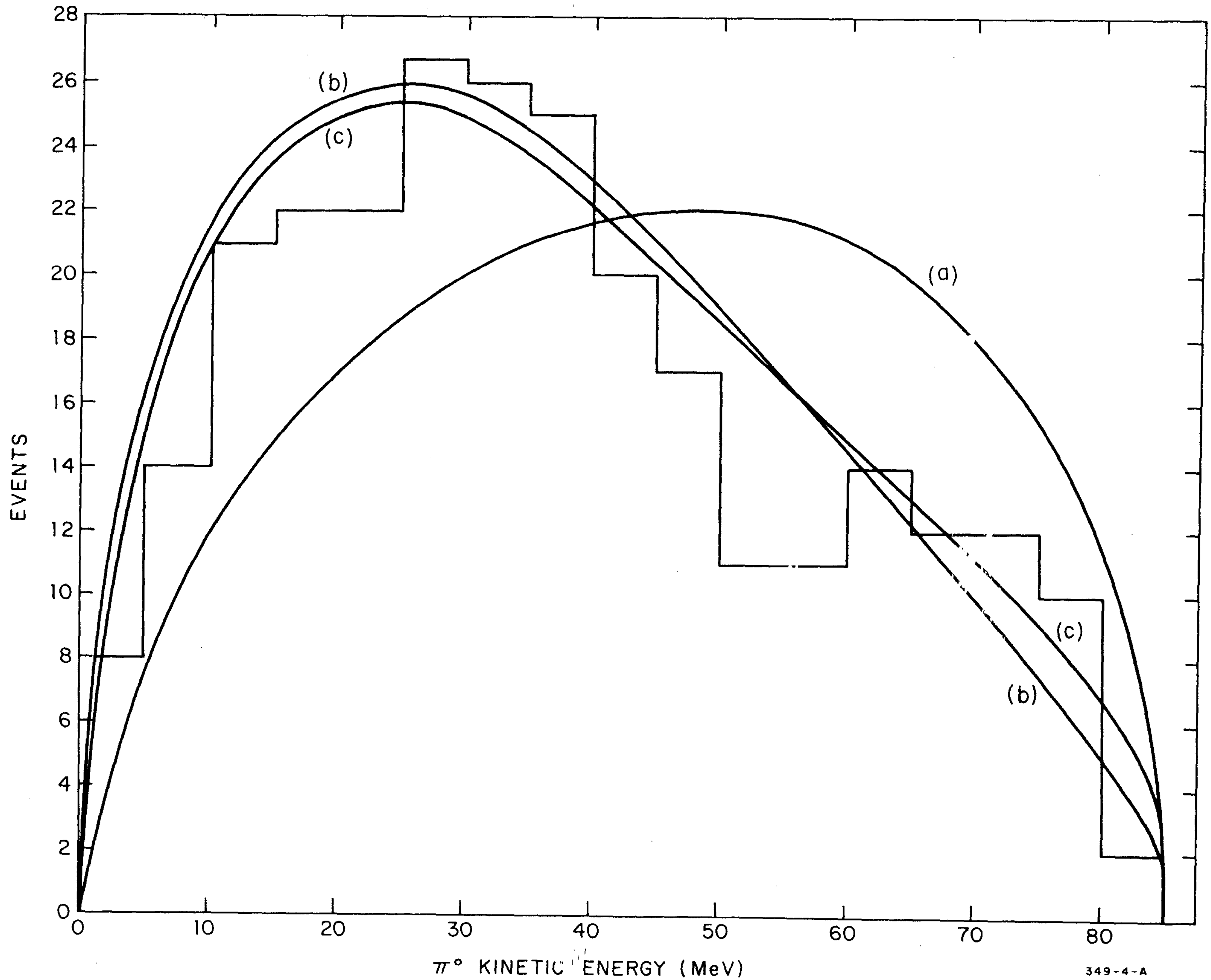

FIG. 4