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GROUP THEORY AND THE HYDROGEN ATOM*

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ABSTRACT

The internal O(4) symmetry group of the non-relativistic hydrogen atom is discussed and used to relate the various approaches to the bound state problems. A more general group O(1,4) of transformations is shown to connect the various levels, which appear as basis vectors for a continuous set of unitary representations of this non compact group.

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I. INTRODUCTION

There has been great interest recently in the possible application of group theory to the strongly interacting particles. Not only do certain systems possess obvious symmetries which allow a classification of their spectra, but it has also been suggested[1] that one look for certain transformations which allow passing from one level to another and thus get a new insight into the structure of the system.

In non-relativistic quantum mechanics, several examples of such behavior are known and it may be worthwhile to investigate in detail a specific one. We have chosen to undertake such a study for the Coulomb potential which seems very well suited for such an investigation.

The classical treatment of the subject consists of solving explicitly the Schrodinger equation in coordinate space by means of hypergeometric functions. In 1926 W. Pauli[2], found the spectrum of the Kepler problem in a very elegant way by the use of the conservation of a second vector besides the angular momentum. A few years later, V. Fock[3] explained the degeneracy of the levels in terms of a symmetry group isomorphic to the one of rotations in a four-dimensional space O_4, and a few months later V. Bargmann[4] related the two approaches explaining further how, in the Coulomb case, the separation of variables in parabolic coordinates was linked with the new, conserved vector — a relation well known in classical mechanics. Later on the rotational invariance was used, for instance, by J. Schwinger to construct the Green function of the problem.

Thus the Coulomb problem is interesting for its O_4 invariance, but it has been recently remarked that one can operate in the Hilbert space of bound states with a still larger group, isomorphic to the de-Sitter group, O(1,4), in such a way that one thus gets an irreducible infinite dimensional unitary representation of this non-compact group.

Our aim has thus been twofold. We first review the symmetry group of the system, describing succinctly the methods discussed above. We note some further relations which were implicit in the works quoted above. Actually, following a remark of Alliluev[6] we shall even generalize the problem to an arbitrary number of dimensions. The larger group, O(1,4), is then introduced in an heuristic way. The new terminology suggested for this kind of superstructure is "Physical Transformation Group." We shall write the explicit realization of this group as a set of unitary operations in the Hilbert space of bound states and prove irreducibility using the infinitesimal generators. Finally, it is suggested that the type of considerations used can be generalized to obtain special types of unitary representations for non-compact groups. This paper is mainly concerned with the problem of bound states. We hope to consider in the future the case of scattering states.

Several recent lectures given at Stanford by Professor Y. Ne'eman were the inspiration for this work. It is a pleasure to thank him for his stimulation. It is clear that many of the results were known to him and certainly to many other physicists. We apologize in advance for giving only a very sketchy bibliography.

II. THE SYMMETRY GROUP

A. The Infinitesimal Method[2]

We want to solve the Schrodinger equation for the Coulomb potential

$$\left(-\frac{\hbar^2\Delta}{2\mu} - \frac{k}{r}\right)\Psi(x) = E\Psi(x) \tag{1}$$

with Δ the Laplacian

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}, \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2},$$

μ is the reduced mass and, in the case of an hydrogen-like atom, $k = Ze^2$. Let $p_i = \frac{\hbar}{i}\frac{\partial}{\partial x_i}$, then due to the invariance of Eq. (1) under spatial rotation the angular momentum

$$L_{ij} = x_i p_j - x_j p_i \qquad L_k = \epsilon_{kij} L_{ij} \tag{2}$$

is conserved, and it is possible to separate the equation using polar coordinates. However, it is known that in the Kepler problem, the following three vector

$$\vec{v} \times \vec{L} - k\frac{\vec{r}}{r}$$

is also a constant of the motion, $\vec{v}$ is the velocity, $\vec{L}$ the angular momentum, and $\vec{r}$ the position vector, $\vec{r} = (x_1, x_2, x_3)$. Pauli simply used the correspondence principle and investigated the commutation relations of the Hermitian part of the previous vector, i.e.,

$$\vec{M} = \frac{1}{2\mu}\vec{p} \times \vec{L} - \frac{1}{2\mu}\vec{L} \times \vec{p} - k\frac{\vec{r}}{r}. \tag{3}$$

The commutation relations of $\vec{L}$, $\vec{M}$ and the Hamiltonian H are

$$
\begin{aligned}
[H, L_i] &= 0 \\
[H, M_i] &= 0 \\
[L_j, L_k] &= i\hbar \epsilon_{jk\ell} L_\ell \qquad (4) \\
[L_j, M_k] &= i\hbar \epsilon_{jk\ell} M_\ell \\
[M_j, M_k] &= \frac{\hbar}{i} \epsilon_{jk\ell} L_\ell \frac{2}{\mu} H
\end{aligned}
$$

and

$$L \cdot M = M \cdot L = 0 \qquad (M^2 - k^2) = \frac{2}{\mu} H(L^2 + \hbar^2) \quad . \qquad (5)$$

Relations (4) suggest the consideration of a subspace belonging to the eigenvalue $E(E < 0)$ of the Hamiltonian as L and M commute with it. In this subspace it is meaningful to introduce the operator

$$\tilde{M}_i = \sqrt{\frac{\mu}{-2E}}\ M_i$$

Then, as a result, $\frac{L + \tilde{M}}{2}$ and $\frac{L - \tilde{M}}{2}$ build up two commuting sets of operators, each one satisfying the commutation relations of ordinary angular momentum; hence $\left(\frac{L + \tilde{M}}{2}\right)^2 = \hbar^2 j_1(j_1 + 1)$ and $\left(\frac{L - \tilde{M}}{2}\right)^2 = \hbar^2 j_2(j_2 + 1)$. But according to Eq. (5), $L \cdot \tilde{M} = \tilde{M} \cdot L = 0$ so that

$$\left(\frac{L + \tilde{M}}{2}\right)^2 = \left(\frac{L - \tilde{M}}{2}\right)^2 \quad ,$$

i.e., $j_1 = j_2$. It is not clear at this point whether $j = j_1 = j_2$ has to be limited to integer values or can also take half-integer values. Assuming for a moment that $2j$ can take any integer value, we derive from Eq. (5), that

$$\frac{2E}{\mu}(\tilde{M}^2 + L^2 + \hbar^2) = -k^2$$

but

$$\tilde{M}^2 + L^2 + \hbar^2 = 4\left(\frac{\tilde{M} \pm L}{2}\right)^2 + \hbar^2 = \hbar^2[4j(j + 1) + 1] = \hbar^2 (2j + 1)^2$$

and

$$E = -\frac{\mu k^2}{2\hbar^2} \frac{1}{(2j + 1)^2}$$

or with $k = Ze^2$

$$E = -\left(\frac{Ze^2}{\hbar}\right)^2 \frac{\mu}{2} \frac{1}{(2j + 1)^2} \quad . \tag{6}$$

If we identify $2j + 1$ with the principal quantum number n, we recognize the familiar expression for the levels in a Coulomb potential. With n taking every integer value from 1 to ∞, we see that j is allowed to take the values $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ Moreover, as $L = (L + \tilde{M})/2 + (L - \tilde{M})/2$, the familiar addition theorem for angular momenta shows that for a given $n = 2j + 1$ the possible values of L are 0, 1, 2, ..., 2j. This procedure shows that the degeneracy of the levels is equal to

$$\sum_{\ell=0}^{2j} (2\ell + 1) = (2j + 1)^2 = n^2 .$$

It is tempting to assume that some group with the Lie algebra of $O_3 \times O_3$ is acting (O_3 is the three-dimensional rotational group). A good candidate is O_4, but when contemplating the actual form of M [Eq. (3)] it is seen that it is essentially a second-order differential operator in coordinate space. However, since the main part is linear in x, there might be some suspicion that it would be interesting to look in p-space for we know that properly parametrized infinitesimal generators are linear differential operators. This explains the second approach to the symmetry due to Fock.

B. The Global Method[3]

We make a Fourier transformation and write the equation in momentum space. The $\frac{1}{r}$ term gives rise to a convolution integral and we find

$$\left(\frac{p^2}{2\mu} - E\right) \Phi\,(p) = \frac{k}{2\pi^2\hbar}\int \frac{d^3q\Phi(q)}{|\vec{q} - \vec{p}|^2} \quad . \tag{7}$$

In fact, it will be of some interest to follow the remark of Alliluev[5] that the method can be generalized to any number of dimensions greater than or equal to 2. The dimension will be denoted by f. We have

$$\frac{1}{r} = \frac{1}{\pi\omega_{f-1}}\int \frac{d^f q}{|\vec{q}|^{f-1}}\, e^{-i\vec{q}\cdot\vec{r}}$$

where ω_n is the area of the unit hypersphere in an n-dimensional space

$$\omega_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$$

For dimension f, Eq. (7) generalized to

$$\left(\frac{p^2}{2\mu} - E\right) \Phi\,(p) = \frac{k}{\pi\omega_{f-1}\hbar}\int \frac{d^f q\Phi(q)}{|\vec{q} - \vec{p}|^{f-1}} \quad . \tag{7'}$$

Let us remark that, in the case of bound states, E is negative. We introduce the quantity

$$p_o^2 = -2mE > 0, \tag{8}$$

and the equation now reads

$$(p^2 + p_o^2)\,\Phi(p) = \frac{k}{\hbar}\,\frac{\Gamma\left(\frac{f-1}{2}\right)}{\pi^{\frac{f+1}{2}}}\int d^f q\;\frac{\Phi(q)}{|\vec{q} - \vec{p}|^{f-1}} \tag{9}$$

In this form the equation seems to exhibit nothing more than the usual f-dimensional rotational invariance. We now perform a change of variable. First, we replace $\vec{p}$ by $\vec{p}/p_o$, then imbed the f-dimensional space in an $f + 1$ dimensional one and perform a stereographic projection on the unit sphere (Fig. 1).

Let $\vec{u}$ be the point on the unit sphere corresponding to $\vec{p}$ and let $\vec{n}$ denote the unit vector from the origin to the north pole of the sphere; we have

$$\vec{u} = \frac{p^2 - p_o^2}{p^2 + p_o^2}\,\vec{n} + \frac{2p_o}{p^2 + p_o^2}\,\vec{p}\ . \tag{10}$$

An immediate calculation shows that

$$\left\{\begin{aligned} d^{f+1}\Omega_u &= 2\delta(u^2 - 1)d^{f+1}u = \frac{(2p_o)^f}{(p_o^2 + p^2)^f}\,d^f p \\ |\vec{p} - \vec{q}|^2 &= \frac{(p^2 + p_o^2)(q^2 + p_o^2)}{(2p_o)^2}\,|\vec{u} - \vec{v}|^2 \end{aligned}\right. \tag{11}$$

if $\vec{v}$ corresponds to $\vec{q}$. Let us also change the wave function by defining

$$\hat{\Phi}(u) = \frac{1}{\sqrt{p_o}}\left(\frac{p_o^2 + p^2}{2p_o}\right)^{\frac{f+1}{2}}\Phi(p) \tag{12}$$

Inserting these values in Eq. (9) we get

$$\Phi(u) = \frac{\mu k}{2p_o \hbar} \frac{\Gamma\left(\frac{f-1}{2}\right)}{\pi^{\frac{f+1}{2}}} \int \frac{d^{f+1}\Omega_v \hat{\Phi}(v)}{\left|\vec{v} - \vec{u}\right|^{f-1}} \tag{13}$$

The great interest of Eq. (13) is to show that the problem is rotationally invariant in an f+1-dimensional space, which in the case of $f = 3$ implies an O_4 symmetry group. Before solving Eq. (13) it is interesting to compare the normalization of $\hat{\Phi}$ and Φ, we have

$$\int \left|\hat{\Phi}(u)\right|^2 d\Omega_u = \int \frac{p_o^2 + p^2}{2p_o^2} \left|\Phi(p)\right|^2 d^f p$$

We can now use the virial theorem which states

$$E \int \left|\Phi(p)\right|^2 d^f p = - \int \frac{p^2}{2\mu} \left|\Phi(p)\right|^2 d^f p$$

to obtain the result that [for a solution of Eq. (13)]

$$\int d^{f+1}\Omega_u \left|\hat{\Phi}(u)\right|^2 = \int d^f p \left|\hat{\Phi}(p)\right|^2 \tag{14}$$

Hence the mapping: $\Phi(p)$, belonging to the eigenvalue $E, \longleftrightarrow \hat{\Phi}(u)$ satisfying Eq. (13) as given by conditions (10) and (12) preserves the scalar products. This mapping can be extended on one side to the Hilbert space of L^2 functions on the sphere - call it $\mathcal{H}_{f+1}$, on the other to the Hilbert space of linear combinations of eigenfunctions (and their limits) corresponding to the discrete spectrum of the Hamiltonian. As the functions corresponding to different eigenvalues of the Hamiltonian are orthogonal and as the same property holds on the sphere for solutions of Eq. (13) corresponding to different eigenvalues of p_o, the extended mapping obtained in that way is one-to-one and isometric that is unitary.

Note that it cannot be given through a geometric transformation of the type (10) which clearly depends on p_o^8. We now solve Eq. (13), using the following remark. In the $f + 1$-dimensional space, the kernel $\frac{1}{|\vec{u} - \vec{v}|^{f-1}}$ is essentially the Green function of the Laplace operator. More precisely,

$$\Delta_u^{f+1} \frac{1}{|\vec{u} - \vec{v}|^{f-1}} = -(f-1)\omega_{f+1}\delta^{f+1}(\vec{u} - \vec{v}) . \tag{15}$$

Moreover, the spherical harmonics defined on the sphere form a complete system of functions in $\mathcal{H}_{f+1}$. They are labelled by an integer λ taking the values 0,1,2,... and an index α whose meaning will be specified in a moment, such that if $Y_{\lambda,\alpha}$ is the spherical harmonic

$$|\vec{u}|^{\lambda} Y_{\lambda,\alpha}\left(\frac{\vec{u}}{|\vec{u}|}\right) = \mathcal{Y}_{\lambda,\alpha}(\vec{u})$$

is an homogeneous polynomial of degree λ in $\vec{u}$ constrained by the condition

$$\Delta^{f+1} \mathcal{Y}_{\lambda,\alpha}(\vec{u}) = 0. \tag{16}$$

The index α allows us to classify the set of solutions of Eq. (16) properly orthonormalized. An arbitrary homogeneous polynomial in $f+1$ variables of degree λ depends on $\binom{\lambda + f}{\lambda}$ constants.

Equation (16) gives $\binom{\lambda + f - 2}{\lambda - 2}$ homogeneous conditions; hence the number of independent spherical harmonics belonging to the same λ, N_λ, is

$$N_\lambda = \begin{pmatrix} \lambda + f \\ \lambda \end{pmatrix} - \begin{pmatrix} \lambda + f - 2 \\ \lambda - 2 \end{pmatrix} = \frac{(\lambda+f-2)!\,(2\lambda+f-1)}{(f-1)!\,\lambda!} \tag{17}$$

(indeed, for $f = 2$ we find $2\lambda + 1$ and for $f = 3$ $(\lambda + 1)^2$, a result which we will use in a moment). Taking into account the fact that $\mathcal{Y}_{\lambda,\alpha}(\vec{v})$ and $\frac{1}{|\vec{v} - \vec{u}|^{f-1}}$ are harmonic in $\vec{v}$ everywhere except at the point $\vec{v} = \vec{u}$, we write Green's formula for $\vec{u}$ on the unit sphere and a surface S_ϵ as shown in Fig. 2:

$$S_\epsilon \equiv \left\{ v: \quad v^2 = 1, |\vec{v} - \vec{u}|^2 \geq \epsilon \right\}$$

$$\cup \quad \left\{ v: \quad v^2 \leq 1, |\vec{v} - \vec{u}|^2 = \epsilon \right\}$$

We have

$$0 = \int_{S_\epsilon} \left[\mathcal{Y}_{\lambda\alpha}(\vec{v}) \frac{d}{dn} \frac{1}{|\vec{v} - \vec{u}|^{f-1}} - \frac{1}{|\vec{v} - \vec{u}|^{f-1}} \frac{d}{dn} \mathcal{Y}_{\lambda\alpha}(\vec{v}) \right] d\sigma$$

The integral splits into two parts. The first one tends smoothly when ϵ goes to zero to an integral evaluated on the whole sphere. The second part taken over a small hemisphere around the point u tends to $\frac{f-1}{2} \omega_{f+1} \mathcal{Y}_{\lambda\alpha}(\vec{u})$ and since $\vec{u}$ is on the unit sphere $\mathcal{Y} \equiv Y$. Moreover, due to the homogeneity of $\mathcal{Y}$, we have $\frac{d}{dn} \mathcal{Y}_{\lambda\alpha}(\vec{v}) \Big|_{v^2=1} = \lambda Y_{\lambda,\alpha}(\vec{v})$ and on the sphere $v^2 = 1$ (with $u^2 = 1$)

$$\frac{d}{dn} \left. \frac{1}{|\vec{v} - \vec{u}|^{f-1}} \right|_{v^2 = u^2 = 1} = - \frac{f-1}{2} \left. \frac{1}{|\vec{v} - \vec{u}|^{f-1}} \right|_{v^2 = u^2 = 1}$$

Hence

$$0 = \frac{f - 1}{2} \omega_{f+1} Y_{\lambda,\alpha}(\hat{u}) + \int \frac{d^{f-1}\Omega_v}{|\hat{v} - \hat{u}|^{f-1}} Y_{\lambda,\alpha}(\hat{u}) \left[- \frac{f-1}{2} - \lambda \right]$$

Using again the formula for the area of the sphere, we get

$$Y_{\lambda,\alpha}(\hat{u}) = \frac{(f - 1 + 2\lambda)}{4\pi^{\frac{f+1}{2}}} \Gamma \left(\frac{f - 1}{2} \right) \int \frac{d^{f+1}\Omega_v}{|\hat{v} - \hat{u}|^{f-1}} Y_{\lambda,\alpha}(\hat{u}) \qquad (18)$$

Equation (13) is now to be compared with Eq. (18). Obviously, due to the completeness of spherical harmonics, we have thus found all the possible levels given by

$$\frac{\mu k}{p_o \hbar} = \frac{f - 1 + 2\lambda}{2}$$

thus

$$E = - \frac{p_o^2}{2\mu} \equiv \frac{\mu}{2} \frac{k^2}{\hbar^2} \frac{1}{\left(\frac{f - 1 + 2\lambda}{2} \right)^2} \qquad (19)$$

If we go back on earth and set $f = 3$, then $\frac{f - 1 + 2\lambda}{2} = \lambda + 1$ and we again get formula (6) with λ now identified with $2j$. The energy levels do not depend on the index α, and thus there are N_λ orthogonal states belonging to the same eigenvalue of the energy. Equation (17) gives the degeneracy in that case, $N_\lambda = (\lambda + 1)^2$.

At the same time we have obtained the eigenfunctions which are to be identified with a set of spherical harmonics on the 4-dimensional sphere (or more generally on an f+1-dimensional sphere). There are several possible ways to label the additional quantum numbers in one level and this will be discussed in the next paragraph. For the moment let us observe that the O(4) symmetry group acts on the eigenfunctions of each level in a very simple way for if o denotes a rotation in f+1-dimensional space

$$Y_{\lambda\alpha}(o\hat{u}) = \sum_{\alpha'} \Delta^{\lambda}_{\alpha\alpha'}(o) \; Y_{\lambda\alpha'}(\hat{u})$$

where Δ denotes an N-dimensional representation of the orthogonal group $O_{(f+1)}$. Remembering that

$$\mathcal{Y}_{\lambda\alpha}(\vec{u}) = |\vec{u}|^\lambda \; Y_{\lambda\alpha}\left(\frac{\vec{u}}{|\vec{u}|}\right)$$

the representation just written is, in fact, obtained by letting the matrix o transform the coordinates of $\vec{u}$ in the form $(ou)_i = \sum_j o_{ij}u_j$ and looking for the corresponding transformation of the symmetric polynomial $\mathcal{Y}_{\lambda\alpha}$. In fact, $\mathcal{Y}_{\lambda\alpha}$ is not an arbitrary symmetric polynomial and the corresponding representation of $O_{(f+1)}$ is the one which, in the language of Young tableaux, is made of a single row of λ boxes. (A harmonic polynomial can essentially be written as

$$\Sigma_{i_1, i_2, \ldots i_\lambda} \; t_{i_1, i_2 \ldots i_\lambda}, \; x_{i_1} x_{i_2} \ldots x_{i_\lambda},$$

with t symmetric in its indices and of zero trace in each pair of them.) In particular, for O(4) these representations when described in terms of two angular momenta are labelled $\mathcal{D}^{j,j}$ with $\lambda = 2j$. Using the classical branching law for the orthogonal group, one readily sees that they split according to the O(3) subgroup in a direct sum of representations with $\ell = 0,1 \ldots, 2j$. This gives us the allowed values of the ordinary angular momentum for a level with principal quantum number $n = \lambda + 1 = 2j + 1$. It is even intuitive that an homogeneous polynomial of degree λ in $f + 1$ variables can be written as a sum of homogeneous polynomials of degrees $0,1,\ldots 2j$ in the first f variables. By choosing them harmonic, one has thus a procedure to compute the wave function. We shall obtain explicitly the wave functions in another way.

Let us finally use Eq. (18) to write an expansion of the Green-kernel. For $\vec{v}$ and $\vec{u}$, not of equal length, one deduces immediately from the fact that in a p-dimensional space

$$|\vec{u}|^{\lambda} Y_{\lambda,\alpha}\left(\frac{\vec{u}}{|\vec{u}|}\right) \quad \text{and} \quad \left(\frac{1}{|\vec{u}|}\right)^{\lambda+p-2} Y_{\lambda,\alpha}\left(\frac{\vec{u}}{|\vec{u}|}\right)$$

are both harmonic

$$\frac{\Gamma\left(\frac{p-2}{2}\right)}{4\pi^{\frac{p}{2}}} \frac{1}{|\vec{u} - \vec{v}|^{p-2}} = \sum_{\lambda} \frac{W_<^{\lambda}}{W_>^{\ell+p-2}} \frac{\sum_{\alpha} Y_{\lambda\alpha}^{(p)}\left(\frac{\vec{u}}{u}\right) Y_{\lambda\alpha}^{(p)}\left(\frac{\vec{u}}{u}\right)}{p - 2 + 2\lambda} \qquad (20)$$

where $W_<$ $(W_>)$ denotes the smaller (greater) of the two quantities $|\vec{u}|$, and $|\vec{v}|$. The superscript on the spherical harmonics recalls the dimension of the space. In this formula they are assumed orthonormalized.

C. Calculation of Wave Functions

We shall now compute the wave functions using another possibility afforded by group theory. We make the remark that the four-dimensional sphere is homeomorphic with the space of parameters of the group SU_2, the uni-modular unitary group in two dimensions (which is the covering group of the ordinary three-dimensional rotation group). Moreover, we know a complete set of functions on this space,[9] namely the matrix elements of the various representations $\mathcal{D}^{j}_{mm'}$, labeled by j, taking the values $0, \frac{1}{2}, 1, \ldots$ and $-j \leq m \leq j$, $-j \leq m' \leq j$. The $\mathcal{D}$ functions were computed by Wigner and we shall give them below. They seem to be good candidates for being spherical harmonics on the sphere if we notice further that, corresponding to a "spin" j, they are $(2j + 1)^2$ in number - precisely the number of spherical harmonics of degree $\lambda = 2j$. We shall prove that it is, indeed, the case. It is honest to point out that this kind of coincidence is very peculiar to the dimension we are precisely interested in.

Let us first recall the correspondence between the sphere and SU_2. The most general unitary unimodular two-by-two matrix can be written:

$$A = u_0 + i\vec{\sigma} \cdot \vec{u} \tag{21}$$

with $(u_0, \vec{u})$ real, $u_0^2 + \vec{u}^2 = 1$, and σ_1, σ_2, σ_3 are the usual Pauli matrices. This parametrization sets a one-to-one correspondence between the two spaces and hence between the functions defined on the two spaces.

Writing the previous matrix A as

$$A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix}, \quad a\bar{a} + b\bar{b} = 1 \qquad a = u_0 + iu_3 \,, \quad b = iu_1 + u_2 \,. \tag{22}$$

An invariant measure on SU_2 is $\delta(a\bar{a} + b\bar{b} - 1)\,\frac{da\,d\bar{a}\,db\,d\bar{b}}{2}$, but up to a constant factor we know that invariant measures are unique on a compact group; hence, this measure is the usual one (up to a factor). It also reads

$$2\delta(u^2 - 1)d^4u = \delta(a\bar{a} + b\bar{b} - 1)\,\frac{da\,d\bar{a}\,db\,d\bar{b}}{2} \tag{23}$$

Consequently, the measure on SU_2 coincides with the usual measure on the sphere and we have

$$\int 2\delta(u^2 - 1)d^4u = \int_{SU_2} \delta(a\bar{a} + b\bar{b} - 1)\,\frac{da\,d\bar{a}\,db\,d\bar{b}}{2} = 2\pi^2 \tag{24}$$

Moreover, we can extend SU_2 to a group $\{R_+\} \times SU_2$ where R_+ is the multiplicative group of real positive numbers and make it in one-to-one correspondence with the four-dimensional space without the origin. This means that we multiply the matrix A by a real positive factor. Including the value 0 extends the correspondence to the whole space. Now let us recall the definition of Wigner's $\mathcal{D}$ functions.

Let B be the most general two-by-two matrix

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Consider polynomials in two variables of degree 2j of which a basis is given by

$$P_{j,m}(\xi,\eta) = \frac{\xi^{j+m}}{\sqrt{(j+m)!}} \quad \frac{\eta^{j-m}}{\sqrt{(j+m)!}} \qquad m = j,\ j-1,\ \ldots.-j\ . \tag{25}$$

$$P_{jm}(a\xi + b\eta,\ c\xi + d\eta) = \frac{(a\xi+b\eta)^{j+m}}{\sqrt{(j+m)\ !}} \quad \frac{(c\xi + d\eta)^{j-m}}{\sqrt{(j - m)\ !}} = \sum_{m'=-j}^{j} \mathfrak{D}^{j}_{mm'}(B)P_{jm'}(\xi,\eta) \tag{26}$$

with

$$\mathfrak{D}^{j}_{mm'}(B) = \sqrt{(j+m)!(j-m)!(j+m')!(j-m')!} \sum_{n_i \geq 0} \frac{a^{n_1}\cdot b^{n_2}\cdot c^{n_3}\cdot d^{n_4}}{n_1!\cdot n_2!\cdot n_3!\cdot n_4!} \tag{27}$$

$$n_1+n_2=j+m,\ n_3+n_4=j-m$$

$$n_1+n_3=j+m',\ n_2+n_4=j-m'$$

The ordinary matrix elements of the irreducible representations of SU_2 corresponding to spin j are obtained by putting in Eq. (27) for B the general element $A \epsilon SU_2$. Formula (27) is suited for computing $\mathcal{L}^{j}_{mm'}(sA)$ when $s \geq o$. Now $\mathcal{L}^{j}_{m,m'}(sA)$ can be considered as a function in the four-dimensional (real) space, and obviously it is homogeneous of degree 2j; that is

$$\mathcal{L}^{j}_{mm'}(sA) = s^{2j}\mathcal{L}^{j}_{mm'}(A) \tag{28}$$

Moreover, we will now show that it satisfies the Laplace equation. Since we obtain for each integer $2j$ a set of $(2j + 1)^2$ linearly independent homogeneous polynomials satisfying the Laplace equation, the $\mathcal{D}^j_{mm'}(A)$ form a complete set of spherical harmonics on the four-dimensional sphere. The Laplace operator is

$$\Delta^4_u = \frac{\partial^2}{\partial u_0^2} + \frac{\partial^2}{\partial u_1^2} + \frac{\partial^2}{\partial u_2^2} + \frac{\partial^2}{\partial u_3^2}$$

We now use the relation

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4 \frac{\partial}{\partial(x + iy)} \frac{\partial}{\partial(x - iy)}$$

hence

$$\begin{aligned} \Delta^4_u &= 4\left[\frac{\partial}{\partial(u_0 + iu_3)} \frac{\partial}{\partial(u_0 - iu_3)} + \frac{\partial}{\partial(iu_1 + u_2)} \frac{\partial}{\partial(-iu_1 + u_2)}\right] \\ &= 4\left[\frac{\partial}{\partial(sa)} \frac{\partial}{\partial(s\bar{a})} + \frac{\partial}{\partial(sb)} \frac{\partial}{\partial(s\bar{b})}\right] \end{aligned} \tag{29}$$

In order to prove that $\Delta^4 \mathcal{L}^j_{mm'} = 0$, it is sufficient to prove that

$$\sum_{m'=-j}^{j} \Delta^4 \mathcal{D}^j_{mm'}(sA) \frac{\xi^{j+m'}}{(j + m')!} \frac{\eta^{j-m'}}{(j - m')!} = 0$$

since the $P_{jm'}(\xi,\eta)$ are linearly independent polynomials; hence we have to compute

$$\Delta^4(sa\xi + sb\eta)^{j+m}(-s\bar{b}\xi + s\bar{a}\eta)^{j-m}$$

Using Eq. (29) we easily find that this quantity is zero. More generally, we can check that

$$\left(\frac{\partial^2}{\partial a \partial d} - \frac{\partial^2}{\partial b \partial c}\right)\mathcal{D}^j_{mm'}(B) = 0$$

Next we study the normalization. We have

$$\int \overline{\mathcal{D}^j_{m_1 m_2}(A)}\,\mathcal{D}^{j'}_{m'_1 m'_2}(A)\,d^4\Omega_u = \frac{2\pi^2}{2j+1}\,\delta_{jj'}\,\delta_{m_1 m'_1}\,\delta_{m_2 m'_2}$$

$$A = u_0 + i\vec{\sigma}\cdot\vec{u} \qquad u_0^2 + \vec{u}^2 = 1 \tag{31}$$

The only point to be verified is the factor $2\pi^2/(2j+1)$, otherwise the orthogonality stems from Schur's Lemma. For that purpose we note that

$$\sum_\mu \mathcal{D}^j_{m_1 \mu}(A)\,\mathcal{D}^j_{\mu m_2}(A) = \delta_{m_1 m_2}$$

Hence

$$\int \sum_\mu \overline{\mathcal{D}^j_{\mu m_1}}(A)\,\mathcal{D}^j_{\mu m_2}(A)\,d^4\Omega_u = \delta_{m_1 m_2}\,2\pi^2$$

where $2\pi^2$ is the surface of the sphere. On the other hand, Eq. (31) gives

$$\frac{2\pi^2}{2j+1}\sum_{n=-j}^{+j} \delta_{m_1 m_2} = 2\pi^2 \delta_{m_1 m_2}$$

A complete set of spherical harmonics properly normalized on the four-dimensional sphere is thus

$$Y^{(4)}_{2j;m_1,m_2}(u) = \sqrt{\frac{2j+1}{2\pi^2}}\, \mathcal{D}^j_{m_1,m_2}(u_o + i\vec{u}\vec{\sigma})\quad 2j = 0,1,\cdots, \; -j \le m_i \le j \quad (32)$$

The $\mathcal{D}^j_{m_1,m_2}$ also afford very quickly the representation of O(4) in the following way. First let U be a generic element of SU_2. Then if we select V and W belonging to SU_2, the correspondence

$$U \to U' = VUW^{-1}$$

is a mapping of SU_2 on itself. It is clear that if we write $U = u_o + i\vec{\sigma}\vec{u}$ the mapping $u \to u'$ is linear; hence, we have obtained an othogonal transformation. The set of pairs (V,W) with the law of multiplication (V',W') (V,W) = (V'V,W'W) forms a group – namely $(SU_2 \times SU_2)$ and we have an homomorphism $(SU_2,SU_2) \to O(4)$ which can readily be seen to cover O(4). This is, of course, well known. The kernel of the mapping consists of the two elements (I,I) and (-I,-I). The diagonal subgroup of pairs of the form (U,U) corresponds to 3-dimensional rotations of $\vec{u}$, and we are going to use it in the following.

The transformations of the type (U,U^{+}), on the other hand (where $U = e^{i\frac{\theta}{2}\vec{\sigma}.\vec{n}}$), correspond to rotations (through angle θ) in the two-plane passing through the 0 axis and the axis $\vec{n}$ (in the 3-dimensional subspace $u_o = 0$). Now we write the general orthogonal transformation as:

$$Y^{(4)}_{\lambda,\alpha}(u) \to Y^{(4)}_{\lambda,\alpha}(o^{-1}u) = \Sigma\Delta_{\alpha'\alpha}(o)Y^{(4)}_{\lambda,\alpha'}(u) o\epsilon O_4 \quad .$$

If $(V,W) \to o$, then $(V,W)^{-1} = (V^{+},W^{+}) \to o^{-1}$; with the set of spherical harmonics given by Eq. (32), we have as a result of the properties of $\mathfrak{D}$ - functions

$$Y^{(4)}_{2j;m_1,m_2}(o^{-1}u) = \mathfrak{D}^{j}_{m_1m'_1}(V^{+})\mathfrak{D}^{j}_{m'_2m_2}(W)Y^{(4)}_{2j;m'_1m'_2}(u)$$

Hence

$$\Delta^{2j}_{m'_1,m'_2;m_1,m_2}[(V,W)] = \mathfrak{D}^{j}_{m'_1,m_1}(W) \tag{33}$$

Of course, (V,W) and $(-V,-W)$ give rise to the same matrix. In particular, since we are to make use of it, we find easily the representation of a rotation through angle θ in the $(0,3)$ plane, namely

$$\Delta^{2j}_{m'_1m'_2;m_1,m_2}\left[\left(e^{i\frac{\theta}{2}\sigma_3}, e^{-\frac{\theta}{2}\sigma_3}\right)\right] = e^{-i(m_1+m_2)\theta}\delta_{m'_1m_1}\delta_{m'_2m_2} \tag{33'}$$

Before giving further interpretation of formula (32) we shall construct the set corresponding to the diagonalization in angular momentum. We remark that rotating $\vec{p}$, the 3-dimensional momentum, amounts to submitting $\vec{u}$, the projection of the 4-vector $u \equiv \left\{u_o,\vec{u}\right\}$, to the same rotation.

If $A(u) = u_o + i\vec{\sigma}\vec{u}$, then

$$A(u_o,R\vec{u}) = RAR^{-1}$$

where for simplicity of the notation, R stands on the right-hand side for the 2×2 unitary matrix which corresponds to the rotation. Our second remark is that if Γ is the two-by-two unitary unimodular matrix

$$\Gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{34}$$

then for any 2×2 unitary unimodular matrix

$$R^{-1}\Gamma = \Gamma R^{T} \tag{35}$$

Consider $\mathfrak{D}^{j}_{m,m}$ $[A(u)\Gamma]$; then

$$\mathfrak{D}^{j}_{mm'}[A(u_0,R\vec{u})\Gamma] = \mathfrak{D}^{j}_{mm'}[RA(u)\Gamma R^{T}] = \mathfrak{D}^{j}_{mm_1}(R)\mathfrak{D}^{j}_{m'_1m'}(R^{T}) \times \mathfrak{D}^{j}_{m_1m'_1}[A(u)\Gamma]$$

But, as is immediate from formula (27), $\mathfrak{D}^{j}(R^{T}) = \mathfrak{D}^{j\,T}(R)$ so that

$$\mathfrak{D}^{j}_{mm'}[A(u_0,R\vec{u})\Gamma] = \mathfrak{D}^{j}_{mm_1}(R)\mathfrak{D}^{j}_{m'm'_1}(R)\mathfrak{D}^{j}_{m_1m'_1}[A(u)\Gamma] \tag{36}$$

This last formula shows that the spherical harmonics of degree 2j form the carrier space of a reducible representation of the 3-dimensional rotation group, and this representation can be reduced to a sum corresponding to angular momenta $L = 0,1,2,\ldots,2j$. If $(j,m;j,m'|LM)$ denotes the usual Clebsch-Gordan coefficient,[10] we recall that

$$\sum_{-L\leq M'\leq L} (jm;jm'|LM')\,\mathfrak{D}^{L}_{MM'}(R) = \sum_{\substack{-j\leq m_1\leq +j \\ -j\leq m'_1\leq +j}} (jm_1;jm'_1|LM)\,\mathfrak{D}^{j}_{m_1m}(R)\,\mathfrak{D}^{j}_{m'_1m'}(R) \tag{37}$$

With the help of Eq. (37) we obtain immediately in u space the properly normalized eigenfunctions of our problem with principal quantum number

$n = 2j + 1$, angular momentum L, and magnetic quantum number M as

$$Y^{(4)}_{\substack{n;L,M \\ n=2j+1}}(u) = \sqrt{\frac{2j+1}{2\pi^2}} \sum_{\substack{-j\le m\le +j \\ -j\le m'\le +j}} (j,m;jm' \,|\, LM) \mathcal{D}^j_{mm'}[(u_o + i\vec{\sigma}\cdot\vec{u})\Gamma] \tag{38}$$

such that

$$Y^{(4)}_{n,L,M}(u_o, R^{-1}\vec{u}) = Y^{(4)}_{n,LM'}(u_o,\vec{u}) \mathcal{D}^j_{M'M}(R) \tag{39}$$

Using the properties of C. G. coefficient one shows that

$$Y^{(4)}_{n;LM}(u) = \sqrt{\frac{2}{\pi}} \frac{i^L \sin \delta^L}{\sqrt{(n^2-1^2)\dots(n^2-L^2)}} \frac{d^L}{d(\cos\delta)^L} \left(\frac{\sin n\delta}{\sin\delta}\right) Y^{(3)}_{LM}\left(\frac{\vec{u}}{|\vec{u}|}\right) \tag{40}$$

where δ is defined through $u_o + i\,\vec{u} = e^{i\delta}$

The derivation of this formula is given in the appendix.

We can easily derive from Eq. (32) or (38) the projection operator on to the space corresponding to principal quantum number $n = 2j + 1$ which appears in several formulas, as for instance in Eq. (20). We have $\mathcal{P}$ standing for this projection operator:

$$\mathcal{P}^{(4)}_{n=2j}(u,v) = \sum_{L,M} Y_{2j;L,M}(u)\, \overline{Y}_{2j;L,M}(v) = \sum_{m,m'} \frac{2j+1}{2\pi^2} \mathcal{D}^j_{mm'}[A(u)] \overline{\mathcal{D}^j_{mm'}[A(v)]}$$

$$= \sum_{m,m'} \frac{2j+1}{2\pi^2} \mathcal{D}^j_{m,m'}[A(u)] \mathcal{D}^j_{m'm}[A^{-1}(v)]$$

$$= \frac{2j+1}{2\pi^2} \sum_m \mathcal{D}^j_{mm}[A(u)A^{-1}(v)]$$

$$= \frac{2j+1}{2\pi^2} \text{Trace}\, \mathcal{D}^j[A(u)A^{-1}(v)] \tag{41}$$

Now we want to compute $A(u)A^{-1}(v)$. We have

$$A(u)A^{-1}(v) = (u_o + i\vec{\sigma}\cdot\vec{u})(v_o - i\vec{\sigma}\vec{v})$$

$$= u_o v_o + \vec{u}\cdot\vec{v} + i\vec{\sigma}(v_o\vec{u} - u_o\vec{v} + \vec{u}\times\vec{v})$$

the unitary unimodular matrix can also be written

$$A(u)A^{-1}(v) = \cos\frac{\varphi}{2} - i\sin\frac{\varphi}{2}\vec{\sigma}\cdot\vec{n} = e^{-i\frac{\varphi}{2}\vec{\sigma}\cdot\vec{n}}$$

If θ is the angle between u and v on the 4-sphere, we have

$$\cos\theta = u_o v_o + \vec{u}\cdot\vec{v} = \cos\frac{\varphi}{2}$$

hence, $\varphi = 2\theta$ (the sign is irrelevant). Since we compute a trace, we can choose our coordinate system as we please. In particular, "quantizing" along the axis $\vec{n}$, we have

$$\text{Trace}\ \mathfrak{L}^j[A(u)A^{-1}(v)] = \sum_{m=-j}^{+j} e^{2im\theta} = \frac{\sin(2j+1)\theta}{\sin\theta}$$

(we recognize the Chebichev polynomials if $\frac{\sin(2j+1)\theta}{\sin\theta}$ is expressed as a polynomial of degree $2j$ in $\cos\theta$). The answer is thus

$$\mathfrak{P}_n^{(4)}(u,v) = \frac{n}{2\pi^2}\frac{\sin n\theta}{\sin\theta} = \frac{1}{2\pi^2}\frac{d}{d\cos\theta}(\cos n\theta) \qquad u\cdot v = \cos\theta, \quad u^2 = v^2 = 1 \qquad (42)$$

Using this result, we can rewrite formula (20) for $f + 1 = 4$ as

$$\frac{1}{4\pi^2}\frac{1}{|\vec{u}-\vec{v}|^2} = \frac{1}{4\pi^2}w_>^{-2}\left(1 + \left(\frac{w_<}{w_>}\right)^2 - \left(\frac{2w_<}{w_>}\right)\cos\theta\right)^{-1}$$

$$= \frac{1}{w_>^2}\sum_{2j=0,1,\ldots}\left(\frac{w_<}{w_>}\right)^{2j}\frac{1}{2(2j+1)}\times\frac{(2j+1)}{2\pi^2}\ \frac{\sin(2j+1)\theta}{\sin\theta}$$

Thus with $t = (w_{<}/w_{>}) < 1$, we find the classical generating functions for Chebichev polynomials

$$\frac{1}{1 + t^2 - 2t\cos\theta} = \sum_{\lambda=0}^{\infty} t^{\lambda} \frac{\sin(\lambda + 1)\theta}{\sin\theta} \tag{43}$$

Comparing this result with the generating function for Legendre polynomials (which arises in our case for $f = 2$ from the Green function corresponding to the Laplace equation in three dimensions):

$$\frac{1}{(1 + t^2 - 2t\cos\theta)^{\frac{1}{2}}} = \sum_{\ell=0}^{\infty} t^{\lambda} P_{\lambda}(\cos\theta)$$

we deduce the relation

$$\frac{\sin(\lambda + 1)\theta}{\sin\theta} = \sum_{\lambda_1+\lambda_2=\lambda} P_{\lambda_1}(\cos\theta)\, P_{\lambda_2}(\cos\theta) \tag{44}$$

More generally, we can compute similar projectors in arbitrary dimensions. Our examples suggest that we distinguish between odd and even dimensions. We have in even dimension $p = 2r$, according to formula (20)

$$\frac{1}{(1 + t^2 - 2t\cos\theta)^{r-1}} = \sum_{0}^{\infty} t^{\lambda} \frac{4(\pi)^r}{2\Gamma(r-1)} \frac{\mathcal{P}_{\lambda}^{(2r)}(\cos\theta)}{r + \lambda - 1}$$

Where $\mathcal{P}_{\lambda}^{(2r)}$ is the projector, i.e., a polynomial of degree λ in $\cos\theta$ which can be obtained simply by differentiating Eq. (43) to give

$$\frac{1}{(1 + t^2 - 2t\cos\theta)^{r-1}} = \frac{1}{2^{r-2}\,\Gamma(r-1)} \sum_{\lambda=0}^{\infty} t^{\lambda} \left(\frac{d}{d\cos\lambda}\right)^{r-2} \frac{\sin(\lambda + r - 1)\theta}{\sin\theta}$$

and

$$\mathcal{P}_{\lambda}^{(2r)}(\cos\theta) = \frac{2(r+\lambda-1)}{(2\pi)^r}\left(\frac{d}{d\cos\theta}\right)^{r-2}\frac{\sin(\lambda+r-1)\,\theta}{\sin\theta}$$

$$\equiv \sum_{\alpha} Y_{\lambda,\alpha}^{(2r)}(u)\ \overline{Y}_{\lambda\alpha}^{(2r)}(v)\ ;\quad r \geq 2 \tag{45}$$

In the odd case the calculation is completely similar and yields:

$$\mathcal{P}_{\lambda}^{(2r+1)}(\cos\theta) = \frac{\left(r+\lambda-\frac{1}{2}\right)}{(2\pi)^r}\left(\frac{d}{d\cos\theta}\right)^{r-1} P_{\lambda+r-1}(\cos\theta)$$

$$\equiv \sum_{\alpha} Y_{\lambda,\alpha}^{(2r+1)}(u)\ \overline{Y}_{\lambda,\alpha}^{(2r+1)}(v)\ ;\quad r \geq 1 \tag{46}$$

Of course, one can express these polynomials in terms of products of Legendre polynomials.

D. Connection Between the Two Approaches; Parabolic Coordinates[4]

In this paragraph we want to show that the generators of the group of symmetry found in the global method coincide essentially with the two vectors, L and M, introduced in Section A, as should be expected. We will also show that the two sets of spherical harmonics that we have found (connected one to another by a unitary fixed transformation), equations (32) and (38), correspond indeed, with the possibility already present in the classical problem, of separating variables into two different systems of coordinates. Classically it is also known that the "accidental degeneracy" is related to this fact.

To generate our group, O(4), we can use six infinitesimal operators - the first three correspond to ordinary rotations in $\vec{p}$-space and lead to the conservation of angular momentum. The next three correspond in u-space to infinitesimal rotations in the $(u_0u_1)(u_0u_2)$ and (u_0u_3) planes. We shall compute the generator in that case. For that purpose let $\Phi(p)$ be a solution of Eq. (7); then the transformation, corresponding to an infinitesimal rotation in the (u_0u_3) plane is

$$\frac{p'^2 - p^2}{p'^2 + p_0^2} = \frac{p^2 - p_0^2}{p^2 + p_0^2} - \epsilon_{03} \frac{2p_0p_3}{p^2 + p_0^2}$$

$$\frac{2p_0p_3'}{p'^2 + p_0^2} = \epsilon_{03} \frac{p^2 - p_0^2}{p^2 + p_0^2} + \frac{2p_0p_3}{p^2 + p_0^2}$$

$$\frac{2p_0p_i'}{p'^2 + p_0^2} = \frac{2p_0p_i}{p^2 + p_0^2} \; ; \quad i = 1,2$$

or

$$\delta p_3 = \epsilon_{03} \left(\frac{p^2 - p_0^2 - 2p_3^2}{2p_0} \right), \; \delta p_2 = -\epsilon_{03} \frac{p_3p_2}{p_0}, \; \delta p_1 = -\epsilon_{03} \frac{p_3p_1}{p_0} \tag{47}$$

Using Eq. (12) this gives the infinitesimal transformation

$$\Phi(p) \to \Phi'(p) = \Phi(p) + \delta\Phi(p)$$

$$\delta\Phi(p) = \frac{\epsilon_{03}}{\left(p^2 + p_0^2\right)^2} \left[\frac{p^2 - p_0^2 - 2p_3}{2p_0} \frac{\partial}{\partial p_3} - \frac{p_3p_1}{p_0} \frac{\partial}{\partial p_4} - \frac{p_3p_2}{p_0} \frac{\partial}{\partial p_2} \right] (p^2 + p_0^2)^2 \Phi(p) \tag{48}$$

The infinitesimal generator when written as

$$\delta\Phi(p) = \frac{-i\mu}{\hbar p_o} \epsilon_{o3} M_{o3} \Phi(p) \tag{49}$$

is (with $x_i = i\hbar \frac{\partial}{\partial p_i}$)

$$M_{o3} = \frac{1}{\left(p^2 + p_o^2\right)^2} \left[\frac{p^2 - p^2}{2\mu} x_3 - \frac{p_3}{\mu} (\vec{p} \cdot \vec{r})\right] (p^2 + p_o^2)^2$$

$$= \frac{p^2 - p_o^2}{2\mu} x_3 - \frac{p_3}{\mu} (\vec{p} \cdot \vec{r}) - 2i \frac{\hbar p_i}{\mu}$$

At this point we recall that M_{oi} acts on an eigenfunction of the Hamiltonian - corresponding to the eigenvalue $E = -\frac{p_o^2}{2m}$. Moving p_o^2 to the right, we can replace it by $-2mH = -2m \left(\frac{p^2}{2m} - \frac{k}{r}\right)$ so that M can now act on any linear combination of eigenfunctions. Clearly the calculation of M_{o2} and M_{o3} is completely analogous. We introduce the vector $\vec{M}$ whose components are M_{o1}, M_{o2}, M_{o3}

$$\vec{M} = \frac{p^2}{2\mu} \vec{r} + \vec{r} \left(\frac{p^2}{2\mu} - \frac{k}{r}\right) - \frac{\vec{p}}{\mu} (\vec{p} \cdot \vec{r}) - 2i\hbar \frac{\vec{p}}{\mu}$$

$$\vec{M} = \left(\frac{p^2}{\mu} \vec{r} - \frac{\vec{p}}{\mu} (\vec{p} \cdot \vec{r}) - i\hbar \frac{\vec{p}}{\mu}\right) - \frac{k}{r} \vec{r} \tag{50}$$

Equation (50) can also be written

$$\vec{M} = \frac{\vec{p}}{2\mu} \times \vec{L} - \vec{L} \times \frac{\vec{p}}{2\mu} - \frac{k}{r} \vec{r}$$

which is seen to coincide with Eq. (3) and leads to the interpretation of the second vector. It merely corresponds to the three generators of rotations in the two planes passing through the fourth-axis introduced in the stereographic projection.

Our second remark has to do with the two systems of spherical harmonics we have used on the sphere S_4. The first one $\{Y_{n;L,M}\}$ clearly corresponds to the usual separation of variables in polar coordinates. It is natural to ask if the second system

$$\left\{ Y_{n;m,m'} = \sqrt{\frac{2j+1}{2\pi^2}}\, \mathcal{D}^{j}_{m,m'} ;\ n = 2j + 1 \right\}$$

corresponds to another natural system of coordinates which allow separation of variables. As expected, we will show that this is related to parabolic coordinates. For that purpose we write in the p space

$$\Phi_{n;m,m'}(\vec{p}) = \sqrt{\frac{2j+1}{\pi^2}} \left[\frac{2p_o}{(p^2+p_o^2)}\right]^2 \sqrt{p_o}\, \mathcal{D}^{j}_{m,m'}(A(\vec{p})) \tag{51}$$

with

$$A(\vec{p}) = \frac{p^2 - p_o^2}{p^2 + p_o^2} + i\, \frac{2p_o}{p^2 + p_o^2}\, \vec{\sigma} \cdot \vec{p}\,.$$

According to Eq. (36)

$$\Phi_{n;m,m'}(R\vec{p}) = \mathcal{D}^{j}_{m,m_1}(R)\, \mathcal{D}^{j}_{m',m'_1}(R^{-1})\, \Phi_{n;m_1 m'_1}(\vec{p})$$

In particular, if R is a rotation of angle ψ around the z axis

$$\mathfrak{D}^{j}_{m,m_1}(R_\psi) = e^{-im\psi}\delta_{m,m_1}$$

so that

$$\Phi_{n;m,m'}(R_\psi) = e^{-i(m-m')\psi}\Phi_{n;m,m'}(p)$$

Hence, $\Phi_{n;m,m'}$ is an eigenfunction of the third component of the angular momentum L_3 corresponding to the eigenvalue $\hbar(m' - m)$. Now, L_3 commutes with $M_3 \equiv M_{03}$ according to Eq. (4). We thus investigate the effect of an infinitesimal "rotation" in the (03) u-plane. For that purpose we use Eq. (33')

$$\mathfrak{D}^{j}_{m,m}(R_{-\epsilon_{03}}u) = e^{-i(m_1+m_2)\epsilon_{03}}\mathfrak{D}^{j}_{m,m'}(u)$$

where $R_{-\epsilon_{03}}$ indicates a rotation through angle $-\epsilon_{03}$ in the (03) plane. Changing from $-\epsilon_{03}$ to $+\epsilon_{03}$ and comparing with Eq. (49), we have immediately the action of M_3. We now have

$$L_3\Phi_{n_1,m,m'}(p) = \hbar(m' - m)\Phi_{n;m,m'}(p)$$

$$M_3\Phi_{n_1m_1m'}(p) = -\frac{\hbar p_0}{u}(m + m')\Phi_{n;m,m'}(p) \qquad (52)$$

On the other hand, we may introduce parabolic coordinates to separate variables in the original Schrodinger Eq. (1). Those are defined in terms of the parameters of two systems of paraboloids with focus at the origin and an azimuthal angle φ in the (x_1, x_2) plane (Fig. 3). Analytically

$$
\begin{aligned}
x_1 &= \sqrt{\lambda_1\lambda_2}\cos\varphi \qquad \lambda_1 \geq 0,\ \lambda_2 \geq 0 \qquad 0 \leq \varphi \leq 2\pi \\
x_2 &= \sqrt{\lambda_1\lambda_2}\sin\varphi \\
x_3 &= \frac{\lambda_1 - \lambda_2}{2}
\end{aligned}
\tag{53}
$$

One has $r = \sqrt{x_1^2 + x_2^2 + x_3^2} = \dfrac{\lambda_1 + \lambda_2}{2}$ so that

$$
\begin{aligned}
\lambda_1 &= r + x_3 \\
\lambda_2 &= r - x_3
\end{aligned}
\tag{54}
$$

The Laplacian takes the form:

$$\Delta = \frac{2}{\lambda_1 + \lambda_2}\left[2\frac{\partial}{\partial\lambda_1}\lambda_1\frac{\partial}{\partial\lambda_1} + 2\frac{\partial}{\partial\lambda_2}\lambda_2\frac{\partial}{\partial\lambda_2} + \frac{1}{2}\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\right)\frac{\partial^2}{\partial\varphi^2}\right]$$

The Schrodinger equation now reads

$$\left(A_1 + A_2 + \frac{2\mu k}{\hbar^2}\right)\psi(\lambda_1, \lambda_2, \varphi) = 0 \tag{55}$$

with

$$A_i = 2\frac{\partial}{\partial\lambda_i}\lambda_i\frac{\partial}{\partial\lambda_i} + \frac{1}{2\lambda_i}\frac{\partial}{\partial\varphi^2} - \frac{p_o^2}{2\hbar^2}\lambda_i, \quad i = 1,2, \quad \frac{p_o^2}{2\mu} = -E$$

We have also written above the third component of the operator $\vec{M}$ as

$$M_3 = \frac{p^2 - p_o^2}{2\mu}x_3 - \frac{p_3}{\mu}(\vec{p}\cdot\vec{r}) - \frac{2i\hbar}{\mu}p_3$$

Using $p_j = \frac{\hbar}{i}\frac{\partial}{\partial x_j}$ and Eq. (54), we easily find

$$M_3 = -\frac{\hbar^2}{2\mu}\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\left(2\frac{\partial}{\partial\lambda_1}\lambda_1\frac{\partial}{\partial\lambda_1} + 2\frac{\partial}{\partial\lambda_2}\lambda_2\frac{\partial}{\partial\lambda_2} + \frac{1}{2}\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\frac{\partial^2}{\partial\varphi^2}\right)$$
$$- \frac{\lambda_1 - \lambda_2}{4\mu}p_o^2 + \frac{2\hbar^2}{\mu}\left(\lambda_1\frac{\partial}{\partial\lambda_1} + \lambda_2\frac{\partial}{\partial\lambda_2}\right)\cdot\frac{1}{(\lambda_1 + \lambda_2)}\left(\lambda_1\frac{\partial}{\partial\lambda_1} - \lambda_2\frac{\partial}{\partial\lambda_2}\right)$$
$$+ 2\frac{\hbar^2}{\mu}\frac{1}{\lambda_1 + \lambda_2}\left(\lambda_1\frac{\partial}{\partial\lambda_1} - \lambda_2\frac{\partial}{\partial\lambda_2}\right)$$

that is, simplifying and comparing with Eq. (55),

$$M_3 = \frac{\hbar}{2\mu}\left(A_1 - A_2\right) \tag{56}$$

Hence, parabolic coordinates where the natural operators to diagonalize are $L_3 = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}$, A_1 and A_2 lead naturally to M_3 and changing the axis of coordinates (or through commutation with $\vec{L}$) to the other components of M as constants of the motion. It is also in this way that V. Bargmann in his quoted paper[4] was naturally led to wave functions on the sphere S_4, essentially identical with the $\mathcal{D}^j$ functions of Wigner.

III. THE LARGER GROUP

We have remarked that the Hilbert space generated by the eigenfunctions of the Schrodinger equation corresponding to bound states is mapped unitarily on the Hilbert space of square-integrable functions on the unit sphere of a 4-dimensional space, which we call $\mathcal{H}_4$. We now want to find a larger group G for which the following conditions are satisfied.

(i) G contains O_p as a maximal compact subgroup $p \geq 2$;

(ii) G acts on the sphere S_p .

(iii) $\mathcal{H}_p$ is an irreducible space for a unitary representation of G ;

$\mathcal{H}_p$ is the space of L^2 functions on a unit sphere in a p-dimensional real space. Instead of giving the answer and verify the previous conditions we indicate a heuristic derivation. Discussing the Fock transformation [Eq. (10)] mapping the three-dimensional space on the sphere we have casually remarked that the transformation depended on the energy (or equivalently on p_o) and we have investigated the effect of varying the energy. The result was a combination of operations which could be described geometrically in terms of inversions and "scale" transformations. Since we are now looking for operations which eventually will help us to relate subspaces corresponding to various energy levels it seems appropriate to investigate the general class of transformations to which the particular ones just mentioned belong. For that purpose, consider the following set of transformations in a p-dimensional space ($\vec{u}$ now stands for the complete p-dimensional vector):

(a)	orthogonal transformations	$\vec{u} \rightarrow O\vec{u}$
(b)	translations	$\vec{u} \rightarrow \vec{u} + \vec{a}$
(c)	scale transformations	$\vec{u} \rightarrow \lambda \vec{u}$
(d)	inversions	$\vec{u} \rightarrow \frac{\vec{u}}{\lvert\vec{u}\rvert^2}$

When combined in all possible manners these transformations generate a group: the conformal group. All these transformations leave the angle between two curves invariant. We can describe this group more precisely as follows:[11] imbed the p-dimensional space in an p + 1 dimensional space. We call z the extra component. Consider the paraboloid $z = \vec{u}^2$; its points are in one to one correspondence with those of the hyperplane $z = 0$ (Fig. 4).

The set of projective transformations which leave the paraboloid invariant, when translated into the $\vec{u}$ space is the conformal group. We obtain it by taking homogeneous coordinates $\frac{z}{t}$, $\frac{\vec{u}}{t}$ so that this group is the homogeneous linear group which leaves invariant the quadratic form $zt - \vec{u}^2$.

It is a pseudo orthogonal group $O(1,p+1)$ since $zt = \frac{1}{4}\left[(z+t)^2-(z-t)^2\right]$. It contains the transformations (i) to (iv) which appear as

$$
\begin{array}{lll}
(a) & \vec{u} \to O\vec{u}, & z \to z, \ t \to t \\
(b) & \vec{u} \to \vec{u} + \vec{a}t, & z \to z + 2\vec{a}.\vec{u} + \vec{a}^2 t, \ t \to t \\
(c) & \vec{u} \to \vec{u}, & z \to \lambda z, \ t \to \frac{1}{\lambda} t \\
(d) & \vec{u} \to \vec{u}, & z \to t, \ t \to z \ .
\end{array}
$$

The general transformation in p-dimensional space is then[12]

$$
u_i \to u_i' = \frac{\alpha_{ij}u_j + \alpha_{iz}\vec{u}^2 + \alpha_{it}}{\left[\alpha_{tj}u_j + \alpha_{tz}\vec{u}^2 + \alpha_{tt}\right]} \qquad (57)
$$

where the matrix

$$\alpha = \begin{pmatrix} \alpha_{tt} & \cdots & \alpha_{tz} & \cdots & \alpha_{iz} \\ \vdots & & \vdots & & \vdots \\ \alpha_{zt} & \cdots & \alpha_{zz} & \cdots & \alpha_{zj} \\ \vdots & & \vdots & & \vdots \\ \alpha_{it} & \cdots & \alpha_{iz} & \cdots & \alpha_{ij} \end{pmatrix}$$

leaves the form $zt - \vec{u}^2$ invariant, that is $\alpha^T \gamma \alpha = \gamma$ with

$$\gamma \equiv \left(\begin{array}{cc|ccccc} 0 & \frac{1}{2} & 0 & \cdots & & & \\ \frac{1}{2} & 0 & 0 & \cdots & & & \\ \hline 0 & 0 & -1 & 0 & & & \\ & & 0 & -1 & 0 & & \\ & & 0 & 0 & -1 & \cdots & \\ & & & & & \ddots & \\ & & & & & & -1 \end{array}\right)$$

We can now ask the following question: what is the subgroup of the conformal group which leaves the unit sphere invariant. In other words, we want z-t to remain invariant. Since $zt - u^2 \equiv \left(\frac{z+t}{2}\right)^2 - \left(\frac{z-t}{2}\right)^2 - u^2$ we see that the remaining subgroup is also a pseudo orthogonal group $O(1,p)$. We will now prove that this group indeed satisfies our criteria. Conditions (i) and (ii) are verified by construction. In order to examine (iii) we will construct explicitly the representations in $\mathcal{H}_p$. Let Λ be an element of $O(1,p)$ that is

$$\Lambda = \begin{pmatrix} a_{oo} & \cdots & a_{oj} & \cdots \\ \vdots & & \vdots & \\ a_{io} & \cdots & a_{ij} & \cdots \\ \vdots & & \vdots & \end{pmatrix} \quad \text{with} \quad \Lambda^T G \Lambda = G = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} \qquad (58)$$

Then the transformation on $\vec{u}$, z, t is

$$\frac{z' + t'}{2} = a_{oo} \frac{z+t}{2} + \sum_j a_{oj} u_j$$

$$\frac{z' - t'}{2} = \frac{z-t}{2}$$

$$u_i' = a_{io} \frac{z+t}{2} + \sum_j a_{ij} u_j$$

or if the initial point $(z = 1, t = 1, \vec{u})$ belongs to the unit sphere then

$$u_i' = a_{io} + \sum_j a_{ij} u_j$$

$$t' = a_{oo} + \sum a_{oj} u_j \quad .$$

Hence on the sphere the transformation induced by Λ is

$$\Lambda : u \to U : U_i = \frac{u_i'}{t'} = \frac{a_{io} + \sum_j a_{ij} u_j}{a_{oo} + \sum_j a_{oj} u_j} \tag{59}$$

One verifies that

$$U^2 - 1 = \frac{u^2 - 1}{(a_{oo} + \sum_j a_{oj} u_j)^2} \tag{60}$$

We now compute the Jacobian of the transformation. For that purpose we extend the transformation (59) to the whole u-space and write the element of area on the unit-sphere as

$$d^p\Omega_U = 2\,\delta\,(U^2-1)\, d^pU \tag{61}$$

From (56) we derive

$$\left(\sum_j a_{oj}\, du_j\right) U_i + (a_{oo} + \sum_j a_{oj}\, u_j)\, dU_i = \sum_j a_{ij}\, du_j$$

$$\frac{D(U_i)}{D(u_j)} = \frac{1}{\left(a_{oo} + \sum_j a_{oj}\, u_j\right)^p} \det\left(a_{ij} - U_i\, a_{oj}\right) \quad . \tag{62}$$

Using the expression of $\vec{U}$ in terms of $\vec{u}$ we get

$$\frac{D(U_i)}{D(u_j)} = \frac{1}{\left(a_{oo} + \sum_j a_{oj}\, u_j\right)^{p+1}} \det \begin{pmatrix} a_{oo} + \sum_j a_{oj}\, u_j \cdots a_{o1} \cdots a_{op} \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ a_{oi} + \sum_j a_{ij}\, u_j \cdots a_{i1} \cdots a_{ip} \\ \vdots \quad\quad \vdots \quad\quad \vdots \end{pmatrix} \tag{63}$$

$$\frac{D(U_i)}{D(u_j)} = \frac{1}{\left(a_{oo} + \sum_j a_{oj}\, u_j\right)^{p+1}} \det \Lambda \quad .$$

We know that $\det \Lambda = \pm 1$. Further we remark that

$$\left(\sum_j a_{oj}\, u_j\right)^2 \leq \left(\sum_j a_{oj}{}^2\right)\left(\sum_j u_j^2\right)$$

We shall only use the preceding expression for $u^2 = \sum_j u_j^2 = 1$.

Then

$$\left(\sum_j a_{oj}\, u_j\right)^2 \leq \left(\sum_j a_{oj}{}^2\right) = a_{oo}^2 - 1$$

Hence for $\vec{u}^{\,2} = 1$ we have $\left|a_{oo} + \sum_j a_{oj}\, u_j\right| \geq \left|a_{oo}\right| - \sqrt{a_{oo}^{\,2} - 1} > 0$

and the Jacobian never vanishes. From (57) and (60) we get

$$d^p\Omega_u = 2\,\delta\,(U^2 - 1)\, d^pU = \frac{1}{\left|\left(a_{oo} + \sum_j a_{oj}\, u_j\right)\right|^{p-1}}\, d^p\,\Omega_u \tag{64}$$

Let $f(U)$ and $g(U)$ be L^2 - functions on the sphere S_p. The previous calculation shows that

$$\int_{S_p} \bar{f}(U)\, g(U)\, d^p\Omega_U = \int_{S_p} \left[\frac{\bar{f}(U(u))}{\left|a_{oo} + \sum_j a_{oj} u_j\right|^{\frac{p-1}{2}}}\right]\left[\frac{g(U(u))}{\left|a_{oo} + \sum_j a_{oj} u_j\right|^{\frac{p-1}{2}}}\right] d^p\Omega_u \tag{65}$$

with $u \to U$ given by (59).

We are now ready to describe the representation of $O(1,p)$ afforded by $\mathcal{H}_p$.[13] Given a $\Lambda \in O(1,p)$ and an element $f \in \mathcal{H}_p$ we set

$$\begin{aligned} & f \to T^{\Lambda} f \\ & [T^{\Lambda}.f](u) = \frac{f(\Lambda^{-1} u)}{\left|a_{oo}(\Lambda^{-1}) + \sum_j a_{oj}(\Lambda^{-1})\, u_j\right|^{\frac{p-1}{2}}} \\ & (\Lambda^{-1}u)_i = \frac{a_{io}(\Lambda^{-1}) + \sum_j a_{ij}(\Lambda^{-1})\, u_j}{a_{oo}(\Lambda^{-1}) + \sum_j a_{oj}(\Lambda^{-1})\, u_j} \end{aligned} \tag{66}$$

The operator T^{Λ} is obviously a linear operator from $\mathcal{H}_p$ to $\mathcal{H}_p$. Equation (62) shows that it is an isometric operator. All what remains to be verified is that the correspondance $\Lambda \to T^{\Lambda}$ is a representation of $O(1,p)$, in other words we want to show

$$T^{\Lambda_1} \cdot T^{\Lambda_2} = T^{\Lambda_1 \Lambda_2} \tag{67}$$

Since obviously $T^I = I$ (the identity operator) Eq. (67) will also show that T^{Λ} has an inverse (and hence is unitary). We will thus get a unitary representation of $O(1,p)$. For that purpose consider

$$\left[\left(T^{\Lambda_1} T^{\Lambda_2} \right) f \right](u) = \frac{f\left(\Lambda_2^{-1}\Lambda_1^{-1} \cdot u\right)}{\left| \left(a_{oo}(\Lambda_2^{-1}) + \sum_j a_{oj}(\Lambda_2^{-1})(\Lambda_1^{-1}u)_j \right) \left(a_{oo}(\Lambda_1^{-1}) + \sum_j a_{oj}(\Lambda_1^{-1})u_j \right) \right|^{\frac{p-1}{2}}} \tag{68}$$

Clearly as $\Lambda_2^{-1}\Lambda_1^{-1} = (\Lambda_1 \Lambda_2)^{-1}$ all we have to show is that the denominator in the preceding equation is

$$a_{oo}\left(\Lambda_2^{-1}\Lambda_1^{-1}\right) + \sum_j a_{oj}\left(\Lambda_2^{-1}\Lambda_1^{-1}\right)u_j , \tag{69}$$

This can be shown by a direct calculation but it is easier to remark that (69) stems from the properties of Jacobians (compare with Eq. (64)). The group law (67) is satisfied. Hence, we have constructed a unitary representation (infinite dimensional) of $O(1,p)$ in the space of L^2 functions on the sphere S_p. In view of the explicit equations of transformations our representation satisfies all the usual continuity requirements. We recall that $O(1,p)$ has four sheets and our representation is a representation of the full-group. However, in the sequel we might as well assume that we deal with the connected part, which allows us to derive the form of the infinitisimal generators. If the representation of the connected subgroup is irreducible the representation of the whole group is a fortiori irreducible.

For the sake of simplicity let us first examine the case where $p = 2$. Our construction leads to a unitary representation of the (real) Lorentz group in three dimensions $O(1,2)$. Restricting ourselves to the connected part of the group, it is generated by three types of transformations:

rotation in the (1,2) plane : generator L

"pure" Lorentz transformation in the (0,1) plane; generator P_1

"pure" Lorentz transformation in the (0,2) plane; generator P_2

the corresponding infinitisimal transformations are [$(\alpha_1\ \alpha_2\ \beta)$ infinitesimal]

$$1 + \alpha_1 P_1 + \alpha_2 P_2 + \beta L$$

$$L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad P_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{70}$$

The commutation relations are

$$[L,P_1] = P_2 \quad [L,P_2] = -P_1 \quad [P_1,P_2] = -L \tag{71}$$

The sphere S_2 is a unit circle on which the spherical harmonics

$$Y_m^{(1)}(\varphi) = \frac{e^{im\varphi}}{\sqrt{2\pi}}$$

constitute a complete basis for $\mathcal{H}_2$ (classical result from the theory of Fourier series); m takes all integer values from $-\infty$ to $+\infty$. In the following we drop the index (1) of $Y_m^{(1)}$. For $\Lambda \simeq 1 + \beta L$ we write $T^\Lambda \simeq I + \beta\, T(L)$, then

$$T(L)\, Y_m = -im\, Y_m \tag{72}$$

Let ψ be equal to $\Lambda^{-1}\varphi$ with $\Lambda \simeq 1 + \alpha_1 P_1$, we get from Eq. (59)

$$\cos\psi \simeq \frac{-\beta + \cos\varphi}{1 - \beta\cos\varphi}$$

$$\sin\psi \simeq \frac{\sin\varphi}{1-\beta\cos\varphi}$$

$$\operatorname{tg}\psi = \frac{\sin\varphi}{-\beta+\cos\varphi}$$

and

$$(T^{\Lambda} f)(\varphi) \approx \frac{1}{(1-\beta\cos\varphi)^{\frac{1}{2}}} \; f\left(\operatorname{arctg}\frac{\sin\varphi}{\cos\varphi-\beta}\right)$$

$$\approx f(\varphi) + \beta\left(\frac{\cos\varphi}{2} f(\varphi) + \sin\varphi \frac{\partial}{\partial\varphi} f(\varphi)\right)$$

$$T(P_1)Y_m = \frac{1}{4}(Y_{m+1}+Y_{m-1}) + \frac{m}{2}(Y_{m+1}-Y_{m-1}) = \frac{1+2m}{4} Y_{m+1} + \frac{1-2m}{4} Y_{m-1} \qquad (73)$$

An analogous calculation yields

$$T(P_2)Y_m = \frac{1}{4i}(Y_{m+1}-Y_{m-1}) + \frac{m}{2i}(Y_{m+1}+Y_{m-1}) = \frac{1+2m}{4i} Y_{m+1} - \frac{1-2m}{4i} Y_{m-1} \qquad (74)$$

The formulas (72), (73) and (74) give us the representation of the Lie algebra of O(1,2) afforded by our construction. One verifies of course the commutation relations (71). Moreover the generators are anti-hermitean as a consequence of the unitarity of the representation. We can prove the irreducibility using the Lie algebra. Indeed constructing $T_{\pm} = T(P_1) \pm iT(P_2)$ we find

$$T_{\pm} Y_m = \frac{1 \pm 2m}{2} Y_{m\pm 1} \qquad (75)$$

Since m is an integer $\frac{1 \pm 2m}{2}$ can never vanish and starting from one vector Y_m by succesive application of $T(L_1) \pm iT(L_2)$ we generate all the others. Hence, the representation is irreducible.

The proof of irreducibility in the general case can be made along similar lines. Let us denote by $L_1, \ldots, L_p$ the generators of "pure Lorentz transformations" in the two-planes (0,1), (0,2), ..., (o,p).

We calculate $T(P_i)$. Let $\Lambda_i \simeq I+\beta P_i = \begin{pmatrix} 1 & o & \cdots & \beta & \cdots \\ o & 1 & & & \\ \vdots & & \ddots & & \\ \beta & & & 1 & \\ o & & & & 1 \\ \vdots & & & & & \ddots \end{pmatrix}$

We have

$$(T^{\Lambda_i}f)(u) \simeq \frac{1}{(1-\beta\, u_i)^{\frac{p-1}{2}}} \; f\left(\frac{u_1}{1-\beta\, u_i}, \frac{u_2}{1-\beta\, u_i}, \ldots, \frac{-\beta+u_i}{1-\beta u_i}, \ldots, \frac{u_p}{1-\beta u_i}\right)$$

Writing $T^{\Lambda_i} = I + \beta\, T(P_i)$

$$T(P_i) = \frac{p-1}{2} u_i + u_i \sum_j u_j \frac{\partial}{\partial u_j} - \frac{\partial}{\partial u_i} \tag{76}$$

This expression can be given an interesting meaning. Let us introduce the generators L_{ij} of rotations in the (ij) planes - according to (66) we have

$$T(L_{ij}) = u_i \frac{\partial}{\partial u_j} - u_j \frac{\partial}{\partial u_i} \tag{77}$$

Let us now commute the Casimir operator

$$L^2 = - \sum_{i<j} T(L_{ij})^2 \tag{78}$$

with the operators u_i $\left(\text{as an operator } u_i \text{ means } f(u) \rightarrow u_i f(u)\right)$. We easily find

$$\left[L^2, u_i\right] = -2\, u^2 \frac{\partial}{\partial u_i} + 2u_i \sum_j u_j \frac{\partial}{\partial u_j} + (p-1)\, u_i \tag{79}$$

But $u^2 = 1$ on the unit sphere hence comparing (76) and (79) we have

$$T(P_i) = \frac{1}{2}\left[L^2, u_i\right] \tag{80}$$

This last equation gives in essence the procedure described by Dothan, Gell-Mann and Ne'eman[1] to generate the "non-compact" operators, as commutators between a Casimir operator of the compact subgroup and a set of abelian operators submitted to auxilliary conditions invariant under the compact subgroup and which transform among themselves under commutation with generators of the subgroup.

We will now prove irreducibility in a fashion similar to the case $p = 2$. It is clear that if by combining linearly the operators $T(P_i)$ we can find two operators which acting on a spherical harmonic of degree λ generate one of degree $\lambda + 1$ and another of degree $\lambda - 1$ the proof of irreducibility will be complete since the set of spherical harmonics of a given degree is the carrier space of an irreducible representation the subgroup of rotations 0_p . Now we recall that

$$|\vec{u}|^{\lambda}\, Y^{(p)}_{\lambda\alpha}(\hat{u})$$

satisfies the Laplace equation

$$\Delta^p \left(|\vec{u}|^\lambda \, Y^{(p)}_{\lambda\alpha}(\hat{u}) \right) = 0$$

But

$$\Delta_p = \frac{p-1}{|\vec{u}|} \frac{\partial}{\partial |\vec{u}|} + \frac{\partial^2}{\partial |\vec{u}|^2} - \frac{1}{|\vec{u}|^2} L^2 \tag{81}$$

with L^2 given above. Hence

$$L^2 \, Y^{(p)}_{\lambda\alpha} = + \lambda(\lambda + p-2) \, Y^{(p)}_{\lambda\alpha}$$

Now consider the special set of spherical harmonics

$$Y^{(p)}_{\lambda} = (u_1 + i \, u_2)^\lambda \tag{82}$$

These are not normalized but this is irrelevant here. For each λ they provide us with one spherical harmonic, the other ones being simply generated by rotations. Now

$$\begin{aligned} \left[T(P_1) + i \, T(P_2)\right] \, Y^{(p)}_{\lambda} (u) &= + \tfrac{1}{2} \, L^2 \, Y^{(p)}_{\lambda+1} \, \tfrac{1}{2} \, (u + i \, u_2) \, L^2 \, Y^{(p)}_{\lambda} \\ &= \tfrac{1}{2} \left[(\lambda+1)(\lambda+p-1) - \lambda(\lambda+p-2) \quad Y^{(p)}_{\lambda+1}\right] \\ &= \left(\lambda + \frac{p-1}{2}\right) Y^{(p)}_{\lambda+1} \end{aligned} \tag{83}$$

Hence

$$\left[T(P_1) + i \, T(P_2)\right]^\lambda \, Y^{(p)}_{o} = \left(\frac{p-1}{2}\right) \left(1 + \frac{p-1}{2}\right) \cdot \cdot \cdot \cdot \left(\lambda - 1 + \frac{p-1}{2}\right) Y^{(p)}_{\lambda} \tag{84}$$

$Y^{(p)}_{o}$ is the unit function, $\left[T(P_1) + i \, T(P_2)\right]$ is a "step" - operator.

Repeated application of this operator generates, starting from the spherical harmonic of degree zero, a spherical harmonic of degree λ. Combining the action of this operator with the generators of rotations we obtain all spherical harmonics. Hence the representation is irreducible.

We have thus completely answered the problems raised at the beginning of this section.

IV. CONCLUSION

The construction of unitary representations of non-compact groups which have the property that the irreducible representation of their maximal subgroup appear at most with multiplicity one is of certain interest for physical applications. The method of construction used here in the Coulomb potential case can be extended to various other cases. The geometrical emphasis may help to visualize things and provide us with a global form of the transformations.

We hope to develop this approach.

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APPENDIX

We want to derive formula (40) from (38). We have

$$Y^{(4)}_{n;L;M}(u) = \sqrt{\frac{n}{2\pi^2}} \sum_{m,m'} (jm;\ jm'|LM) \overline{\mathcal{L}^{j}_{mm'}\left[(u_o + i\vec{\sigma}\vec{u})\ \Gamma\right]}$$

$$u \equiv (u_o, \vec{u}) \quad 2j + 1 = n \quad .$$

Let us first remark that from Eq. (27) $\overline{\mathcal{L}^{j}_{mm'}\left[(u_o + i\vec{\sigma}\vec{v})\ \Gamma\right]}$ where $\vec{v}$ is a vector along the z-axis, vanishes unless $m + m' = 0$. Now any $\vec{u}$ can be written $R_{\alpha,\beta,o}\,\vec{v}$ with α, β the polar angles of $\vec{u}$. From the very construction of $Y^{(4)}$ [Eq. (39)] we have

$$Y^{(4)}_{n,L,M}(u_o, \vec{u}) = Y^{(4)}_{n,LM}(u_o, R_{\alpha,\beta,o}\,\vec{v}) = Y^{(4)}_{n,L,o}(u_o, \vec{v})\,\mathcal{L}^{L}_{oM}(R^{-1}_{\alpha\beta}) \ ,$$

$$Y^{(4)}_{n,LM}(u_o, \vec{u}) = Y^{(4)}_{n,L,o}(u_o, \vec{v})\,\overline{\mathcal{L}^{L}_{Mo}(R_{\alpha\beta})} \quad .$$

In this last equality the second factor is, except for a factor $\sqrt{4\pi/(2L + 1)}$, the usual three dimensional spherical harmonic; hence

$$Y^{(4)}_{n,L,M}(u_o) = \sqrt{\frac{4\pi}{2L + 1}}\ Y^{(4)}_{n,L,o}(u_o, \vec{v})\ Y^{(3)}_{LM}\left(\frac{\vec{u}}{|\vec{u}|}\right) . \qquad \text{(A-1)}$$

The vector $\vec{v}$ is along the z-axis and its length is given by $|\vec{v}| = |\vec{u}| = \sqrt{1 - u_o^2}$. We shall write $u_o = \cos\delta$, $|\vec{v}| = \sin\delta$. It remains to study the first factor on the right hand side of this equation. For convenience we write

$$Y^{(4)}_{n;L,o}(u_o, \vec{v}) = i^L \sqrt{\frac{2L + 1}{2\pi^2}}\ T_{n,L}(\delta) \quad . \qquad \text{(A-2)}$$

From Eq. (38) and (27) it now follows that

$$T_{n,L}(\delta) = \sqrt{\frac{2j+1}{2L+1}}\ \frac{1}{i^L} \sum_m (j,m;\ j,-m|L,o)\overline{\mathcal{L}^{\,j}_{m,-m}(u_o + i\vec{v}\vec{\sigma})\ \Gamma}\ ,$$

$$T_{n,L}(\delta) = \sqrt{\frac{2j+1}{2L+1}}\ \frac{1}{i^L} \sum_m (j,m;\ j,-m|L,o)\ (-1)^{j-m}\ e^{-2im\delta}\ .$$

We shall now simply use known properties of the Clebsch-Gordan coefficients in order to express $T_{n,L}$ in a simpler way. First we note that

$$(j,m;\ j,-m|L,o) = (-1)^{2j-L}\ (j,-m;\ j,m|L,o)\ .$$

Hence

$$\overline{T_{nL}(\delta)} = T_{nL}(\delta)$$

$$T_{nL}(-\delta) = (-1)^L T_{nL}(\delta)\ .$$

That was the reason for introducing the factor i^L.

Then using $(j,m;\ j,-m|o,o) = (-1)^{j-m}\sqrt{2j+1}$ we find

$$T_{n,o}(\delta) = \sum_{m=-\frac{n-1}{2}}^{m=+\frac{n-1}{2}} e^{-2im\delta} = \frac{\sin n\delta}{\sin\delta} \qquad \text{(A-3)}$$

Furthermore

$$2m(j,m;\ j,-m|Lo) = (L+1)\sqrt{\frac{n^2-(L+1)^2}{(2L+1)(2L+3)}}\ (j,m;\ j,-m|L+1,o)$$

$$+\ L\sqrt{\frac{n^2-L^2}{(2L-1)(2L+1)}}\ (j,m;\ j,-m|L-1,o)\ .$$

Hence

$$\frac{d}{d\delta} T_{n,L}(\delta) = \frac{L+1}{2L+1} \sqrt{n^2 - (L+1)^2}\, T_{n,L+1}(\delta) + \frac{L}{2L+1} \sqrt{n^2 - L^2}\, T_{n,L-1} \qquad \text{(A-4)}$$

Using a similar technique we find, with the help of

$$(j,m;\ j,-m|L,o) + (j,m+1,\ j,-m-1|L,o) = \sqrt{\frac{L(L+1)}{j(j+1) - m(m+1)}}\ (j,m+1,\ j,-m|L,1)$$

and

$$(2m+1)\ (j,m+1;\ j,-m|L,1) = \sqrt{\frac{L(L+2)(n^2-(L+1)^2)}{(2L+1)(2L+3)}}\ (j,m+1;\ j,-m|L+1,1)$$

$$+ \sqrt{\frac{(L+1)(L-1)(n^2 - L^2)}{(2L+1)(2L-1)}}\ (j,m+1;\ j,-m|L-1,1)\ ,$$

that

$$\frac{d}{d\delta} \sin\delta\, T_{n,L}(\delta) = \sin\delta \left(-\frac{L}{2L+1} \sqrt{n^2-(L+1)^2}\, T_{n,L+1}(\delta) + \frac{L+1}{2L+1} \sqrt{n^2-L^2}\, T_{n,L-1}(\delta) \right)$$

$$\text{(A-5)}$$

Relations (A-4) and (A-5) are equivalent to

$$\begin{cases} \left(-\frac{d}{d\delta} + L \operatorname{cotg}\delta\right) T_{n,L}(\delta) = \sqrt{n^2 - (L+1)^2}\, T_{n,L+1} \\ \left(\frac{d}{d\delta} + (L+1) \operatorname{cotg}\delta\right) T_{n,L}(\delta) = \sqrt{n^2-(L)^2}\, T_{n,L-1} \end{cases} \qquad \text{(A-6)}$$

Using the fact that

$$-\frac{d}{d\delta} + L \operatorname{cotg}\delta \equiv \sin\delta^{L+1} \frac{d}{d\cos\delta} \frac{1}{\sin^L\delta}\ ,$$

we deduce from (A-3) that

$$T_{n,L}(\delta) = \frac{1}{\sqrt{(n^2-1^2)(n^2-2^2)\ldots(n^2-L^2)}} \sin\delta^L \left(\frac{d}{d\cos\delta}\right)^L \frac{\sin n\delta}{\sin\delta}\ . \qquad \text{(A-7)}$$

Putting this expression in (A-2) we get the desired result. The functions $T_{n,L}(\delta)$ are, of course, well known[4]. Our calculation relates them in a very simple way to the Clebsch-Gordan coefficients through

$$T_{nL}(\delta) = \frac{1}{\sqrt{(n^2-1^2)(n^2-2^2)\ldots(n^2-L^2)}} \sin\delta^L \left(\frac{d}{d\cos\delta}\right)^L \frac{\sin(n\delta)}{\sin\delta}$$

$$\equiv \sqrt{\frac{n}{2L+1}} \; \frac{1}{i^L} \sum_{m=-j}^{+j} (j,m;\, j,-m \,|\, Lo)\, (-1)^{j-m}\, e^{-2im\delta}$$

$$2j+1 \equiv n$$

It is not the place here to discuss further properties of these functions.

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6. Alliluev, J.E.T.P. 6, 156 (1958).

7. The first relation in Eq. (5) has an obvious geometrical meaning in the Kepler problem where L is orthogonal to the elliptical orbit and M is along the main axis with a length given by the excentricity of the ellipse times k. The expression for the energy differs from the classical one only by the $\hbar^2$ term.

8. If we vary p_o in Eq. (10) we obtain a mapping of the sphere onto itself of a type which will be of interest in the next section. It can be geometrically described as follows. First perform an inversion of radius $\sqrt{2}$ with center at the north pole of the unit sphere (this projects the sphere on the plane of Fig. 1); then a "scale" transformation ($\vec{u} \rightarrow \lambda\vec{u}$), finally the inversion again. The whole operation leaves the sphere invariant and the north and south poles do not move. It turns out that we can get the same result as the product of two inversions with respect to two spheres orthogonal to the unit one with centers on the north-south axis.

9. This is the content of Peter-Weyl theorem.

10. See for instance, A. Messiah, Mecanique Quantique, Dunod; Paris (1960) - Especially Tome II, Appendix C.

11. This is a familar device when using the conformal group and has been used also for non definite metric. We learned it from Dr. R. Stora.

12. It is interesting to note that the conformal group is a general invariance group for the Laplace equation in the following sense. If $f(u')$ is a solution of $\Delta'^{p} f(u') = 0$ then $g(u) = (\alpha_{tt} + \alpha_{tz} u^2 + \sum_j \alpha_{tj} u_j)^{(2-p)/2} f(u')$ with u' given by Eq. (57) satisfies also Laplace equation $\Delta^p g(u) = 0$. This result can be quickly obtained by checking it for the four types of transformations. The only non straightforward case (d) gives rise to the fact that if $Y_\lambda^{(p)}$ is a spherical harmonic then both $r^\lambda Y_\lambda^{(p)}$ and $r^{-(\lambda+p-2)} Y_\lambda^{(p)}$ are solutions of the Laplace equation; a property which we have already used.

13. The quantity $|a_{oo} + \Sigma\, a_{oj} u_j|$ is real positive; it is clear that to satisfy the unitarity condition we can more generally set

$$[T^\Lambda f](u) = \frac{f(\Lambda^{-1} u)}{\left| a_{oo}(\Lambda^{-1}) + \sum_j a_{oj}(\Lambda^{-1}) u_j \right|^{\frac{p-1}{2} + i\gamma}}$$

where γ is real and say non negative. Different γ define inequivalent representations. This shows that the representation of the group G is not uniquely defined. The minor changes in adding λ are all very simple and we do not include them in the text. The only formula for which this does not go through as simply is Eq. (80) which is the basis of the introduction of "non-compact" operators in reference 1.

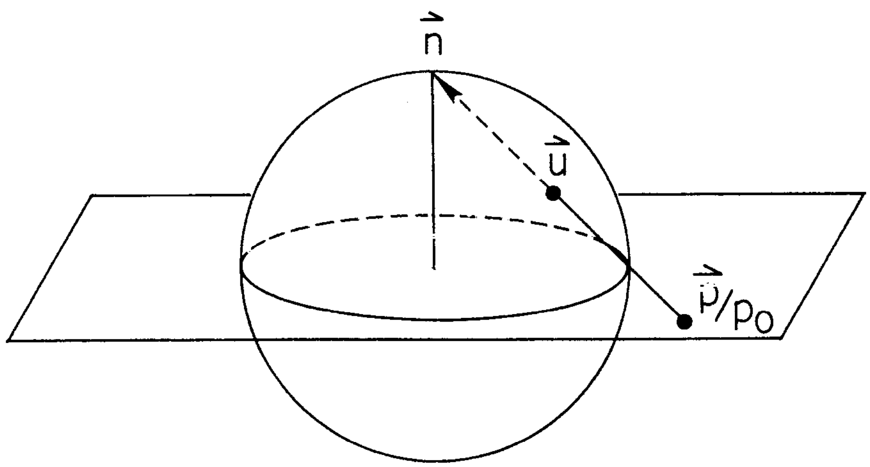
FIGURE CAPTIONS

Fig. 1. Stereographic projection of the f dimensional space to the unit sphere in f + 1 dimensions.

Fig. 2. The surface of integration in Green's formula.

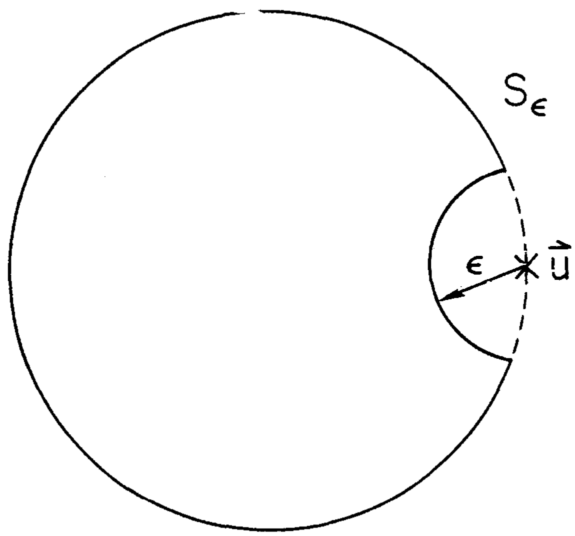
Fig. 3. Parabolic coordinates.

Fig. 4. Projection of a p dimensional space on a paraboloid in p + 1 dimensions.



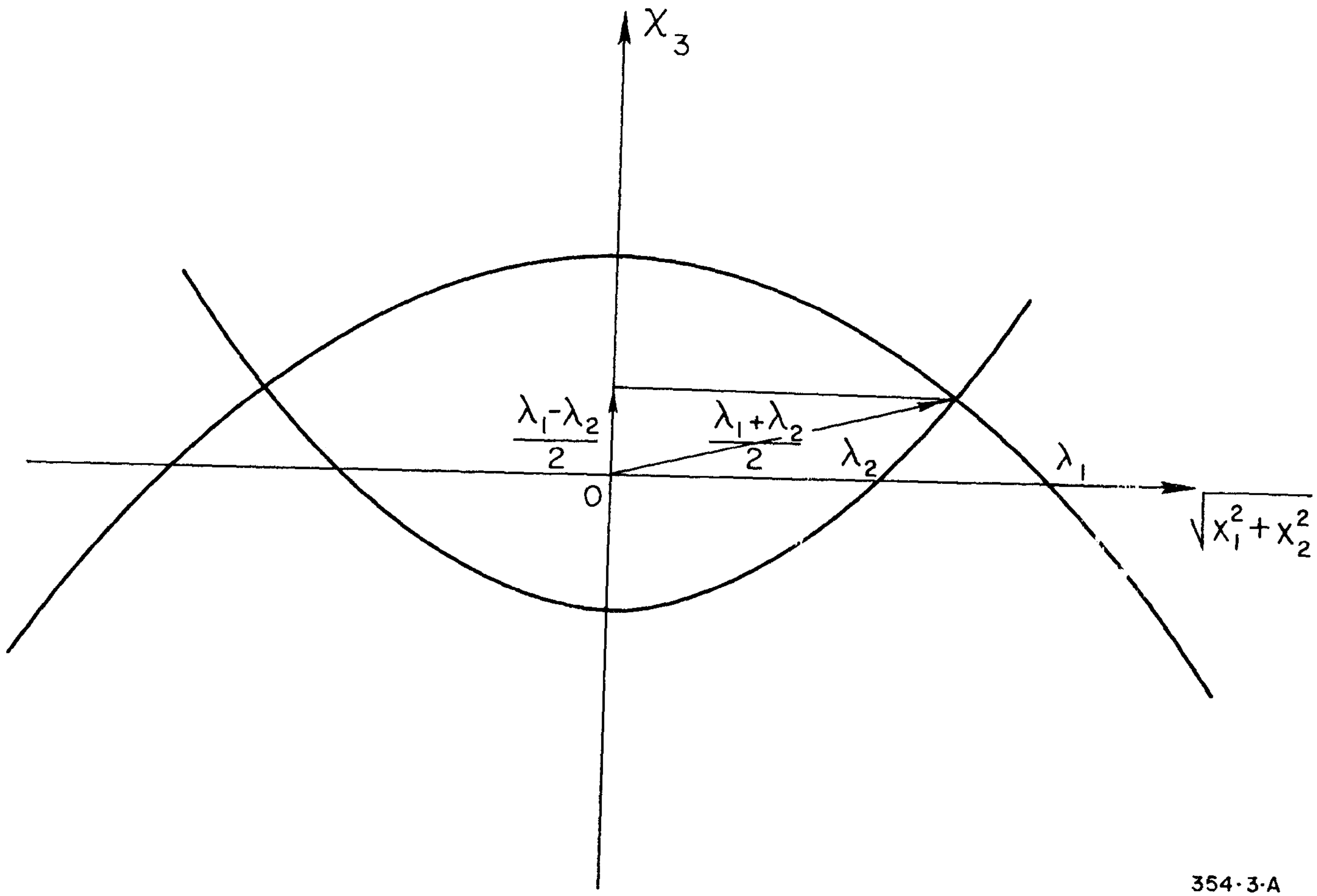
354·1·A

FIG.- 1



354·2·A

FIG.-2



X_3
$\frac{\lambda_1-\lambda_2}{2}$
$\frac{\lambda_1+\lambda_2}{2}$
λ_2
λ_1
0
$\sqrt{x_1^2+x_2^2}$
354·3·A

FIG. - 3

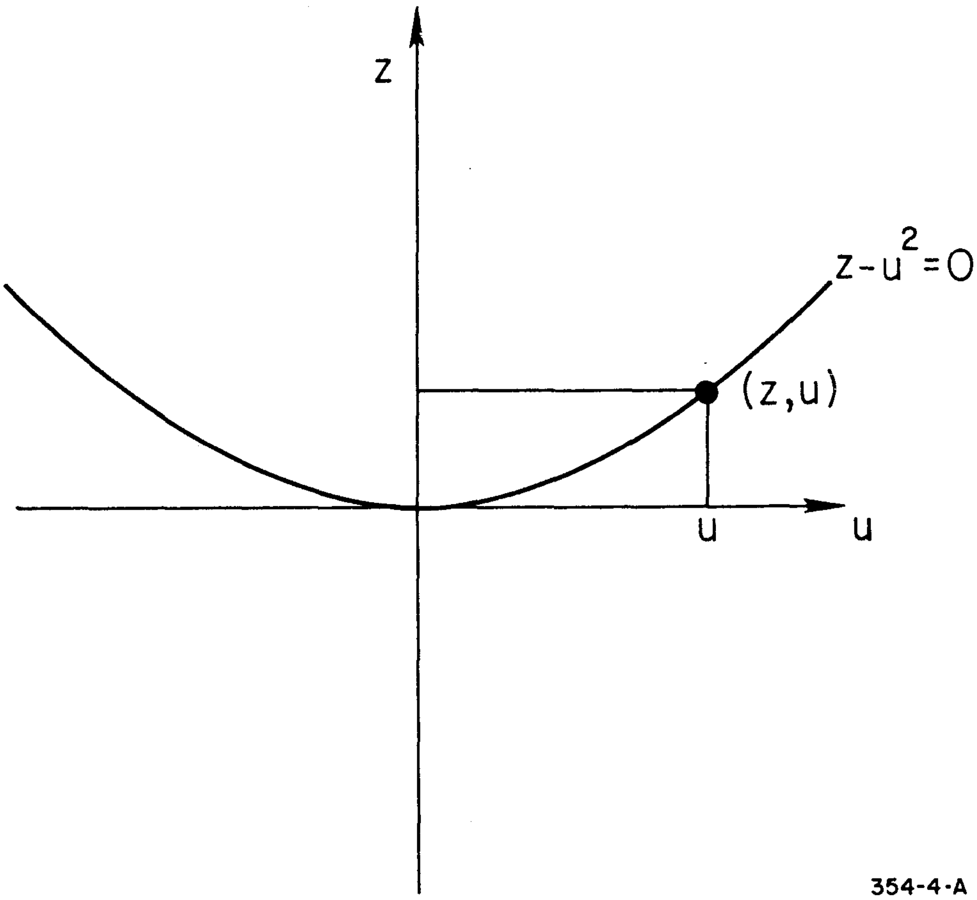


FIG.-4