ERRATA

SLAC-PUB-108 - "Renormalization of the Weak Axial-Vector Coupling Constant"

by

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The following corrections should be made on your copy of the above publication:

1. Equation 5, Page 3 should read:

$$\partial^{\mu} A_{\mu}^{i}(x) = \frac{i \sqrt{2}\, M \mu^2}{g_{\pi n} K_{\pi n n}(0)} \frac{G_a}{G_v} \varphi_{\pi}^{i}(x) \tag{5}$$

2. Equation 9, Page 4 should read:

$$\begin{aligned} < P(p_2) \left| Q_a^+ \right| \alpha_{out} > &= \frac{-i < P(p_2) \left| \dot{Q}_a^+ \right| \alpha_{out} >}{E_p - E_\alpha} \\ &= \left[\frac{\sqrt{2}\, M \mu^2}{g_{\pi n} K_{\pi n n}(0)} \right] \frac{G_a}{G_v} \int d^3x \; \frac{< P(p_2) \left| \varphi_\pi^+ (x) \right| \alpha_{out} >}{E_p - E_\alpha} \end{aligned} \tag{9}$$

3. Equation 13, Page 6 should read:

$$\begin{aligned} &\int_{k_{o\,min}}^{\infty} \frac{dk_o}{k_o^2} \sum_{\alpha \neq N} \left| <_{out} \alpha \left| \varphi_\pi^-(o) \right| P(p) > \right|^2 \delta^{(3)}(\vec{p}-\vec{p}_\alpha)\, \delta(E_p + k_o - E_\alpha) = \\ &\int_{k_{o\,min}}^{\infty} \frac{dk_o}{k_o^2} \left\{ \sum_{\alpha \neq N} \frac{\left| <_{out} \alpha \left| j_\pi^-(o) \right| P > \right|^2 \delta^{(3)}(\vec{p}-\vec{p}_\alpha)\, \delta(E_p + k_o - E_\alpha)}{(k_o^2 - \mu^2 + i\epsilon)(k_o^2 - \mu^2 - i\epsilon)} \right. \\ &\qquad - \frac{\delta(k_o - \mu)}{(2\pi)^{3/2} \sqrt{2\mu}} \left[\frac{<_{out} P(p)\, \pi^- (\vec{k} = 0) \left| j_\pi^- (o) \right| P(p) >}{k_o^2 - \mu^2 + i\epsilon} \right. \\ &\qquad \left. \left. + \frac{< P \left| j_\pi^+(o) \right| P(p)\, \pi^-(\vec{k} = 0)\ _{out}>}{k_o^2 - \mu^2 - i\epsilon} \right] \right\} . \end{aligned} \tag{13}$$

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RENORMALIZATION OF THE WEAK AXIAL-VECTOR COUPLING CONSTANT*

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It has been strongly suggested by M. Gell-Mann[1] that, "the integrals of the time components of the vector and the axial-vector current octets ... generate, under equal time commutation, the algebra of $SU(3) \otimes SU(3)$," and that these algebraic relations are preserved even though the axial-vector and the strangeness changing vector currents are not conserved. These non-linear commutation relations fix the relative scale of the vector and axial-vector matrix elements measured in the weak interactions.

In this letter these ideas are combined with that of a partially conserved $\Delta Y = 0$ axial-vector current to obtain an expression in terms of π-proton total cross sections, (24), for $|G_a/G_V|$, the absolute ratio of renormalized axial-vector and vector coupling constants of ordinary β-decay. A numerical evaluation using experimental data for strong interaction π-nucleon scattering yields

$$|G_a/G_V| = 1.16 \tag{1}$$

The present experimental value[2] is

$$G_a/G_V = -1.18 \pm 0.02 .$$

We consider the charges defined by

$$\begin{aligned} I^i &= \int d^3x \, V_o^i \\ Q_a^i &= \int d^3x \, A_o^i \end{aligned} \qquad i = 1,2,3 \tag{2}$$

where V_o^i, A_o^i are the time-components of the isovector members of the vector and axial-vector current octets. $\vec{V}_\mu$ is, in fact, the conserved isotopic spin current[3] so that $\vec{I}$ is the total isotopic spin operator. $\vec{Q}_a$ is the isotopic chirality. The effective interaction for $\Delta Y = 0$ leptonic decays of the hadrons is taken as

$$- L_{eff} = \frac{G_v}{\sqrt{2}} j^{\mu}_{lept} (V^{+}_{\mu} \pm A^{+}_{\mu}) + h.a.,$$
$$V^{\pm}_{\mu} = V^{1}_{\mu} \pm iV^{2}_{\mu} \quad . \tag{3}$$

For matrix elements between physical proton and neutron states of equal momentum it follows that

$$< P(p) \left| A^{+}_{\mu}(x) \right| N(p) > = \frac{(M/E_p)}{(2\pi)^3} \frac{G_a}{G_v} \bar{u}(p) \gamma_{\mu}\gamma_5 u(p) \tag{4}$$

which defines G_a , the renormalized axial-vector coupling constant.

By partial conservation of the axial-vector current, (P.C.A.C.)[4,5], we mean

$$\partial^{\mu} A^{i}_{\mu}(x) = \frac{i \sqrt{2} M \mu^2}{g_{\pi n} K_{\pi nn}(0)} \varphi^{i}_{\pi}(x) \tag{5}$$

where $\varphi_{\pi}(x)$ is the renormalized Heisenberg field of the π-mesons; M = nucleon mass, μ = pion mass, $g^2_{\pi n}/4\pi = 14.6$, and $K_{\pi nn}(0)$ is the invariant π-nucleon vertex function evaluated at zero pion mass.

The commutation rule which we use is

$$2I_3 = \left[Q^{+}_{a} , Q^{-}_{a} \right] . \tag{6}$$

Adapting the method of Fubini and Furlan,[6] we take matrix elements of (6) between physical one-proton states which gives

$$\delta^{(3)}(\vec{p}_2 - \vec{p}_1) = < P(p_2) \left| \left[Q^{+}_{a}, Q^{-}_{a} \right] \right| P(p_1) > \tag{7}$$

We introduce a complete set of physical intermediate states in the right hand side of (7) and isolate the contribution of one-neutron states

$$\delta^{(3)}(\vec{p}_2-\vec{p}_1) = (G_a/G_v)^2 \left[1 - \left(\frac{M}{E_p}\right)^2\right] \delta^{(3)}(\vec{p}_2-\vec{p}_1)$$

$$+ \overline{\sum_{\alpha\neq N}} < P(p_2) \left|Q_a^+\right| \alpha >_{out} <_{out}\alpha \left| Q_a^-\right| P(p_1) > \tag{8}$$

$$- \overline{\sum_{\beta}} < P(p_2) \left|Q_a^-\right| \beta >_{out} <_{out}\beta \left| Q_a^+\right| P(p_1) > .$$

From (5)

$$< P(p_2) \left|Q_a^+\right| \alpha_{out} > = \frac{-i < P(p_2) \left|\dot{Q}_a^+\right| \alpha_{out} >}{E_p - E_\alpha}$$

$$= \left[\frac{\sqrt{2}\, M\, \mu^2}{g_{\pi n} K_{\pi nn}(0)}\right] \int d^3x \; \frac{< P(p_2) \left| \varphi_\pi^+ (x) \right| \alpha_{out} >}{E_p - E_\alpha} \tag{9}$$

We then obtain for the second term on the right side of (8)

$$\sum_{\alpha\neq N} < P(p_2) \left|Q_a^+\right| \alpha_{out} > <_{out}\alpha \left| Q_a^-\right| P(p_1) > = (2\pi)^6 \, \delta^{(3)}(\vec{p}_2-\vec{p}_1) \left[\frac{\sqrt{2}\, M\, \mu^2}{g_{\pi n} K_{\pi nn}(0)}\right]^2$$

$$(G_a/G_v)^2 \int_{k_{o\,min}}^{\infty} \frac{dk_o}{k_o^{\,2}} \overline{\sum_{\alpha\neq N}} \left| <_{out}\, \alpha \left| \varphi_\pi^- (o) \right| P(p_1) > \right|^2 \times \delta^{(3)}(\vec{p}_1-\vec{p}_\alpha)$$

$$\delta(E_p + k_o - E_\alpha) \tag{10}$$

with

$$k_{o\,min} = \left[(M + \mu)^2 + \left|\vec{p}_1\right|^2\right]^{\frac{1}{2}} - E_p \, ,$$

$$0 \leq k_{o\,min} \leq \mu \, .$$

The multiplicative factor, $\delta^{(3)}(\vec{p}_2 - \vec{p}_1)$, appears in all terms of (9) and will be dropped. We put $\vec{p}_2 = \vec{p}_1 = \vec{p}$. The matrix elements occuring in (10) can be classified into two types of Feynman diagrams: (a) connected graphs which correspond to scattering from an initial state of the proton and the off-mass-shell pion, in the rest frame of the pion, to the final state, $\langle_{out}\, \alpha |$; (b) disconnected graphs corresponding to propagation of the proton without interaction. For graphs of type (a)

$$\langle_{out}\, \alpha | \varphi_\pi^-(0) | P(p) \rangle_{con.} = - \frac{\langle_{out}\, \alpha | j_\pi^-(o) | P(p) \rangle}{k_o^2 - \mu^2 + i\epsilon} \tag{11}$$

where

$$j_\pi(x) \equiv (\Box + \mu^2)\, \varphi_\pi(x) \ .$$

For graphs of type (b) we write

$$\langle_{out}\, \alpha | \varphi_\pi^-(o) | P(p) \rangle_{disc.} = \delta^{(3)}(\vec{p} - \vec{p}\,') \langle_{out}\, \alpha' | \varphi_\pi^-(o) | 0 \rangle$$

where $\vec{p}\,'$ is the momentum of the free proton in the state $\langle_{out}\, \alpha |$.

When taking the absolute square of the matrix elements in (10) there will be contributions from squares of connected graphs, squares of disconnected graphs, and cross-terms. All terms from squared disconnected graphs can be neglected since they will be exactly cancelled by corresponding contributions from the other term of the commutator in (8). The cross-terms should be dominated by $|\alpha' \rangle = | \pi \rangle$, a physical one-pion state. Other states have higher thresholds in k_o , and their contribution should be damped strongly in the k_o - integration of (10). This assumption is in the basic spirit of the P.C.A.C. hypothesis. One then obtains

$$\int_{k_{o\,min}}^{\infty} \frac{dk_o}{k_o^{\,2}} \sum_{\alpha \neq N} \left| \langle_{out}\, \alpha \,|\, \varphi_\pi^-(o) \,|\, P(p) \rangle \right|^2 \delta^{(3)}(\vec{p}-\vec{p}_\alpha)\, \delta(E_p + k_o - E_\alpha) =$$

$$\int_{k_{o\,min}}^{\infty} \frac{dk_o}{k_o^{\,2}} \Bigg\{ \frac{\left| \langle_{out,}\, \alpha \,|\, j_\pi^-(o) \,|\, P \rangle \right|^2 \delta^{(3)}(\vec{p}-\vec{p}_\alpha)\, \delta(E_p + k_o - E_\alpha)}{(k_o^2 - \mu^2 + i\epsilon)\,(k_o^2 - \mu^2 - i^{\epsilon})}$$

$$- \frac{\delta(k_o - \mu)}{(2\pi)^{3/2}\, \sqrt{2\mu}} \left[\frac{\langle_{out}\, P(p)\, \pi^-(\vec{k}=0) \,|\, j_\pi^-(o) \,|\, P(p) \rangle}{k_o^2 - \mu^2 + i\epsilon} \right.$$

$$\left. + \frac{\langle P \,|\, j_\pi^+(o) \,|\, P(p)\, \pi^-(\vec{k}=0)\ {}_{out}\rangle}{k_o^2 - \mu^2 - i\epsilon} \right] \Bigg\}. \qquad (13)$$

The frightening looking singularities in the above expression will be cancelled by unitarity. Since the T-matrix elements in (13) are multiplied by $\delta(k_o - \mu)$, they can be continued in the pion-mass from μ to k_o. That is,

$$\frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\mu}} \langle_{out}\, P(p)\, \pi^-(\vec{k}=0) \left| j_\pi^-(o) \right| P(p) \rangle \xrightarrow[\mu \to k_o]{} \frac{1}{(2\pi)^3 (2k_o)}\, T_{\pi^- p}(k_o, E_p).$$

$$T_{\pi^- p}(k_o, E_p) = i \int d^4y\, e^{ik_o y_o} \langle P(p) \left| T\left(j_\pi^+(y)\ j_\pi^-(o)\right) \right| P(p) \rangle \qquad (14a)$$

$$+ \text{(equal time commutators independent of } k_o) .$$

$T\left(j_\pi^+(y)\ j_\pi^-(o)\right)$ denotes the time-ordered product of the pion currents. Similarly,

$$\frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\mu}} \langle P \left| j_\pi^+(o) \right| P\, \pi^-(o)\ {}_{out}\rangle \xrightarrow[\mu \to k_o]{} \frac{1}{(2\pi)^3 (2k_o)}\, T_{\pi^- p}^{\dagger}(k_o, E_p) \qquad (14b)$$

$T_{\pi^- p}^{\dagger}$ has a representation similar to that for $T_{\pi^- p}$ above, except in terms of an anti-time-ordered product. $T_{\pi^- p}(k_o, E_p)$ is the π^-proton forward elastic scattering amplitude in reference system where the off-mass-shell pion is at rest and the proton has momentum p. If we now use

$$\delta(k_o - \mu) = \frac{k_o}{\pi i} \left[\frac{1}{k_o^2 - \mu^2 - i\epsilon} - \frac{1}{k_o - \mu^2 + i\epsilon} \right] ,$$

and substitute these results into (13), it can be seen that the coefficient of the dangerous double pole pinch vanishes due to the unitarity condition.[7]

Performing the same manipulations for the $\sum_{\beta}$ term in (9), one arrives at the result

$$1/(G_a/G_v)^2 = 1 - \left(\frac{M}{E_p}\right)^2 + (2\pi)^3 \left[\frac{2 M \mu^2}{g_{\pi n} K_{\pi nn}(o)}\right]^2 \left(\frac{1}{2\pi i}\right) \int_{k_{o\,min}}^{\infty} \frac{dk_o}{k_o^2}$$

$$\left[\frac{T_{\pi^- p}(k_o,E_p)}{(k_o^2-\mu^2+i\epsilon)^2} - \frac{T_{\pi^- p}^{\dagger}(k_o,E_p)}{(k_o^2-\mu^2-i\epsilon)^2} - \frac{T_{\pi^+ p}(k_o,E_p)}{(k_o^2-\mu^2+i\epsilon)^2} + \frac{T_{\pi^+ p}^{\dagger}(k_o,E_p)}{(k_o^2-\mu^2-i\epsilon)^2} \right] \tag{15}$$

To evaluate the integral above, we return to the expression (14a) for $T_{\pi - p}(k_o, E_p)$, insert a complete set of intermediate states in the time ordered product and obtain a Low equation for T

$$T_{\pi^- p}(k_o, E) = (2\pi)^3 \left\{ \sum_{\alpha} \frac{\left| < P(p) \right| j_\pi^+ \left| \alpha > \right|^2 \delta^{(3)}(\vec{p} - \vec{p}_\alpha)}{k_o + E_p - E_\alpha + i\epsilon} \right.$$

$$\left. - \sum_{\beta} \frac{\left| < P(p) \right| j_\pi^- \left| \beta > \right|^2 \delta^{(3)}(\vec{p} - \vec{p}_\alpha)}{k_o - E_p + E_\alpha - i\epsilon} \right\} . \tag{16}$$

All the k_o dependence of $T_{\pi^- p}$ is in the denominators. For fixed physical $E_p, T_{\pi^- p}(k_o, E_p)$ is analytic in the complex k_o-plane with branch cuts from $k_{o\,min}$ to $+\infty$ and from $-k_{o\,min}$ to $-\infty$. The one-nucleon intermediate state does not contribute a pole term at $k_o = 0$ since the residue is zero for a pseudoscalar pion. Let $\mathcal{T}(z, E_p)$ denote the analytic continuation of $T_{\pi^- p}(k_o, E_p)$. $T_{\pi^- p}(k_o, E_p)$ is the limit of $\mathcal{T}(z, E_p)$ on the top of the right hand cut and the bottom of the left hand cut. It can be shown in the same manner that $T^{\dagger}_{\pi^- p}(k_o, E_p)$ is the limit of $\mathcal{T}(z, E_p)$ on the other sides of the cuts. Using crossing symmetry the integral[8] in (15) can then be evaluated as

$$\frac{1}{2\pi i} \oint_C \frac{\mathcal{T}(z, E_p)}{(z^2-\mu^2)^2} \frac{dz}{z^2} = \frac{1}{\mu^4} \frac{d}{dk_o} T_{\pi^- p}(k_o, E_p) \Big|_{k_o = 0} \tag{17}$$

where C is the contour indicated in Fig. 1.

Crossing symmetry further implies that

$$\frac{d}{dk_o} T_{\pi^- p}(k_o, E_p) \Big|_{k_o=0} = \frac{d}{dk_o} T^-(k_o, E_p) \Big|_{k_o=0} \tag{18}$$

where T^- is the coefficient of the anti-symmetric isospin function in the conventional decomposition for π-nucleon scattering.[9]

We seek to reduce our answer to an expression involving on-the-mass-shell quantities only. The forward scattering amplitudes satisfy dispersion relations[9] in the variable ν, the pion energy in the "laboratory system" where the nucleon is at rest, and these dispersion relations can be continued in the pion mass to $k^2 = 0$. Since $k_o E_p = M\nu$,

$$\frac{d}{dk_o} T^-(k_o, E_p)\Big|_{k_o=0} = \left(\frac{\partial}{\partial k_o} + \frac{E}{M}\frac{\partial}{\partial \nu}\right) T^-(k_o,\nu)\Big|_{k_o=\nu=0,\ \frac{\nu}{k}=\frac{E}{M}} . \tag{19}$$

From the dispersion relations

$$(2\pi)^3 \frac{d}{dk_o} T^-(k_o,E_p)\Big|_{k_o=0} = \left[\frac{g_{\pi n} K_{\pi nn}(o)}{\sqrt{2}\, E_p}\right]^2 + \frac{2}{\pi}\int_{\mu+\mu^2/2M}^{\infty} \frac{d\nu}{\nu^2}\ \mathrm{Im}\,\mathcal{M}^-(k_o = 0,\nu) \tag{20}$$

where $\mathcal{M}$ is the invariant forward scattering amplitude. The energy dependent term contribution comes from the Born term. We now write for (15)

$$1/(G_a/G_v)^2 = 1 + (1/\pi)\left[(2M)/g_{\pi n}K_{\pi nn}(o)\right]^2 \int_{\mu+\mu^2/2M}^{\infty} \frac{d\nu}{\nu^2}\ \mathrm{Im}\,\mathcal{M}^-(k_o = 0,\nu) \tag{21}$$

<u>The contribution of the Born term from the dispersion relations has completely cancelled the original $(M/E_p)^2$ factor from the one-neutron intermediate state</u>, and we are left with a covariant answer. To put the answer in a final useful form, first let $\nu_L = \nu - (\mu^2/2M)$. In the region of integration[10,11]

$$\mathcal{M}^-\left[k_o = 0,\ \nu_L + (\mu^2 - k_o^2)/2M\right] \simeq K^2_{\pi nn}(o)\,\mathcal{M}^-(\mu,\nu_L) \tag{22}$$

$$\mathrm{Im}\,\mathcal{M}^-(\mu,\nu_L) = q\sigma^-_{tot}(\nu_L) \tag{23}$$

where q is the magnitude of the 3-momentum of the pion in the laboratory system. We obtain finally an answer in terms of experimentally measured total cross-sections

$$\left|G_a/G_v\right| = \left[1 + (2/\pi)\,(M/g_{\pi n})^2 \int_{\mu}^{\infty} \frac{q\,d\nu_L}{\nu_L^2}\left(\sigma^{\pi^-p}_{tot}(\nu_L) - \sigma^{\pi^+p}_{tot}(\nu_L)\right)\right]^{-\frac{1}{2}} . \tag{24}$$

Evaluation of (24) using experimental cross-sections[12] gives the result quoted in (1).[13]

The author is indebted to his colleagues in the theoretical group at SLAC for their friendly and stimulating interest in this calculation. For helpful discussions, he is particularly grateful to Professors J. D. Bjorken and S. D. Drell and Drs. A. C. Finn and J. D. Sullivan. In addition he wishes to thank Professor Drell for a critical reading of this manuscript.

After completing this work, the author was informed that similar results have been independently obtained by S. L. Adler.[11]

REFERENCES

1. M. Gell-Mann, Phys. Rev. 125, 1067 (1962); Physics 1, 63 (1964).

2. C. S. Wu, private communication.

3. Electromagnetic violations of I-spin and electromagnetic mass-splittings are neglected throughout this calculation.

4. M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960).

5. S. L. Adler, Phys. Rev. 137, B1022 (1965); and to be published.

6. S. Fubini and G. Furlan, CERN preprint (unpublished).

7. The cancellation is exact at least at the double pole and in the region of small denominators where the unitarity condition can be continued off-the-mass-shell.

8. This use of the Low equation to manipulate the integration contours of the off-mass-shell scattering amplitude is similar to the approach of W. N. Cottingham, Ann. Phys. (N.Y.) 25, 424 (1963).

9. G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. 106, 1337 (1957). Our normalization and metric conventions differ from this reference.

10. E. Ferrari and F. Selleri, Nuovo Cimento 21, 1028 (1961).

11. For a more detailed treatment of possible off-mass-shell corrections, the reader is referred to the letter of S. L. Adler.

12. The relevant data has been tabulated by C. Hohlen, C. Ebel, and J. Giesecke, Z. Physik 180, 430 (1964). The author thanks Dr. M. Bander for his help in programming the numerical integrations.

13. According to (24) it is the effect of the (3,3) resonance which makes $|G_a| > |G_V|$. In fact the (3,3) resonance contribution alone gives $|G_a/G_V| \simeq 1.3$, and the higher energy T = 1/2 resonances reduce this value. The convergence of the integral depends on the validity of the Pomeranchuk Theorem, but a $\sigma^-_{tot} = \frac{C}{\nu^\alpha}$ fit to the data above 5 GeV with $\alpha = 0.5$ to 0.7 gave a -0.02 contribution which has been included in the result.

FIGURE CAPTION

Figure 1 The contour of integration, C, for the integral in (17).

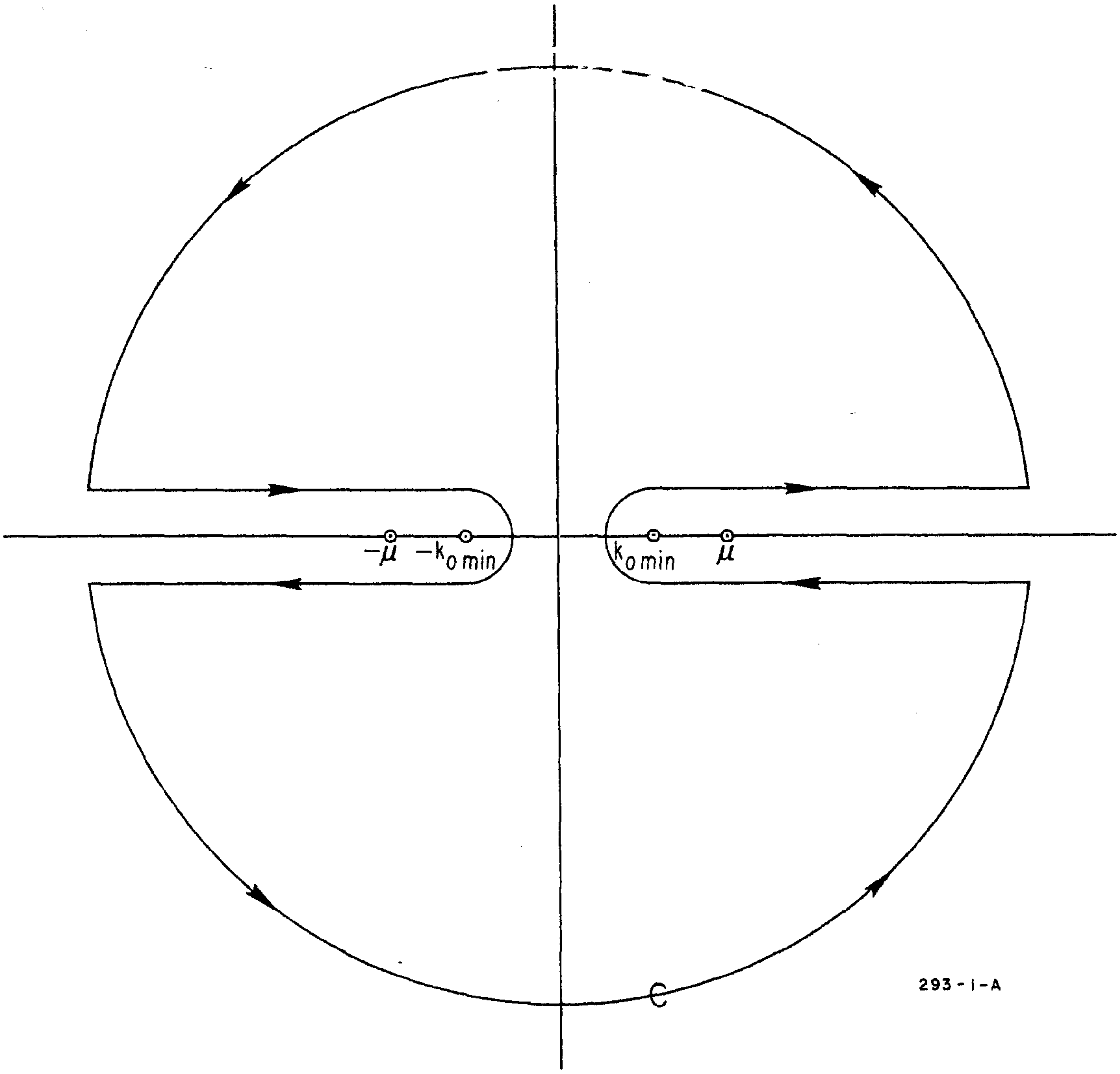

FIG. 1