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APPROXIMATE SOLUTION TO THE MULTICHANNEL ND^{-1} EQUATIONS*

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ABSTRACT

By making a pole approximation to the spectral integral over the kinematical factor $\rho_\ell(z)$ it is shown that the partial wave matrix ND^{-1} integral equations are reduced to algebra. The approximation depends only on the particular partial wave and not the dynamics of the reaction and it admits of systematic improvement. The resulting scattering amplitude $T_\ell(z)$ is symmetric, independent of the subtraction point for the D function, has the correct-discontinuities on the right and left hand cuts and can moreover be explicitly expressed as an algebraic function of the driving term $B_\ell(z)$. This last feature enables us to directly inspect the relation between the driving force and the scattering amplitude and establishes the general usefulness of the method. We find, for example, that the solution imposes general conditions on $B_\ell(z)$ for the existence of bound states, resonances or possible ghosts. The self-consistency property of bootstrap calculations imposes additional explicit restrictions on acceptable $B_\ell(z)$ for the existence of the bootstrap.

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$x^{(\nu)}$

I. INTRODUCTION

In partial wave dispersion relations for multichannel processes the matrix ND^{-1}method[1,2] has come to assume a major role in obtaining solutions to the non-linear integral equations arising out of the demand of unitarity on the partial wave scattering amplitude $T_\ell(z)$. More recently many of the dynamical models[3-10] based on the bootstrap hypothesis[11] have appealed to the ND^{-1} integral equation formalism as a means for obtaining self-consistent solutions to the coupled integral equations which arise from the dynamics imposed by unitarity and analyticity.[12,13] Although some understanding of the nature of the exact solutions to the ND^{-1} equations has been obtained,[14,15,16] particularly in the single channel case, no general solutions to the ND^{-1} equations has been found. Consequently one is motivated to find approximations to the integral equations which admit of systematic improvement towards the exact solutions. Various approximation schemes begining with the determinental method of Baker[17] and including the methods of Shaw,[18] Fulton[19] and Martin[20] have been proposed each of which enjoys undesirable features. It is known that the exact solution $T_\ell(z)$ is independent of the subtraction point for the D function[20] and is symmetric,[20,21] a reflection of time reversal invariance, for a symmetric driving term $B_\ell(z)$. The determinental method yields solutions which have neither of these features. The other approximations methods while they have the property of independence of the subtraction point and symmetry of $T_\ell(z)$ proceed by approximating and modifying the driving term $B_\ell(z)$. This last feature implies that a different approximation must be found for each scattering problem with a different $B_\ell(z)$

and no general insight as to the relation between $T_\ell(z)$, the scattering amplitude, and the driving term, representing the forces, is obtained. Hence much of the physics can remain obscure.

In this paper we present another approximation scheme which admits of systematic improvement. It has the desired features of providing a solution that is independent of the subtraction point and has a symmetric $T_\ell(z)$ for a symmetric input $B_\ell(z)$. Moreover since we approximate only a spectral integral over $\rho_\ell(z)$, the kinematical factor, on the left by putting a pole on the right, the dynamical term $B_\ell(z)$ remains unchanged and we are able to establish an explicit algebraic expression for $T_\ell(z)$ in terms of $B_\ell(z)$, Eq. (21), which has the correct discontinuities across the right- and left-hand cuts. With such an explicit algebraic solution available we may investigate directly the dependence of the scattering matrix on the parameters appearing in $B_\ell(z)$ as the coupling constants and mass ratios. For applications such as bootstrap calculations, this method, since it involves only algebra and no integrations, provides a reduction of computing time and alleviates the characteristic "curse of dimensionality."[22] More importantly it offers a very direct method for obtaining physical insight into the bootstrap mechanism.

II. APPROXIMATION METHOD

Our integral equation formalism is based on the ND^{-1} matrix formulation of Bjorken[2] and we assume an unsubtracted dispersion relation for the symmetric scattering amplitude $T_\ell(z)$

$$T_\ell(z) = B_\ell(z) + \frac{1}{\pi} \int_R dz' \frac{T^*_\ell(z')\rho_\ell(z')T_\ell(z')}{z' - z} \qquad (1)$$

Here R stands for the integration over the right hand unitarity cuts, z is the energy variable, $\rho_\ell(z)$ is a diagonal matrix of kinematical factors containing the θ-functions for the two particle thresholds and $B_\ell(z)$ represents the forces arising from the dynamical singlarities on the left. We assume $B_\ell(z)$ admits of the Hilbert representation,

$$B_\ell(z) = \frac{1}{\pi} \int_L \frac{dz' \ \mathrm{Im}B_\ell(z')}{z' - z} \tag{2}$$

where the integral extends over the left hand cuts. Dropping the subscript ℓ we assume next that the solutions to Eq. (1) are of the form

$$T(z) = N(z) \ D^{-1}(z) \tag{3}$$

where $D(z)$ has cuts only on the right and $T(z)$ has no CDD poles.[23] Since both $N(z)$ and $D(z)$ can be multiplied by an arbitary constant matrix without effecting the solutions $T(z)$ we have the freedom to normalize $D(z)$ to 1 at some point $z = s_o$. $N(z)$ is assumed to have cuts only on the left and we assume $N(z) \to 0$ as $|z| \to \infty$. We thus obtain from these demands and the unitarity condition, $\mathrm{Im}T^{-1}(z+i\epsilon) = -\rho(z)$, the representations[17]

$$D(z,s_o) = 1 - \frac{z-s_o}{\pi} \int_R \frac{dx\rho(x)N(x,s_o)}{(x-s_o)\ (x-z)} \tag{4}$$

$$N(z,s_o) = \frac{1}{\pi} \int_L \frac{dx \ \mathrm{Im}N(x,s_o)}{x - z} \tag{5}$$

where

$$\mathrm{Im}N(x,s_o) = \mathrm{Im}B(x) \ D(x,s_o) \ . \tag{6}$$

The solutions to this inhomogeneous linear system of integral equations will then be a solution of the original non-linear Eq. (1).

Substituting Eq. (5) in (4) and interchanging orders of integration, which causes no additional terms to appear since the integrations are over different ranges, one obtains

$$D(z,s_o) = 1 + \frac{z-s_o}{\pi} \int_L dy K(z,y,s_o)\ \mathrm{Im}N(y,s_o) \qquad (7)$$

where the kernal depends only on the kinematical factor and is symmetric in z, y and s_o :

$$K(z,y,s_o) = \frac{1}{\pi} \int_R \frac{dx\ \rho(x)}{(x-s_o)(x-z)(x-y)} . \qquad (8)$$

Using

$$\frac{1}{x-z}\ \frac{1}{x-s_o} = \frac{1}{z-s_o} \left(\frac{z}{x(x-z)} - \frac{s_o}{x(x-s_o)} \right)$$

we may write for $K(z,y,s_o)$

$$K(z,y,s_o) = \frac{zF(z)}{(z-y)(z-s_o)} + \frac{yF(y)}{(y-z)(y-s_o)} + \frac{s_oF(s_o)}{(s_o-z)(s_o-y)} \qquad (9)$$

where the diagonal matrix

$$F_\ell(z) = \frac{z}{\pi} \int_R \frac{dx\rho_\ell(x)}{x^2(x-z)} \qquad (10)$$

depends only on z and the kinematical factor $\rho_\ell(x)$. Substituting Eq. (9) in (7) and doing algebra there results

$$D(z,s_o) = 1 - zF(z)N(z,s_o) + s_oF(s_o)N(s_o,s_o) + \frac{1}{\pi} \int_L dy F(y)\mathrm{Im}N(y,s_o) \left(\frac{z}{y-z} - \frac{s_o}{y-s_o} \right) \qquad (11)$$

In obtaining Eq. (11) no approximations have been made. Our approximation now follows from the observation that to obtain D in terms of N we need to know $F(y)$ given by Eq. (10) on the left hand cuts. The function $H_\ell(z) = F_\ell(z)/z$ is a spectral integral over the positive definite kinematical factor, $\rho_\ell(z) > 0$, and hence on the left will have all its derivatives positive, $H_\ell^{(n)}(z) > 0$, and can be quite accurately approximated on the left by a pole on the right

$$F_\ell(z)/z = H_\ell(z) \simeq C_\ell/(z-a_\ell) \tag{12}$$

where C_ℓ and a_ℓ are constants which are chosen to reproduce $H_\ell(z)$, which is known exactly once $\rho_\ell(z)$ is given, as best as possible. The constants C_ℓ and a_ℓ are completely determined once the partial wave is specified. The one pole approximation is rather good (see Fig. 2) reproducing the exact $H_\ell(z)$ within 8% over a large range. If greater accuracy is desired one may add more pole terms or double poles to Eq. (12) more closely approximating the exact $H_\ell(z)$. However for most applications, for which one assumes the nearby singularities dominant, Eq. (12) will suffice and reproduces the main features of the exact solution. Of course, no finite number of pole terms can ever reproduce the exact $H_\ell(z)$ with its branch cut. This is the only approximation we shall make and it does not presume a modified form for $B(z)$. This approximation, moreover, has the advantage of reducing the ND^{-1} equations to algebra.

Substitution of Eq. (12) in Eq. (11) yields D in terms of N:

$$D(z,s_o) = g(s_o) - zF(z)N(z,s_o) + \frac{Cz}{z-a}\left(zN(z,s_o) - aN(a,s_o)\right) \tag{13}$$

where

$$g(s_o) = 1 + s_o F(s_o) N(s_o,s_o) - \frac{Cs_o}{s_o - a}\left(s_o N(s_o,s_o) - a\, N(a,s_o)\right) \qquad (14)$$

$$g(o) = 1$$

This expression (Eq. (13)) has the properties of the original integral Eq. (4): $D(z,s_o) - 1 = - D(s_o,z) + 1$ and unitarity $\mathrm{Im}D(z,s_o) = - \rho(z) N(z,s_o)$. Using $F(z) \simeq Cz/(z-a)$ Eq. (13) becomes

$$D(z,s_o) = g(s_o) - \frac{Caz}{z-a}\, N(a,s_o) \qquad (15)$$

which, within the limits of our approximation, is an accurate expression for $D(z)$ along the left cut. This is what is required to solve for $N(z)$ from $\mathrm{Im}N(z,s_o) = \mathrm{Im}B(z) D(z,s_o)$ and Eq. (5). From Eqs. (5), (6), (15) and the identity

$$\frac{x}{(x-a)(x-z)} = \frac{z}{(z-a)(x-z)} - \frac{a}{(z-a)(x-a)}$$

there results

$$N(z,s_o) = B(z) g(s_o) - \left(zB(z) - aB(a)\right) \frac{CaN(a,s_o)}{z-a} \qquad (16)$$

for $N(z)$ in terms of $B(z)$. Since $D(z,s_o)$ given by Eq. (13) and $N(z,s_o)$ given by Eq. (16) imply $T(z) = N(z,s_o) D^{-1}(z,s_o)$ is independent of s_o (Appendix I) and symmetric $T(z) = T^T(z)$ (Appendix II) we may set $s_o = 0$ without loss of accuracy and a gain in simplicity.

Then from Eq. (16)

$$N(z) = B(z) - \left(zB(z) - aB(a)\right) \frac{CaN(a)}{z-a} \qquad (17)$$

and Eq. (13)

$$D(z) = 1 - zF(z)N(z) + \frac{Cz}{z-a}\left(zN(z)-aN(a)\right) \tag{18}$$

and one finds from Eq. (17) setting $z=a$

$$N(a) = \left[1 + Ca\left(B(a) + aB'(a)\right)\right]^{-1} B(a) \tag{19}$$

where $B'(a) = dB(a)/da$. For S waves, $\ell=0$, because of the asymptotic behavior of kinematical factor we find that it is better to approximate $F(z) = C/(z-a)$ instead of $H(z)$ (see Appendix III). Then Eq. (18) is replaced by

$$D(z) = 1 - zF(z)N(z) + \frac{Cz}{z-a}\left(N(z) - N(a)\right)_{\ell=0} \tag{20}$$

and Ca replaced by C in Eq.(17) and (19). The addition of more pole terms and double poles to our approximation does not effect the method for obtaining $N(z)$ and $D(z)$ as explicit algebraic functions of $B(z)$ although the inversion going from Eq. (17) to (19) would now involve the solving of a linear system of algebraic matrix equations which can be solved using standard techniques.

Our solution for $T(z)$, the scattering amplitude, as an explicit function of the driving terms, $B(z)$, is

$$T(z) = \left[B(z) - \left(zB(z) - aB(a)\right)\frac{Ca}{z-a}N(a)\right] \cdot$$

$$\left[1 - \frac{Cza}{z-a}N(a) - z\left(F(z) - \frac{Cz}{z-a}\right)\left(B(z) - \left(zB(z) - aB(a)\right)\cdot\frac{Ca}{z-a}N(a)\right)\right]^{-1} \tag{21R}$$

where $N(a)$ is given by Eq. (19). On the left cut Eq.(21R) becomes

since $F(z) \simeq Cz/(z-a)$

$$T(z) = \left[B(z) - \left(zB(z) - aB(a)\right) \frac{Ca}{z-a} N(a)\right] \left[1 - \frac{Cza}{z-a} N(a)\right]^{-1} \qquad (21L)$$

This solution has the correct discontinuities since (21R) implies $\mathrm{Im}T^{-1}(z) = -\rho(z)$ and (21L) implies $\mathrm{Im}T(z) = \mathrm{Im}B(z)$. Hence we expect many of the features of the exact solution are reproduced by Eq. (21).

In obtaining these solutions we have assumed that all integrals are convergent. Our proposed approximate solution Eq.(21) for $T(z)$ in the low energy region depends only on $B(z)$ in that region and we may conjecture that the above solution is valid irrespective of $B(z)$ in the high energy region simply on the basis that it has the correct discontinuities.

III. BOUND STATE AND BOOTSTRAP CONDITIONS

For simplicity we restrict our attention to the single channel S wave case and we scale the energy so that the threshold is at $z=4$. Let us suppose the forces are attractive in this channel and sufficiently strong to produce a bound state with energy $0 < z_o < 4$. In this energy region, to the left of the unitarity cut, we approximate $F(z) \simeq C/z-a$ and from Eq. (20) and Eq. (19) (with Ca replaced by C) we have

$$D(z) = 1 - \frac{Cz}{z-a} N(a)$$

$$= 1 - \frac{Cz\ B(a)}{(z-a)\left(1 + C\left(B(a) + aB'(a)\right)\right)} . \qquad (22)$$

Factoring the coupling constant from the driving term $B(z) \to g^2 B(z)$ the condition that $T(z)$ have a simple pole at $z = z_o$ corresponding to the bound state implies $D(z_o) = 0$ which imposes the restriction on the form of $B(z)$:

$$\frac{1}{g^2} = \frac{Ca}{z_o - a} \left(B(a) - (z_o - a)\, B'(a) \right) \tag{23}$$

This condition has its basis in the unitary condition; it is understood to be approximate.

A similar condition may be obtained for the existence of a resonance using Eq. (20) for $D(z)$ instead of Eq. (22). Since $g^2 > 0$, $C < 0$, $0 < z_o < 4 < a$, Eq. (23) implies that for there to exist a bound state

$$B(a) - (z_o - a)\, B'(a) \geq 0 \tag{24}$$

the equality holding in the limit of infinite coupling. Equation (23) may admit solutions with $z_o < 0$ which implies the existence of a ghost state. One may, of course, use Eq. (23) to obtain the location of the bound state in terms of g^2, $B(z)$.

If we furthermore employ the bootstrap hypothesis then this bound state corresponds to the exchanged particle producing the force $B(z, z_o)$ and the residue of $T(z)$ at $z = z_o$ is the coupling constant $-g^2$. From the condition $\frac{1}{g^2} = - D'(z_o)/N(z_o)$ and Eq. (22) and (23) one obtains

$$\frac{1}{g^2} = - Ca \left[B^2(a, z_o) + B'(a, z_o) \right] \tag{25}$$

along with Eq. (23);

$$\frac{1}{g^2} = \frac{Ca}{z_o - a} \left[B(a, z_o) - (z_o - a)\, B'(a, z_o) \right] . \tag{26}$$

If the bootstrap is to work and the exchanged particle is to correspond to the bound state we have an additional restriction from Eqs. (25) and (26)

$$B(a,z_o)(a - z_o) = 1 \tag{27}$$

These bootstrap conditions for the single channel case have appropriate generalization to the multichannel case. If we use an improved approximation for $F(z)$ including additional pole terms then the conditions do not take the simple form above; in fact, there may be more than one solution for g^2 .[24]

From Eqs. (25) and (26) one can obtain the self-consistent solutions for g^2 and z_o if the bootstrap conditions are satisfied. One picks z_o so that Eq. (27) is satisfied and g^2 can then be obtained from either Eq. (25) or (26). For S waves a good one pole fit to $F(z)$ is obtained with $C = -1.8/\pi$, $a = 7.6$ (see Fig. 1 and Appendix III). In the bootstrap example of two pseudoscalar mesons of mass = 1 scattering in the S state via the exchange of a scalar meson of $(\text{mass})^2 = z_o$ to produce a bound S state with $(\text{mass})^2 = z_o$, the Born term is given by

$$B(z,z_o) = \frac{2}{(z-4)} \ln\left(1 + \frac{z-4}{z_o}\right) . \tag{28}$$

We find that condition (27) is satisfied for $z_o \simeq 4.5$ and (25) implies that the self-consistent value for the coupling is $g^2 \simeq 2.7$.

The multichannel pole approximation method can be expected to reduce self-consistency type calculations to matrix algebra prior to any

specification of the input forces in the crossed channels. The driving terms can then be solved in a self consistent manner.[25]

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APPENDIX I

Here we demonstrate the independence of $T(z)$ on the subtraction point. Our solution is

$$N(z,s_o) = B(z)\, g(s_o) - \Big(zB(z) - aB(a)\Big)\, \frac{CaN(a,s_o)}{z-a} \qquad (I.1)$$

$$D(z,s_o) = g(s_o) - zF(z)N(z,s_o) + \frac{Cz}{z-a}\Big(zN(z,s_o) - aN(a,s_o)\Big) \qquad (I.2)$$

Equation (I.1) with $z = s_o$ implies

$$N(a,s_o) = R(a)\, g(s_o) \qquad (I.3)$$

with $R(a)$ independent of s_o. Then Eq. (I.3) and (I.1) imply

$$N(z,s_o) = R(z)\, g(s_o) \qquad (I.4)$$

with $R(z)$ independent of s_o. Substitution of Eq. (I.3) and (I.4) into (I.2) implies

$$D(z,s_o) = Q(z)\, g(s_o) \qquad (I.5)$$

with $Q(z)$ independent of s_o and hence from (I.4) and (I.5) $T(z) = N(z,s_o)\, D^{-1}(z,s_o) = R(z)\, Q^{-1}(z)$ independent of s_o.

APPENDIX II

Here we shall show that $T(z) = T^T(z)$ for a symmetric driving term $B(z) = B^T(z)$. Since the solution is independent of s_o choose $s_o = z$ so that $D(z,z) = 1$ from (I.2) and $T(z) = N(z,z)$. Hence we must demonstrate $N(z,z) = N^T(z,z)$. Writing

$$g(z) = 1 + A(z)$$

where

$$A(z) = zN(z,z)F(z) - \frac{Cz}{z-a}\left(zN(z,z) - aN(a,z)\right) \qquad \text{(II.1)}$$

and $F(z) = F^T(z)$ and $C = C^T$ are diagonal matrices, we find from (I.1)

$$N(z,z) = B(z) + B(z)\,A(z) - \left(zB(z) - aB(a)\right)\frac{Ca}{z-a}\,N(a,z) \qquad \text{(II.2)}$$

the first term of which is obviously symmetric. Solving Eq. (II.2) for the first term $B(z)$, taking the transpose of the resulting equation and substituting this expression for $B(z)$ into the second two terms of (II.2) there results:

$$\begin{aligned} N(z,z) = B(z) &+ N^T(z,z)\left(zF(z) - \frac{Cz^2}{z-a}\right)N(z,z) + N^T(z,z)\,\frac{Cza}{z-a}\,N(a,z) \\ &- A^T(z)B(z)A(z) + N^T(a,z)\,\frac{Ca}{z-a}\left(zB(z) - aB(a)\right)\left(zF(z) - \frac{Cz^2}{z-a}\right)N(z,z) \\ &- z\Bigg\{N^T(z,z) - \left[zN^T(z,z)\left(F(z) - \frac{Cz}{z-a}\right) + N^T(a,z)\,\frac{Cza}{z-a}\right]B(z) \\ &+ N^T(a,z)\,\frac{Ca}{z-a}\left(zB(z) - aB(a)\right)\Bigg\}\,\frac{Ca}{z-a}\,N(a,z) + aB(a)\,\frac{Ca}{z-a}\,N(a,z) \\ = s(z) &- aN^T(a,z)\,\frac{Ca}{z-a}\,B(a)\left(zF(z) - \frac{Cz^2}{z-a}\right)N(z,z) + aB(a)\,\frac{Ca}{z-a}\,N(a,z) \end{aligned}$$

(II.4)

where $s(z)$ is a symmetric matrix consisting of terms of the form $W^T BW$, $W^T Bx + X^T BW$. We must still show that the remaining terms of Eq. (II.4) are symmetric. Setting $z=a$ and $s_o=z$ in Eq. (I.1) there results

$$Q(a)N(a,z) = B(a)\, g(z) \tag{II.5}$$

with $Q(a) = Q^T(a) = 1 + Ca\left(B(a) + aB'(a)\right)$. Multipling by $N^T(a,z)$ on both sides of Eq. (II.5) and using Eq. (II.1)

$$\begin{aligned} N^T(a,z)Q(a)N(a,z) = N^T(a,z)B(a) + N^T(a,z)B(a)\left(zF(z) - \frac{Cz^2}{z-a}\right)N(z,z) \\ + N^T(a,z)B(a)\,\frac{Cza}{z-a}\,N(a,z) \end{aligned} \tag{II.6}$$

when multiplied by $Ca^2/(z-a)$ and substituded in (II.4) yields

$$\begin{aligned} N(z,z) = s(z) + \Bigg[N^T(a,z)\,B(a) + B(a)N(a,z) + N^T(a,z)\,B(a)N(a,z)\,\frac{Cza}{z-a} \\ - N^T(a,z)Q(a)N(a,z)\Bigg]\,\frac{Ca^2}{z-a} \end{aligned}$$

so that $N(z,z) = N^T(z,z)$ where we have used the fact that C is diagonal and commutes with any matrix.

APPENDIX III

Here we consider the details of the pole approximation to the spectral integral. The function $F_\ell(z)$ depends only on the kinematical factor

$$F_\ell(z)/z = H_\ell(z) = \frac{1}{\pi} \int_R \frac{dx \rho_\ell(x)}{x^2(x-z)} . \qquad \text{(III.1)}$$

Setting the beginning of the unitarity cut at $z = 4$ and writing for the kinematical factor

$$\rho_o(z) = \left(\frac{z-4}{z}\right)^{\frac{1}{2}} ; \quad \rho_\ell(z) = \frac{(z-4)^{\frac{2\ell+1}{2}}}{z^{\frac{2\ell-1}{2}}} \qquad \text{(III.2)}$$

$$\ell=0 \qquad\qquad \ell \geq 1$$

we find for $H_\ell(z)$ from Eq. (III.1)

$$H_1(z) = \frac{1}{z\pi} \left[\frac{2}{3} + 2a^2 \left(1 - \frac{a}{2} \log \frac{a+1}{a-1}\right)\right]$$

$$H_{\ell+1}(z) = a^2 H_\ell(z) + \frac{2}{z(2\ell+3)\pi} \qquad \text{(III.3)}$$

$$a^2 = 1 - \frac{4}{z} \qquad\qquad \ell \geq 1$$

From Eq. (III.3) we have $H_1(z) \to \log z/z$; $z \to -\infty$. For S waves we shall use $F(z)$ instead of $H(z)$:

$$F_o(z) = \frac{2}{\pi z} \left[1 - \frac{a}{2} \log \frac{a+1}{a-1}\right] + \text{const} \qquad \text{(III.4)}$$

where we may set const = 0 since the addition of a constant to $F(z)$ does not change the solution. The single pole approximation then consists of

writing

$$H_\ell(z) \simeq C_\ell/(z-a_\ell) \quad \ell \geq 1$$

$$F_o(z) \simeq C_o/(z-a_o) \quad \ell = 0$$

and the constants C_ℓ and a_ℓ are chosen to reproduce the behavior of $H_\ell(z)$ and $F_o(z)$ given by (III.3-4) for $-60 < z \leq 4$.

For S and P waves the results of a single pole fit are shown in Fig. 1 and Fig. 2. We reproduce the exact function to the accuracy shown with $C_o = -1.8/\pi$ $a_o = 7.6$ and $C_1 = -1.71/\pi$, $a_1 = 14.3$.

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FIGURE CAPTIONS

1. Pole fit to the S wave spectral function $F_o(z)$. The solid line is $4\pi F_o(z)$ and the dotted line is the single pole fit with $a_o = 7.6$ and $C_o = -1.8/\pi$.

2. Pole fit to the P wave spectral function $H_1(z)$. The solid line is $4\pi F_o(z)$ and the dotted line is the single pole fit with $a_1 = 14.3$ and $C_1 = -1.71/\pi$.

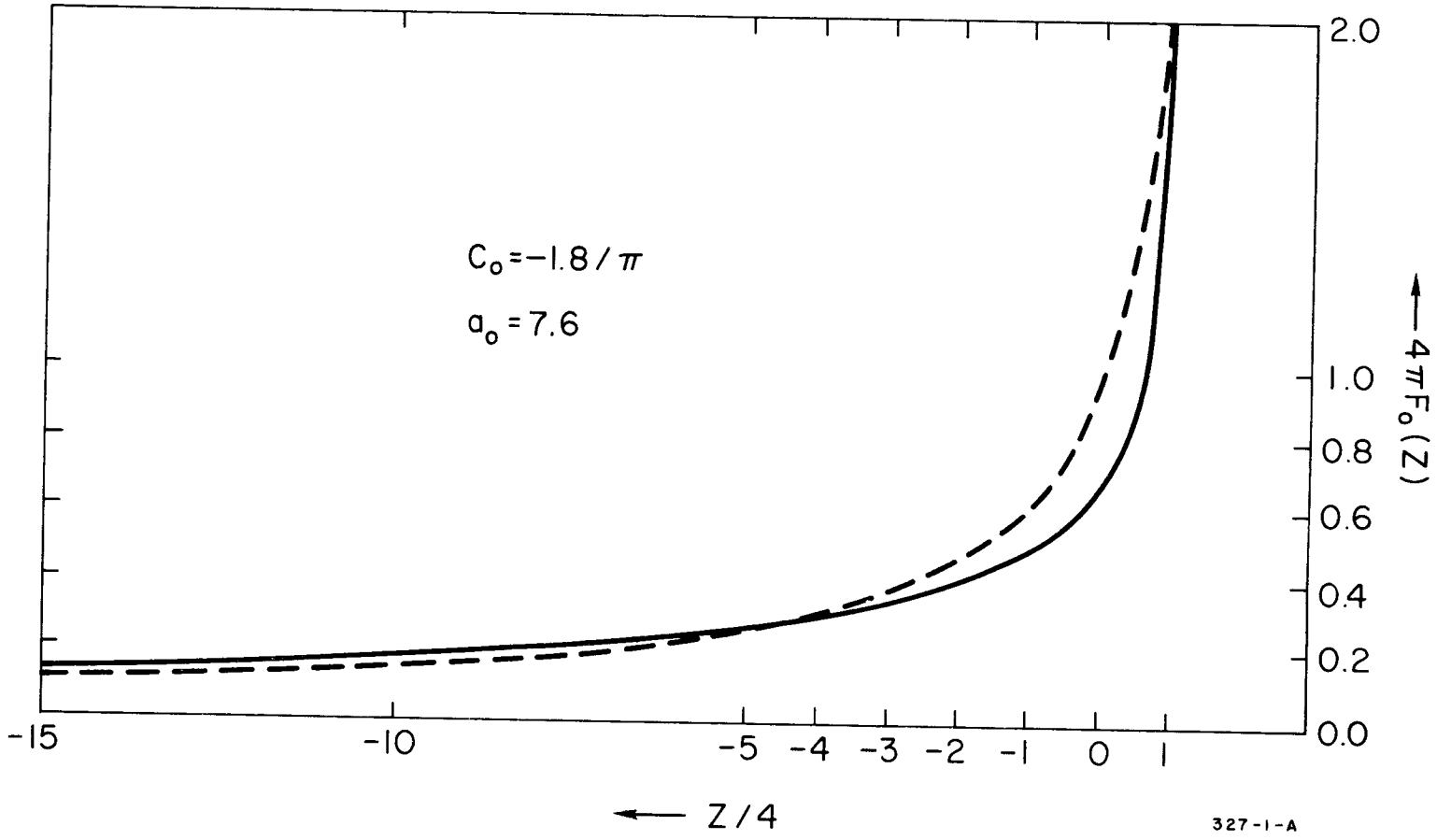
$C_o = -1.8/\pi$
$a_o = 7.6$
2.0
1.0
0.8
0.6
0.4
0.2
0.0
$4\pi F_o(Z)$
-15
-10
-5
-4
-3
-2
-1
0
1
$\leftarrow Z/4$
327-1-A

Fig. 1

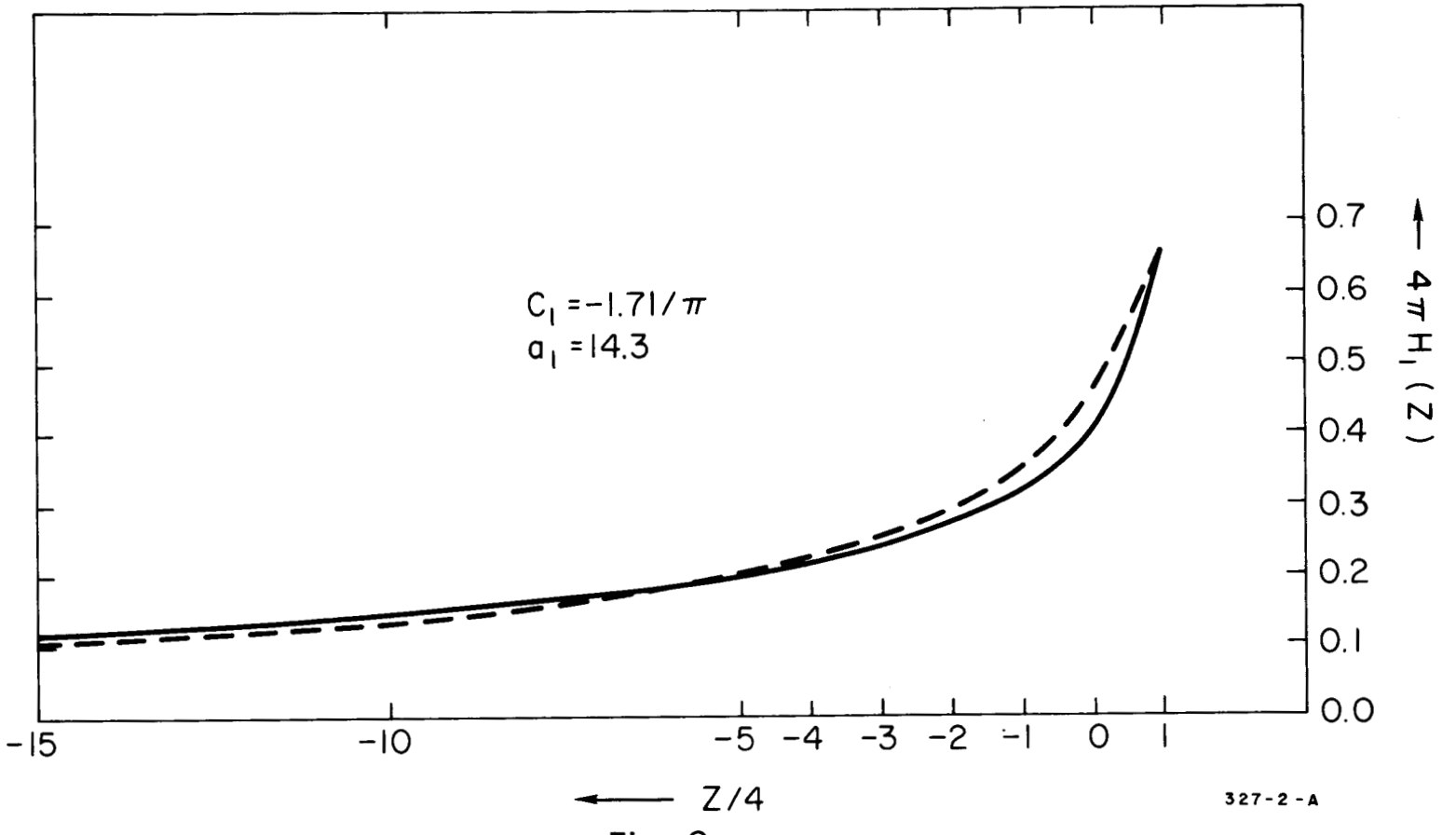
$C_1 = -1.71/\pi$
$a_1 = 14.3$
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
$4\pi H_1(Z)$
-15
-10
-5
-4
-3
-2
-1
0
1
Z/4
327-2-A

Fig. 2