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SYSTEMATICS OF ANGULAR AND POLARIZATION DISTRIBUTIONS IN THREE BODY DECAYS*

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ABSTRACT

A general method is proposed for the measurement of the polarization and alignment of a particle of arbitrary spin from the analysis of its three body decays. This method provides a procedure for the determination of spin and parity of the decaying system which is independent of the dynamics of the decay process. The procedure is closely related to the one currently used for two-body reactions except that the normal to the decay-plane replaces the center-of-mass momentum as an analyzer. The general formalism is developed and illustrated by two examples; three pion decays and baryon-two pion decays.

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I. INTRODUCTION

The description of three interacting bodies is a well seasoned and familiar problem which has received a revived interest by particle physicists during the past few years.[1,2] The fact that increasing numbers of particles (or resonances) of high mass are being experimentally discovered which have appreciable three body decay modes behooves us to examine the three body problem from the standpoint of a decaying system. However, we do not consider the dynamics of the decay process but merely make use of the consequences of rotational and inversion invariance. The treatment presented here is therefore completely general, exhibiting the kind of angular and polarization distributions which are consistent with a system of arbitrary spin decaying into three particles with spin. Such distributions, when compared with experiment, provide a possible determination of the spin and parity of the decaying particle and eventually a means to measure its polarization and alignment, quantities of great interest for the understanding of its production mechanism.[3] Our method applied to three body decays is closely related to the one currently used in the analysis of two body decays except that the normal to the decay plane replaces the center-of-mass momentum as an analyzer of the polarization. Formulae giving the angular and polarization distributions in terms of the decaying particle density matrix are in fact written in a very similar form for both cases.

As is well known, the description of a three body system requires five variables. A convenient choice of these variables consists of two energies and three angles. The two energies are taken to be the center-of-mass energy of two decay particles whose domain of variation defines a Dalitz plot. The three angles can be chosen as those which define completely the orientation of the decay plane. In the treatment presented here we consider only the orientation of the decay plane and sum over all energy configurations, or, in some cases, separately over different regions of the Dalitz plot. In this sense, the distributions presented here are the complement of the Dalitz plot distribution where all angular configurations are averaged over, and where the three body system is studied in terms of its energy distribution.[4]

The analysis of the energy distribution in terms of a Dalitz plot has the advantage of giving useful information even if the decaying particle is neither polarized nor aligned. Nevertheless, its practical interest is bound to the dominance of a very small number of independent amplitudes. In many cases the general analysis suggested here, which does not rely on any dynamical assumptions governing the decay process, can be used to determine the spin and parity of a decaying state via its three-body decay alone. When the system has in addition a two-body decay mode the combined analysis of both two and three body modes can be applied in unison in order to obtain improved and more accurate knowledge of the system's quantum numbers.[5] In all cases it could be used in order to get information about the production mechanisms by means of polarization and alignment analyzation.

The angular distribution of the normal to the decay plane is readily obtained when three free relativistic particle states of well defined angular momentum J and parity are constructed using the general projection method of Wigner.[6] The angular dependence of the decay amplitude is given as a linear combination of rotation matrix elements corresponding to the $2J + 1$ dimensional representation of the rotation group: $D^{J}_{m'm}(\alpha,\beta,\gamma)$. The arguments are three Euler angles, which can be chosen as the azimuthal and polar angles of the normal to the decay plane and a third angle, γ, referring to a rotation of the decay plane around the normal. These angles then completely specify the orientation of the decay plane. This is a straightforward extension to three particles of a procedure already used to construct two particle states.[7]

The general formalism is presented in Section II, and a general expression for the angular distribution of the normal to the decay plane is obtained.[8] The simplifications due to parity conservation and possible identity of two of the particles are also discussed. The formalism is then applied in Section III to the problem of the decay into three spin zero particles and in Section IV to the problem of the decay into two spin zero and one spin 1/2 particle. The distribution of the polarization of the decay spin 1/2 particle is discussed in detail and we stress the analogy between the formulae obtained and the ones currently used for two body decays into a spin zero and a spin 1/2 particle. In both Sections III and IV we also discuss decays into a pion and a resonance which eventually decays into two pions or a pion and a hyperon depending on its quantum numbers.

In addition to giving the general formalism the most simple cases are explicitly treated. In Section III angular distributions are given for the decay of spin $1^{\pm}$ and $2^{\pm}$ into three pions. In Section IV angular distribution of the normal to the decay plane, as well as polarization distributions for the decay spin 1/2 hyperon are given for the decaying state having spin 1/2 and spin 3/2.

The $D^J_{m'm}$ functions required for explicit calculations with spins less than or equal to 3 are given in an appendix.

II. GENERAL FORMALISM

Three Particle States

A quantum state containing 3 free particles is completely defined by the momentum and polarization of each particle. Such a state may be constructed as the direct product of three one-particle states $|\vec{q}_i, \lambda_i >$ where $\vec{q}_i$ and λ_i stand respectively for the momentum and helicity of the i-th particle. To be more precise we define the state $|\vec{q}_i, \lambda_i >$ as done in Reference (7), namely that

$$|\vec{q}_i,\ \lambda_i > = R_{\varphi_i \theta_i 0}\ |\vec{Q}_i,\ \lambda_i > \tag{1}$$

where $|\vec{Q}_i,\ \lambda_i >$ is an helicity state with eigenvalue λ_i and momentum Q_i along the positive z axis ($|\vec{Q}_i| = |\vec{q}_i|$). $R_{\varphi_i \theta_i 0}$ stands for the rotation operator, with Euler angles $\varphi_i, \theta_i, 0$. φ_i and θ_i are respectively the azimuthal and polar angles of $\vec{q}_i$ with respect to the fixed coordinate system x y z (Figure 1). The helicity, i.e., the component of the total angular momentum of the particle along its momentum, is obviously invariant under rotation.

A three particle state is written as[10]

$$\left| \vec{q}_1, \lambda_1; \vec{q}_2, \lambda_2; \vec{q}_3, \lambda_3 \right> \tag{2}$$

It is convenient to describe the decay in the center-of-mass system where

$$\vec{q}_1 + \vec{q}_2 + \vec{q}_3 = 0 \tag{3}$$

The three momenta then form a triangle in a plane, the normal of which is defined as a unit vector along $\vec{q}_1 \times \vec{q}_2$. The conservation of energy gives the further restriction

$$\sqrt{q_1^2 + m_1^2} + \sqrt{q_2^2 + m_2^2} + \sqrt{q_3^2 + m_3^2} = m_o \tag{4}$$

where m_o is the mass of the decaying particle.

A more convenient description of this state is in terms of a different set of quantum numbers which are the energies ω_1, ω_2 and ω_3 of the three particles restricted by (4) – and three Euler angles α, β, γ which specify the orientation of the momentum triangle in space (Figure 2).

The rotation angles are defined by starting from a standard position where the triangle is in the x-y plane. As a convention we take $\vec{q}_1 + \vec{q}_2$ along the x axis and the normal $\vec{q}_1 \times \vec{q}_2$ along the z axis. The angles

α and β are respectively chosen as the azimuthal and polar angles of the normal to the decay plane. The angle γ refers to a rotation around the normal and is illustrated in Figure 2.. All helicities remain unchanged through these three successive rotations. We then write a three particle state thus defined as

$$|\omega_1 \lambda_1; \omega_2 \lambda_2; \omega_3 \lambda_3; \alpha, \beta, \gamma > \tag{5}$$

With the set of states (2) the density of final states $d\rho_F$ for the three body decay is written as

$$d\rho_F = \frac{d^3 \vec{q}_1 d^3 \vec{q}_2 d^3 \vec{q}_3 \ \delta\ (\vec{q}_1 + \vec{q}_2 + \vec{q}_3)\ \delta\ (\omega_1 + \omega_2 + \omega_3 - m_o)}{2\omega_1 2\omega_2 2\omega_3}$$

or as usually done

$$d\rho_F = \frac{1}{8}\ d\omega_1 d\omega_2 d\varphi_1 d\cos\theta_1\ d\varphi_{12}\ \delta\left(\cos\theta_{12} - \frac{(m_o - \omega_1 - \omega_2)^2 - q_1^2 - q_2^2 - m_3^2)}{2\ q_1 q_2}\right) \tag{6}$$

where φ_{12} and θ_{12} are the azimuthal and polar angles of $\vec{q}_2$ with respect to $\vec{q}_1$. Integration with respect to $\cos\theta_{12}$, φ_{12}, $\cos\theta_1$ and φ_1 gives a density distribution in the ω_1, ω_2 plane. This is the Dalitz plot.

With the states (5), the density of states is obtained by replacing $d\varphi_1\ d\cos\theta_1\ d\varphi_{12}$ by $d\ \alpha\ d\cos\beta\ d\ \gamma$ in (6). The Jacobian determinant is equal to 1.[11]

In their center-of-mass system the three decay particles are in a state of well defined angular momentum and, if we consider only decays via strong or electromagnetic interactions, also parity. The total angular momentum is equal to the spin j of the decaying particle. Such a state is written as

$$|\omega_1\lambda_1;\ \omega_2\lambda_2;\ \omega_3\lambda_3;\ j\ m\ M> \tag{7}$$

where m is the eigenvalue of the component of angular momentum operator J along a fixed axis chosen as the z axis; M is the eigenvalue of angular momentum along the normal to the decay plane, which can be used together with the other observables J^2 and J_z to specify the state.

The angular distribution of the normal to the decay plane, obtained from a pure state of definite m and M such as (7), is given by

$$\frac{dN}{d\Omega} = \int |A|^2\, d\gamma\, d\omega_1 d\omega_2 \tag{8}$$

where $d\Omega = \sin\beta\ d\beta\ d\alpha$ and where

$$A = <\omega_1\lambda_1;\ \omega_2\lambda_2;\ \omega_3\lambda_3;\ \alpha\ \beta\ \gamma\ |\omega_1\lambda_1;\ \omega_2\lambda_2;\ \omega_3\lambda_3;\ j\ m\ M>$$

In order to continue further we need the relationship between a state of definite angular momentum such as (7) and a state described in terms of Euler angles. To achieve this we follow the procedure of Wigner[6] and write

$$|\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3;j\ m\ M> = \int D^{j*}_{mM}(\alpha\ \beta\ \gamma)\ |\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3;\alpha\ \beta\ \gamma>\ d\alpha\ \sin\beta\ d\beta\ d\gamma \tag{9}$$

where the integration is performed over all rotations, namely

$$0 \leq \alpha \leq 2\pi \qquad 0 \leq \beta \leq \pi \qquad 0 \leq \gamma \leq 2\pi$$

As is well known these angles can be defined as in Figure 2 or just as well γ may be considered as the angle of the third rotation performed around the normal to the decay plane. As easily checked using the group property of the D functions, (9) transforms under rotations as a state of total angular momentum j with z component m and component M along the normal to the decay plane, a rotationally invariant quantity. The energy and helicity of each particle are invariant under rotations and their same eigenvalues appear on both sides of (9). It should be remarked that we do not obtain in this way the most convenient orthonormal set of states for three free particles similar to the case of the two body problem.[6] Such states have been explicitly constructed by Wick[1] in coupling two particles together and then coupling the third one to the system constructed from the first two. A quantum state with eigenvalues j m and M will be in general described by a wave function of ω_1 and ω_2 which multiplies the angular wave function (10). The angular distribution of the normal which is obtained by integration over the Dalitz plot (8) will average over all configurations the final state interaction of two of the decay particles in a particular angular momentum state.

Using the angular momentum eigenstate (9) we have that

$$A = D^{j*}_{mM} (\alpha \beta \gamma) \tag{10}$$

A normalization coefficient could appear in (10). It is however independent of m and M and therefore irrelevant for our purposes.

The Normal to the Decay Plane as an Analyzer

We now turn to the decay of a particle of spin j whose state is not pure but rather a statistical mixture of states described by a density matrix $\rho_{mm'}$. The eigenvalues m and m' run from -j to +j in integer steps and refer to the z axis. The angular distribution of the normal to the decay plane can be obtained for each set of eigenvalues of the final particle helicities. Using (10) the angular distribution reads as

$$\left(\frac{dN}{d\Omega}\right)_{\lambda_1,\lambda_2,\lambda_3} = \Sigma_{M,M'}\Sigma_{m,m'}\rho_{mm'} \int D^{j*}_{mM} (\alpha\ \beta\ \gamma)\ D^{j}_{m'M'} (\alpha\ \beta\ \gamma)\ d\gamma\, \mathcal{F}_{MM'}$$

$$\text{where } \mathcal{F}_{MM'} = \int d\omega_1 d\omega_2\ F_M(\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3)\ F^*_{M'}(\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3)$$

The phenomenological decay amplitudes F_M which have been introduced are functions of rotationally invariant quantities only. They depend in general on M but not on m.

Since the γ dependence of a D function is simply a factor $e^{-iM\gamma}$ interference between different F_M amplitudes vanishes in the normal angular distribution when it is integrated over γ.

If everything else but the direction of the normal to the decay plane is summed over, a simple relation is obtained for the angular distribution of the normal

$$\frac{dN}{d\Omega} = \Sigma_{m,m'}\ \rho_{mm'}\ \Sigma_M\ D^{j*}_{mM} (\alpha\ \beta\ 0)\ D^{j}_{m'M} (\alpha\ \beta\ 0)\ |R_M|^2 \qquad (11)$$

where

$$\left|R_M\right|^2 = 2\pi \Sigma_{\lambda_1,\lambda_2,\lambda_3} \int d\omega_1 d\omega_2 \left|F_M (\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3)\right|^2$$

Equation (11) relates the angular distribution of the normal to the density matrix of the initial particle in terms of the $2j + 1$ decay parameters R_M.

This also shows that the maximum number of independent decay amplitudes, as far as the orientation of the decay plane is considered, is actually $2j + 1$ for each set of final helicities. Conservation of parity in the decay process further reduces this number as will be shown later. This number of independent decay amplitudes is also equal to the maximum number of linearly independent tensors, built with the particle momenta, in terms of which the decay amplitudes can also be written.

In order to use (11) one may calculate the required D functions. Alternatively, use of the Clebsch-Gordan series allows (11) to be written as

$$\frac{dN}{d\Omega} = \Sigma_{mm'} \rho_{mm'} \sum_M \sum_{\ell} C(jj\ell \,|\, m',-m)\; C(jj\ell \,|\, M,-M)(-1)^{M-m} \sqrt{\frac{4\pi}{2\ell+1}}\; Y^{\ell *}_{m'-m} (\beta,\alpha) \left|R_M\right|^2 \tag{12}$$

where we have introduced standard Clebsch-Gordan coefficients.[12]

The angular distribution is thus given by a sum of spherical harmonics with highest order $2j$. This generalizes the well known theorem on the complexity of the angular distribution in 2-body reactions to the case of 3 bodies in terms of the normal to the decay plane.

It is convenient to group together terms with opposite values of M and to write (11) as

$$\frac{dN}{d\Omega} = \sum_{M \geq 0} \left\{ \sum_{m\,m'} \left(\mathrm{Re}\ \rho_{mm'} \cos(m-m')\alpha - \mathrm{Im}\ \rho_{mm'} \sin\ (m-m')\alpha \right) \left[R_M^{+}\ Z_{mm'}^{jM+}(\beta) + R_M^{-}\ Z_{mm'}^{jM-}(\beta) \right] \right\} \tag{13}$$

where we have introduced the notations

$$Z_{mm'}^{jM\pm}\ (\beta) = d_{mM}^{j}\ (\beta)\ d_{m'M}^{j}(\beta) \pm d_{m-M}^{j}\ (\beta)\ d_{m'-M}^{j}\ (\beta)$$

and $R_M^{\mp} = \frac{1}{2}\left(\left|R_M\right|^2 \pm \left|R_{-M}\right|^2 \right)$, $R^{+} \geq 0$ and R^{-} may be either positive or negative. The D functions have been written[12] as

$$e^{-im'\alpha}\ d_{m'M}^{j}\ (\beta)\ e^{-iM\gamma}$$

As follows from their definition, and the relation

$$d_{m'm}^{j}\ (\beta) = (-1)^{j+m'}\ d_{m'-m}^{j}\ (\pi-\beta)$$

the Z functions satisfy the relation

$$Z_{mm}^{jM\pm}\ (\beta) = \pm\ (-1)^{m-m'}\ Z_{mm'}^{jM\pm}\ (\pi-\beta)$$

If we invert the direction of the normal which, in terms of the Euler angles means the following transformation $\alpha \quad \pi + \alpha$, $\beta \rightarrow \pi - \beta$, then the angular function which goes with R_M^{+} is unchanged while the function which goes with R_M^{-} changes sign as is obvious from (13). Hence the normal direction is not determined when two particles are identical and when the summation over all available energies is performed according to (8). In that case, all terms proportional to R_M^- will vanish identically. In order to keep the direction of the normal well defined it is necessary to sum independently on parts of the Dalitz plot, for instance separately for $\omega_1 > \omega_2$ and $\omega_1 < \omega_2$.

We can further group together terms with opposite values of both m and m' and write the angular distribution of the normal as

$$\frac{dN}{d\Omega} = \sum_{M \geq 0} \frac{1}{2} \sum_{m\, m'} \Big[\cos(m-m')\,\alpha \Big(\mathrm{Re}\;\rho_{mm'} + (-1)^{m-m'}\,\mathrm{Re}\;\rho_{-m-m'}\Big)$$
$$- \sin(m-m')\,\alpha \Big(\mathrm{Im}\;\rho_{mm'} - (-1)^{m-m'}\,\mathrm{Im}\;\rho_{-m-m'}\Big)\Big] Z^{jM+}_{mm'}(\beta)\, R^+_M$$
$$+ \Big[\cos(m-m')\,\alpha \Big(\mathrm{Re}\;\rho_{mm'} - (-1)^{m-m'}\,\mathrm{Re}\;\rho_{-m-m'}\Big) \qquad (14)$$
$$- \sin(m-m')\,\alpha \Big(\mathrm{Im}\;\rho_{mm'} + (-1)^{m-m'}\,\mathrm{Im}\;\rho_{-m-m'}\Big)\Big] Z^{jM-}_{mm'}(\beta)\, R^-_M$$

Due to the hermiticity of the density matrix, and the definition of the Z functions, terms where m and m' are interchanged give the same contribution. As follows from their definition $Z^{jM-}_{m-m}(\beta) \equiv 0$, for integer j and $Z^{jM+}_{m-m}(\beta) \equiv 0$, for half integer j.

Parity Conservation

If parity is conserved in the decay we have to replace (7) by an eigenstate of the parity operator with the proper eigenvalue. We then consider the action of the parity operator P on an angular momentum eigenstate (9). We have that

$$P \,|\, \omega_1\lambda_1, \omega_2\lambda_2, \omega_3\lambda_3; j\ m\ M > = \int D^{j*}_{mM}(\alpha\beta\gamma)\, R_{\alpha\beta\gamma}\, P \,|\, \omega_1\lambda_1, \omega_2\lambda_2, \omega_3\lambda_3, 0,0,0 > d\alpha\, \sin\beta d\beta d\gamma$$

since the parity operator P commutes with the rotation operator. We now use the fact that the parity operation can be defined as the product of a reflection with respect to a plane times a rotation of angle π around a normal to that plane. The plane chosen is the decay plane of the reference state.

$$|\omega_1\lambda_1,\omega_2\lambda_2,\omega_3\lambda_3,0,0,0> \text{ i.e., the x-y plane (Figure 2).}$$

We denote by Y the reflection operator with respect to that plane and write $P = e^{+i\pi J_z} Y$. The action of Y changes the sign of all helicities. In fact the following relation holds[13]

$$Y\,|\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3;0,0,0> \; = \eta_1\eta_2\eta_3 \; (-1)^{S_1-\lambda_1+S_2-\lambda_2+S_3-\lambda_3}\,|\omega_1-\lambda_1;\omega_2-\lambda_2;\omega_3-\lambda_3;0,0,0>$$

where S and η stand for the spin and intrinsic parity of each particle. It follows that

$$P\,|\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3;j\;m\;M> \; = \eta_1\eta_2\eta_3(-1)^{S_1-\lambda_1+S_2-\lambda_2+S_3-\lambda_3}$$

$$\int D^{j*}_{mM}(\alpha\beta\gamma)\;R_{\alpha\beta\gamma}\;e^{+i\pi J_z}\;|\omega_1-\lambda_1;\omega_2-\lambda_2;\omega_3-\lambda_3,0,0,0>\,d\alpha\,\sin\beta d\beta d\gamma$$

In order to express the state after the parity operation in terms of the original states (9) we use $R_{\alpha\beta\gamma} = e^{-i\alpha Jz}\,e^{-i\beta Jy}\,e^{-i\gamma Jz}$ and simply add $-\pi$ to the first rotation angle thus replacing $D^{j*}_{mM}(\alpha\;\beta\;\gamma)$ by $(-1)^M\;D^{j*}_{mM}(\alpha,\beta,\gamma+\pi)$. In this manner one obtains that

$$P\,|\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3;jmM> \; = (-1)^M(-1)^{S_1-\lambda_1+S_2-\lambda_2+S_3-\lambda_3}\eta_1\eta_2\eta_3\;|\omega_1-\lambda_1;\omega_2-\lambda_2;\omega_3-\lambda_3;jmM> \tag{15}$$

Applying (15) to a 3-pion state we find the relation that

$$P \mid \omega_1,\omega_2,\omega_3;j\ m\ M > = (-1)^{M+1} \mid \omega_1,\omega_2,\omega_3;j\ m\ M > \tag{16}$$

This yields an important result for 3-pion decays, namely that if the parity of the decaying particle is even (odd) only odd (even) values of M contribute.

For a one baryon and two pion state we take $\lambda_1 = \frac{1}{2}$ and get

$$P \mid \omega_1\ \frac{1}{2}\ ,\ \omega_2,\omega_3;j\ m\ M > = (-1)^M \epsilon \mid \omega_1\ -\frac{1}{2}\ ,\ \omega_2,\omega_3;j\ m\ M >$$

where ϵ is the relative parity between the initial and final baryon. Whereas only even or odd values of M contribute to a final 3π state, for a two pion one baryon state, all values of M contribute. However we consider the proper parity states given by

$$\frac{1}{\sqrt{2}} \left(\mid \omega_1,\lambda_1;\omega_2;\omega_3;j\ m\ M > \pm (-1)^M \mid \omega_1,-\lambda_1;\omega_2;\omega_3;j\ m\ M > \right) \tag{17}$$

Either parity case will give the same angular distribution since states of different helicities are orthogonal.

<u>One of the Momentum as an Analyzer</u>

The basic quantum states (5), which we have introduced, are labelled by Euler angles which refer to the direction of the normal. We could just as well consider these 3 angles as defining the direction of one of the three momenta, q_1 say, and a further rotation of q_2 around q_1. We can follow the same steps and obtain a formula identical to (11) for the angular distribution of one of the momenta. The functions R_M will of course be different. Equation (14) is still valid and gives the polarization of the decaying particle in terms of the distribution of one of the momenta.[14] If the analysis in terms of the normal turns out to be a little easier to work through, it is due to the simple form in which parity conservation is expressed. For a three pion decay, we simply had to eliminate either even or odd values of M. When the three Euler angles refer to one momentum it is found that (16) has to be replaced by the following relation:

$$P \left| \omega_1 \omega_2 \omega_3 ; j \; m \; M \right> = (-1)^{j+M+1} \left| \omega_1 \omega_2 \omega_3 ; j \; m {-M} \right> \tag{15'}$$

If the parity of the decaying particle is $(-1)^j$, the decay amplitude, R_M and R_{-M} are equal (opposite) if M is odd (even) and there is no M = 0 amplitude. If the parity is $-(-1)^j$ the opposite assignment holds. For each M value, both parity states give the same angular distribution.

Identical Particles

The identity of two (or all three) particles will imply further relations among the decay amplitudes. In the examples considered in Sections III and IV for instance, they will apply when two π-mesons have the same charge or are in an eigenstate of isotopic spin. If two identical particles are produced, the decay state has to be symmetrical (antisymmetrical) with respect to the exchange of the two particles according to their Bose-Einstein (Fermi-Dirac) statistics. In order to construct states with such permutation property, we introduce a permutation operator P_{12} (exchange of particle 1 and 2 leaving 3 unchanged) and apply it on both sides of (9)

$$P_{12} \,|\, \omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3;j,m,M > = \int d\alpha \sin\beta d\beta d\gamma\, D^{j*}_{mM}(\alpha\beta\gamma)\, P_{12}\,|\,\omega_1\lambda_1;\omega_2\lambda_2;\omega_3\lambda_3;\alpha\beta\gamma > \tag{18}$$

$$= \int d\alpha \sin\beta d\beta d\gamma\, D^{j*}_{mM}(\alpha\beta\gamma)\,|\,\omega_2\lambda_2;\omega_1\lambda_1;\omega_3\lambda_3;\alpha\beta\gamma >$$

The set of angles $\alpha\ \beta\ \gamma$ which now appears in the Ket vector no longer refers to the normal to the decay plane but rather to a final rotation of negative angle around an axis opposite to the normal. Since we are defining the set of angles with respect to the normal this actually corresponds to a new set, namely $\alpha + \pi$, $\pi - \beta$ and $2\pi - \gamma$. The rotation defined by the set of angles $\alpha + \pi$ and $\pi - \beta$ brings $\vec{q}_1 + \vec{q}_2$ in a direction identical to the one obtained using α and β. A rotation of angle $2\pi - \gamma$ around the new normal then gives the same configuration as the one obtained with the set of angles α, β, and γ. Since we integrate over all rotation angles, we may replace the arguments of the D function and write (18) as

$$\int d\alpha \sin\beta d\beta d\gamma\, D^{j*}_{mM}(\alpha - \pi, \pi - \beta, 2\pi - \gamma)\,|\,\omega_2\lambda_2,\omega_1\lambda_1,\omega_3\lambda_3;\alpha\ \beta\ \gamma >$$

$D^{j*}_{mM}(\alpha - \pi, \pi - \beta, 2\pi - \gamma)$ is the new angular part of the wave function describing the orientation of the normal to the decay plane.

Transforming the D functions and using the definition of our state (9), we rewrite (18) as $\xi\,|\,\omega_2\lambda_2,\omega_1\lambda_1,\omega_3\lambda_3;j,m,-M >$ where $\xi = (-1)^{j+2M}$.

We note that, as a consequence of our convention, the application of P_{12} twice is equivalent to a rotation of 2π.

The decay states being symmetrical (antisymmetrical) with respect to the exchange of the two particles have amplitudes $F_M(\omega_1\lambda_1,\omega_2\lambda_2)$ which will satisfy the relation

$$\xi\, F_M\,(\omega_1\lambda_1,\omega_2\lambda_2) = +\,(-)\; F_{-M}\,(\omega_2\lambda_2,\omega_1\lambda_1) \tag{19}$$

When the identical particles are spin zero mesons the helicity indices are suppressed and we have in both cases

$$\left|F_M\,(\omega_1,\omega_2)\right|^2 = \left|F_{-M}\,(\omega_2,\omega_1)\right|^2 \tag{20}$$

When integration over the whole Dalitz plot is performed according to (8), we find that opposite values of M give the same angular distribution for the normal to the decay plane, and therefore R_M^- does not contribute.

III. DECAY INTO 3 SPINLESS PARTICLES

We now consider in more detail the decay of a particle of arbitrary integer spin j into three non identical spinless particles. At first we do not take into account any restrictions resulting from possible isotopic spin configurations.

The 2j + 1 a priori independent decay amplitudes are reduced by parity conservation according to (15) and we obtain the maximum number of independent amplitudes as shown in Table I.

In the most simple cases we have: one amplitude for 0^- and 1^-; two independent amplitudes for 1^+ and 2^+; three independent amplitudes for 2^- and 3^-, etc. This result may be obtained by other approaches, but not in such a simple way. We can, for example, exhibit sets of independent amplitudes written in terms of cartesian tensors and which for the spin 1 and 2 cases take the form

$$
\begin{array}{ll}
1^- & G\, \epsilon_{\mu\nu\rho\sigma} q_1^\nu q_2^\rho q_3^\rho \\[2ex]
1^+ & G_1(q_1 + q_2)_\mu + G_2(q_1 - q_2)_\mu \\[2ex]
2^+ & \left(G_1(q_1 + g_2)_\mu + G_2(q_1 - q_2)_\mu\right) \epsilon_{\nu\rho\sigma\alpha} q_1^\rho q_2^\sigma q_3^\alpha \\[2ex]
2^- & G_1(q_1^\mu q_1^\nu + q_2^\mu q_2^\nu) + G_2(q_1^\mu q_1^\nu - q_2^\mu q_2^\nu) + 2G_3\, q_1^\mu q_2^\nu
\end{array}
\tag{21}
$$

The G's which are the coefficients of the independent tensors are Lorentz invariant quantities. They are assumed to be analytic functions of s, t and u, the center-of-mass energy squared of the three particles taken two by two, i.e.,[15]

$$
s = (q_3 + q_1)^2, \quad u = (q_3 + q_2)^2, \quad t = (q_1 + q_2)^2 \tag{22}
$$

The functions $|R_M|^2$ defined above will in general be linear combinations of products of two of the tensor invariants G_i with coefficients that are functions of s, t, or u.

Taking account of the conservation of parity we next give the explicit expressions for the angular distribution of the normal to the decay plane. For the case of the decaying particle having spin and parity 1^- we have only the $M = 0$ amplitude and $R_0^- = 0$. R_0^+ is the common factor to the angular distribution which following (14) takes the form

$$\frac{dN}{d\Omega} = R_0^+ \left\{ \left(\rho_{11} + \rho_{-1-1}\right) Z^{10}_{11}(\beta) + \rho_{00} Z^{10}_{00}(\beta) \right.$$
$$+ 2\left[\cos\alpha\left(\mathrm{Re}\,\rho_{10} - \mathrm{Re}\,\rho_{-10}\right) - \sin\alpha\left(\mathrm{Im}\,\rho_{10} + \mathrm{Im}\,\rho_{-10}\right)\right] Z^{10}_{10}(\beta)$$
$$\left. + 2\left[\cos 2\alpha\left(\mathrm{Re}\,\rho_{1-1}\right) - \sin 2\alpha\left(\mathrm{Im}\,\rho_{1-1}\right)\right] Z^{10}_{1-1}(\beta) \right\} \qquad (23)$$

We readily get the Z functions from the table of d functions given in the Appendix and obtain

$$\frac{dN}{d\Omega} = R_0^+ \left\{ 2\cos^2\beta\,\rho_{00} + \sin^2\beta(\rho_{11} + \rho_{-1-1}) \right.$$
$$- 2\sqrt{2}\sin\beta\cos\beta\left(\left(\mathrm{Re}\,\rho_{10} - \mathrm{Re}\,\rho_{-10}\right)\cos\alpha - \left(\mathrm{Im}\,\rho_{10} + \mathrm{Im}\,\rho_{-10}\right)\sin\alpha\right)$$
$$- 2\sin^2\beta\,(\mathrm{Re}\,\rho_{1-1}\cos 2\alpha - \mathrm{Im}\,\rho_{1-1}\sin 2\alpha) \qquad (24)$$

This is a well known result. The angular distribution determines six quantities (including the trace $\rho_{11} + \rho_{00} + \rho_{-1-1}$) of the spin 1 density matrix(usually known as the tensorial polarization) but leaves undetermined the three other terms (the vectorial polarization). The fact that the vectorial polarization is not determined is because there is only one decay amplitude. The observation of the γ distribution would give nothing new.

We now turn to the pseudo-vector (1^+) case. Where there are two decay amplitudes corresponding to $M = \pm 1$ and the angular distribution is a function of two terms, one proportional to R_1^+ and one proportional to R_1^-. It reads

$$\frac{dN}{d\Omega} = R_1^+ \Big\{ (\rho_{11} + \rho_{-1-1}) Z_{11}^{11+}(\beta) + \rho_{00} Z_{00}^{11+}(\beta)$$
$$+ 2 \Big[\cos\alpha \, (\mathrm{Re}\,\rho_{10} - \mathrm{Re}\,\rho_{-10}) - \sin\alpha \, (\mathrm{Im}\,\rho_{10} + \mathrm{Im}\,\rho_{-10}) \Big] Z_{10}^{11+}(\beta)$$
$$+ 2 \Big[\cos 2\alpha \, \mathrm{Re}\,\rho_{1-1} - \sin 2\alpha \, \mathrm{Im}\,\rho_{1-1} \Big] Z_{1-1}^{11+}(\beta) \Big\}$$
$$+ R_1^- \Big\{ (\rho_{11} - \rho_{-1-1}) Z_{11}^{11-}(\beta)$$
$$+ 2 \Big[\cos\alpha \, (\mathrm{Re}\,\rho_{10} + \mathrm{Re}\,\rho_{-10}) - \sin\alpha \, (\mathrm{Im}\,\rho_{10} - \mathrm{Im}\,\rho_{-10}) \Big] Z_{10}^{11-}(\beta) \Big\}$$

The Z functions are easily calculated yielding the explicit expression

$$\frac{dN}{d\Omega} = R_1^+ \Big\{ (\rho_{11} + \rho_{-1-1}) \frac{1 + \cos^2\beta}{2} + \rho_{00} \sin^2\beta$$
$$+ \sqrt{2} \sin\beta \cos\beta \Big((\mathrm{Re}\,\rho_{10} - \mathrm{Re}\,\rho_{-10}) \cos\alpha - (\mathrm{Im}\,\rho_{10} + \mathrm{Im}\,\rho_{-10}) \sin\alpha \Big)$$
$$+ \sin^2\beta \, (\cos 2\alpha \, \mathrm{Re}\,\rho_{1-1} - \sin 2\alpha \, \mathrm{Im}\,\rho_{1-1})$$
$$+ R_1^- \Big\{ (\rho_{11} - \rho_{-1-1}) \cos\beta + \sqrt{2} \sin\beta \Big[\cos\alpha \, (\mathrm{Re}\,\rho_{10} + \mathrm{Re}\,\rho_{-10}) - \sin\alpha$$
$$(\mathrm{Im}\,\rho_{10} - \mathrm{Im}\,\rho_{-10}) \Big] \Big\} . \tag{25}$$

Provided the two decay amplitudes $R_1^{\mp}$ are both different from zero, the vectorial polarization can now be completely determined. One needs only the ratio of their absolute values.

The angular distribution of the normal to the decay plane for a spin 2 particle is obtained in the same way. The pertinent d^2 functions are given in the Appendix. For the 2^+ case where there are two independent decay amplitudes we obtain for the normal angular distribution

$$
\begin{aligned}
\frac{dN}{d\Omega} = \; & R_1^{+} \Big\{ (\rho_{22} + \rho_{-2-2}) \frac{\sin^2\beta}{2} (1 + \cos^2\beta) + (\rho_{11} + \rho_{-1-1}) \\
& \frac{1}{2} (\cos^2\beta + \cos^2 2\beta) + \rho_{00} \frac{3}{4} \sin^2 2\beta - \Big(\cos\alpha \operatorname{Re} (\rho_{21} - \rho_{-2-1}) - \sin\alpha \\
& \operatorname{Im} (\rho_{21} + \rho_{-2-1})\Big) \sin 2\beta \cos^2\beta - \Big(\cos 2\alpha \operatorname{Re} (\rho_{20} + \rho_{-20}) - \sin\alpha \operatorname{Im} (\rho_{20} - \rho_{-20})\Big) \\
& \times \frac{1}{2} \sqrt{\frac{3}{2}} \sin^2 2\beta + (\cos 2\alpha \operatorname{Re} \rho_{1-1} - \sin 2\alpha \operatorname{Im} \rho_{1-1})(\cos^2\beta - \cos^2 2\beta) \\
& + \Big(\cos\alpha \operatorname{Re} (\rho_{10} - \rho_{-10}) - \sin\alpha \operatorname{Im} (\rho_{10} + \rho_{-10})\Big) \frac{1}{2} \sqrt{\frac{3}{2}} \sin 4\beta \\
& + \Big(\cos 3\alpha \operatorname{Re} (\rho_{-21} - \rho_{2-1}) - \sin 3\alpha \operatorname{Im} (\rho_{-21} + \rho_{2-1})\Big) \sin 2\beta \sin^2\beta \\
& - \Big(\cos 4\alpha \operatorname{Re} \rho_{2-2} - \sin 4\alpha \operatorname{Im} \rho_{2-2}\Big) \sin^4\beta \Big\} \\
& + R_1^{-} \Big\{ (\rho_{22} - \rho_{-2-2}) \sin^2\beta \cos\beta + (\rho_{11} - \rho_{-1-1}) \cos\beta \cos 2\beta \\
& - \Big(\cos\alpha \operatorname{Re} (\rho_{21} + \rho_{-2-1}) - \sin\alpha \operatorname{Im} (\rho_{21} - \rho_{-2-1})\Big) \sin\beta \, (3 \cos^2\beta - 1) \\
& + \Big(\cos 2\alpha \operatorname{Re} (\rho_{20} - \rho_{-20}) - \sin 2\alpha \operatorname{Im} (\rho_{20} + \rho_{-20})\Big) \sqrt{\frac{3}{2}} \sin 2\beta \sin\beta \\
& + \Big(\cos\alpha \operatorname{Re} (\rho_{10} + \rho_{-10}) - \sin\alpha \operatorname{Im} (\rho_{10} - \rho_{-10})\Big) \sqrt{\frac{3}{2}} \sin 2\beta \cos\beta \\
& - \Big(\cos 3\alpha \operatorname{Re} (\rho_{-21} + \rho_{2-1}) - \sin 3\alpha \operatorname{Im} (\rho_{-21} - \rho_{2-1})\Big) \sin\beta \sin^2\beta \Big\}
\end{aligned}
\tag{26}
$$

For 2^- we have 3 decay amplitudes corresponding respectively to $M = \pm 2$ and 0 and thus the decay distribution will be a three parameter expression. We use (14) and the d^2 functions given in the Appendix and obtain

$$
\begin{aligned}
\frac{dN}{d\Omega} = \; & R_2^+ \Big\{ (\rho_{22} + \rho_{-2-2}) \left[\frac{1}{8} \sin^4\beta + \cos^2\beta \right] \\
& + (\rho_{11} + \rho_{-1-1}) \frac{1}{2} \sin^2\beta \, (1 + \cos^2\beta) + \rho_{00} \frac{6}{8} \sin^4\beta \\
& + \Big(\cos\alpha \, \mathrm{Re}\, (\rho_{21} - \rho_{-2-1}) - \sin\alpha \, \mathrm{Im}\, (\rho_{21} + \rho_{-2-1}) \Big) \frac{1}{4} \sin 2\beta \left(3 + \cos^2\beta \right) \\
& + \Big(\cos 2\alpha \, \mathrm{Re}\, (\rho_{20} + \rho_{-20}) - \sin 2\alpha \, \mathrm{Im}\, (\rho_{20} - \rho_{-20}) \Big) \frac{\sqrt{6}}{4} \sin^2\beta \, (1 + \cos^2\beta) \\
& + \Big(\cos 2\alpha \, \mathrm{Re}\, \rho_{1-1} - \sin 2\alpha \, \mathrm{Im}\, \rho_{1-1} \Big) \sin^4\beta \\
& + \Big(\cos\alpha \, \mathrm{Re}\, (\rho_{10} - \rho_{-10}) - \sin\alpha \, \mathrm{Im}\, (\rho_{10} + \rho_{-10}) \Big) \frac{\sqrt{6}}{4} \sin 2\beta \sin^2\beta \\
& - \Big(\cos 3\alpha \, \mathrm{Re}\, (\rho_{-21} - \rho_{2-1}) - \sin 3\alpha \, \mathrm{Im}\, (\rho_{-21} + \rho_{2-1}) \Big) \frac{1}{4} \sin 2\beta \sin^2\beta \\
& + \Big(\cos 4\alpha \, \mathrm{Re}\, \rho_{2-2} - \sin 4\alpha \, \mathrm{Im}\, \rho_{2-2} \Big) \frac{\sin^4\beta}{4} \Big\} \\
& + R_2^- \Big\{ (\rho_{22} - \rho_{-2-2}) \frac{1}{2} \cos\beta \, (1 + \cos^2\beta) + (\rho_{11} - \rho_{-1-1}) \sin^2\beta \cos\beta \\
& + \Big(\cos\alpha \, \mathrm{Re}(\rho_{21} + \rho_{-2-1}) - \sin\alpha \, \mathrm{Im}\, (\rho_{21} - \rho_{-2-1}) \Big) \frac{1}{2} \sin\beta \, (1 + 3\cos^2\beta) \\
& + \Big(\cos 2\alpha \, \mathrm{Re}(\rho_{20} - \rho_{-20}) - \sin 2\alpha \, \mathrm{Im}(\rho_{20} + \rho_{-20}) \Big) \sqrt{\frac{6}{4}} \sin^2\beta \cos\beta + \cos\alpha \\
& + \Big(\cos\alpha \, \mathrm{Re}(\rho_{10} + \rho_{-10}) - \sin\alpha \, \mathrm{Im}(\rho_{10} - \rho_{-10}) \Big) \frac{\sqrt{6}}{2} \sin^3\beta \; + \\
& + \Big(\cos 3\alpha \, \mathrm{Re}(\rho_{-21} + \rho_{2-1}) - \sin 3\alpha \, \mathrm{Im}(\rho_{-21} - \rho_{2-1}) \Big) \frac{1}{2} \sin^3\beta \Big\} \\
& + \frac{1}{2} R_0 \Big\{ (\rho_{22} + \rho_{-2-2}) \frac{6}{8} \sin^4\beta + (\rho_{11} + \rho_{-1-1}) \frac{3}{4} \sin^2 2\beta \\
& + \rho_{00} \left(2 + \frac{9}{2} \sin^4\beta - 6 \sin^2\beta \right) \Big\}
\end{aligned}
$$

$$- \left(\cos\alpha \, \mathrm{Re}\, (\rho_{21} - \rho_{-2-1}) - \sin\alpha \, \mathrm{Im}\, (\rho_{21} + \rho_{-2-1}) \right) \frac{3}{4} \sin 2\beta \sin^2\beta$$

$$+ \left(\cos 2\alpha \, \mathrm{Re}\, (\rho_{20} + \rho_{-20}) - \sin 2\alpha \, \mathrm{Im}\, (\rho_{20} - \rho_{-20}) \right) \sqrt{\frac{6}{4}} \sin^2\beta \, (3\cos^2\beta - 1)$$

$$- \left(\cos 2\alpha \, \mathrm{Re}\, \rho_{1-1} - \sin 2\alpha \, \mathrm{Im}\, \rho_{1-1} \right) 6 \sin^2\beta \cos^2\beta$$

$$- \left(\cos\alpha \, \mathrm{Re}\, (\rho_{10} - \rho_{-10}) - \sin\alpha \, \mathrm{Im}\, (\rho_{10} + \rho_{-10}) \right) \frac{\sqrt{6}}{2} \sin 2\beta \, (3\cos^2\beta - 1)$$

$$- \left(\cos 3\alpha \, \mathrm{Re}\, (\rho_{-21} - \rho_{2-1}) - \sin 3\alpha \, \mathrm{Im}\, (\rho_{-21} + \rho_{2-1}) \right) \frac{3}{2} \sin^2\beta \sin 2\beta$$

$$+ \left(\cos 4\alpha \, \mathrm{Re}\, \rho_{2-2} - \sin 4\alpha \, \mathrm{Im}\, \rho_{2-2} \right) \frac{3}{2} \sin^4\beta \Bigg\} \qquad (27)$$

When two particles are identical, integrating over the Dalitz plot averages to zero those terms proportional to R^- and the resulting expressions reduce to those given by Dennery and Krzywicki.[16] It is however possible to average separately over parts of the Dalitz plot ($\omega_1 > \omega_2$ and $\omega_2 > \omega_1$, say) and thereby allow for non-zero contributions from terms proportional to R^-.

Should resonances with higher spin be observed, explicit angular distributions of the normal to the decay plane could be readily obtained from the Legendre polynomial of order j, P_j (cos β) using the following relations.[12]

$$d^j_{m,m'\pm1}(\beta) = \frac{1}{\sqrt{(j\pm m' + 1)(j\mp m')}} \left\{ \frac{-m}{\sin\beta} + m' \cot\beta \mp \frac{\partial}{\partial\beta} d^j_{m'm}(\beta) \right\}$$

$$d^j_{00}(\beta) = P_j(\cos\beta) \quad ; \quad d^j_{m'm}(\beta) = (-1)^{m'-m} d^j_{-m'-m}(\beta) = (-1)^{m'-m} d^j_{mm'}(\beta) \qquad (28)$$

Relations (24), (25), (26) and (27) are somewhat more complicated than necessary since they correspond to the most general density matrix. In many practical cases the production mechanism is such that the density matrix has many symmetries when referred to particular axes and many of the terms written in (24-27) will not appear. On the other hand, the observation, or absence of particular terms in (24-27) would give information on the production process.[3] In this respect we recall the relations which express parity conservation in a two body production process, when the initial beam and target are not polarized. If the z axis to which the density matrix is referred is chosen normal to the production plane parity conservation in the production process yields

$$\rho_{m'm} = 0 \quad \text{if} \quad m-m' \quad \text{odd} \tag{29}$$

If the z axis is along the resonance momentum in the center-of-mass system, parity conservation in the production process yields[17]

$$\rho_{m'm} = (-1)^{m'-m} \rho_{-m'-m} \tag{30}$$

This last choice of density matrix has the advantage of being invariant under special Lorentz transformations along the resonance momentum i.e., when one passes from the production CM system to the decay CM system.[18]

We now consider the implication of the identity of the π-mesons. If two of the π-mesons are identical, i.e., have the same charge or are in a state of well-defined isotopic spin, we have shown in the preceding section that

$$F_M(\omega_1,\omega_2,\omega_3) = \pm F_{-M}(\omega_2,\omega_1,\omega_3)$$

whether they are symmetrical or antisymmetrical with respect to the exchange of the two particles' charges. It follows that $R_M^+ = \frac{1}{2}\left(|F_M|^2 + |F_{-M}|^2\right)$ and $R_M^- = \frac{1}{2}\left(|F_M|^2 - |F_{-M}|^2\right)$ are respectively symmetric and antisymmetric functions of $(\omega_1 - \omega_2)$ or of $(s - u)$. An antisymmetric function does not contribute when the distribution is integrated over the Dalitz plot (8). In order to observe terms proportional to R_M^{-2}, and determine all parts of the decaying particle density matrix, it is necessary to define the normal to the decay plane according to the different energies of the two identical particles. As mentioned above this corresponds to summing twice over half of the Dalitz plot with $\omega_1 > \omega_2$ and $\omega_1 < \omega_2$.

In many cases the symmetric function will be dominant since the simplest symmetric function is 1 while the simplest antisymmetric one is $(s - u)/M^2$, where M is a phenomenological parameter with the dimension of a mass. In any reliable model this mass would be of the order of the inverse range of the interaction. If the range is short, i.e., if vector mesons play a dominant role,[19] the average energy of each particle could be less than the inverse range (depending, of course, on how heavy the decay particle is) and the antisymmetric term would then be quenched by centrifugal barrier effect as opposed to the dominant symmetric one.

Furthermore, when the decay amplitude is written in terms of Cartesian tensors such as (22) as is usually the case when dealing with a particular model, the antisymmetric term vanishes when the different tensor amplitudes have the same phase, i.e., are relatively real. This can be seen as follows: If the spin is j, the decay amplitude is written as a Cartesian tensor or order j. It is constructed with the two linearly independent vectors available, for instance $q = q_1 - q_2$ and $p = q_1 + q_2$ where q_1 and q_2 are the momenta of the two identical pions. The decay amplitude is a linear combination of monomial expression of the type

$$G_k \, p_{i_1} \, p_{i_2} \cdots p_{i_n} \, q_{i_{n+1}} \cdots q_{i_J} \tag{31}$$

The density matrix element constructed in tensor form $\rho_{i_1 \cdots i_J, j_1 \cdots j_J}$ contributes to the angular distribution a term

$$\sum_{k\ell} G_k \, G_\ell^* \, p_{i_1} \cdots q_{i_J} \, p_{j_1} \cdots q_{j_J} \, \rho_{i_1 \cdots i_J, j_1 \cdots j_J}$$

where the indices of the sets $\{i\}$ and $\{j\}$ running from 1 to 3 refer either to p or q components depending on the subscript $k, \ell \ldots$. We can apply the Hermitian property of the density matrix to write the decay distribution as

$$\frac{1}{2} \sum_{k\ell} \Big\{ \mathrm{Re} \left\{ G_k G_\ell^* \right\} \left(\left(p_{i_1} \cdots q_{i_J} \cdot p_{j_1} \cdots q_{j_J} \right) + \left(p_{j_1} \cdots q_{j_J} \right) \left(p_{i_1} \cdots q_{i_J} \right) \right) \mathrm{Re} \, \rho_{i_1 \cdots i_J, j_1 \cdots j_J}$$

$$- \mathrm{Im} \left\{ G_k G_\ell^* \right\} \left(\left(p_{i_1} \cdots q_{i_J} \, p_{j_1} \cdots q_{j_J} \right) - \left(p_{j_1} \cdots q_{j_J} \right) \left(p_{i_1} \cdots q_{i_J} \right) \right) \mathrm{Im} \, \rho_{i_1 \cdots i_J, j_1 \cdots j_J}$$

Using the fact that the whole decay amplitude is symmetrical with respect to the exchange of the two identical particles we have that if G_k is symmetrical (antisymmetrical), the associated tensor contains a component of q an even (odd) number of times. Inspection then shows that odd powers of components of the normal to the decay plane, i.e., terms of the form $n_k = p_i q_j - q_i p_j$, which correspond to terms linear in $\cos\beta$ or $\sin\beta$ in the angular distribution are obtained only in the terms proportional to $\mathrm{Im}\left\{G_k, G_\ell^*\right\}$.

In order to fully determine the decaying particle density matrix we see that it is necessary to have amplitudes of different phases. This is necessarily the case in 3 pion decays when a two pion resonance (the ρ meson) can actually be produced.

To illustrate this point we consider the decay of a pseudo vector particle A into a $\rho\,\pi$ state with the subsequent decay of the ρ into two pions (Figure 3). We introduce the unsymmetrized $A\,\rho\,\pi$ decay amplitude as

$$g_1\,\epsilon_A\,\epsilon_\rho + g_2(\epsilon_A\cdot q_2)(\epsilon_\rho\cdot q_2)$$

and a $\rho\pi\pi$ decay amplitude

$$g\,\epsilon_\rho(q - q_1)$$

where ϵ_A and ϵ_ρ respectively stand for the linear polarization vectors of the A and ρ mesons. The $A\,3\pi$ decay amplitude can be expressed after proper symmetrization of pions 1 and 2, as

$$\epsilon_A\,g\left\{\frac{g_1(q - q_1) + g_2 q_2\,q_2(q - q_1)}{(q + q_1)^2 - m_\rho^2} + \frac{g_1(q - q_2) + g_2 q_1\,q_1(q - q_2)}{(q + q_2)^2 - m_\rho^2}\right\}$$

This last expression is of the form

$$G_1(s,t,u)(q_1 + q_2) + G_2(s,t,u)(q_1 - q_2) \tag{32}$$

where $G_1 (G_2)$ are symmetrical (antisymmetrical) functions with respect to the exchange of s and u. In (32) the mass of the ρ is actually complex and we write m_ρ^2 as $M_\rho^2 + 2i\, M_\rho \Gamma_\rho$ where M_ρ and Γ_ρ are the ρ mass and width. In terms of the coupling constants g_1 and g_2 one finds for the interference term the covariant expression

$$\mathrm{Im}\left\{G_1^* G_2\right\} = \frac{2M_\rho \Gamma_\rho (s-u)}{\left|(s-m^2)(u-m^2)\right|} \left\{g_2^2 \left((K\cdot q)^2 - (K\cdot P - 2q_1\cdot q_2 - \mu^2)^2\right)\right.$$

$$\left. + 3g_1^2 + 2g_1 g_2 (K\cdot P - 2q_1 \cdot q_2 - \mu^2)\right\}$$

where $K = q + q_1 + q_2$ is the A meson momentum. The term R_1^- in Eq. (25) is proportional to the interference term $\mathrm{Im}\,(G_1 G_2^*)$. The interference term will be non-negligible as compared to a symmetric $\left|G_1^+\right|^2$ term on the ρ bands, except on that part of the ρ bands which actually cross-over within the Dalitz plot. The non-cross-over ρ bands contain the events useful for determining the vectorial polarization of the A particle.

Vector Meson-Pion Decay

Since meson resonances appear to play a dominant role in elementary particle interactions a three meson decay may often be considered as two successive two body decays, two of the mesons being the decay products of a meson resonance produced together with the third one. Decays of this type have been already observed[5] and we now consider in some detail an example of such a process (Figure 3).

To illustrate the argument we consider a parity conserving decay where the intermediate two meson resonance is a vector meson and where the initial decaying state has a definite angular momentum. In order to construct a state of well-defined parity we use the result of applying the parity operator on a two body helicity state given by Equation (41) of Reference 7, i.e.,

$$P \mid j\ m\ \lambda > = \eta_1 \eta_2\ (-1)^{j-S_1-S_2} \mid j\ m,\ -\lambda > \tag{33}$$

Therefore a decay state of well-defined parity can be expressed as[21]

$$\sum_{\lambda \geq 0} F_\lambda \frac{1}{\sqrt{2}} \Big(\mid j\ m\ \lambda > + \ \epsilon\ (-1)^j \mid j\ m,\ -\lambda > \tag{34}$$

where j is the spin of the parent decaying particle, m its component on a fixed axis, λ is the helicity of the vector meson and ϵ is the relative parity of the vector meson and parent decaying particle. The sum in (34) extends over only two values of λ; $\lambda = 1$ (or -1) and 0.

It follows from (33) that for either choice of parity a vector meson helicity of ± 1 is allowed while the helicity 0 state is allowed only when $\epsilon = (-1)^j$. If the vector meson is a ρ (negative parity) the helicity state $\lambda = 0$ is allowed for the assignments, $1^+, 2^-, 3^+ \ldots$, etc., for the parent decaying particle. Turning now to the two spinless particle decay mode of the vector meson we see that states with $\lambda = \pm 1$ and 0 give different angular distributions. When the angular distribution is referred to the vector meson line of flight as a polar axis and averaged azimuthally one finds respectively for the cases $\lambda = \pm 1$ and $\lambda = 0$ (in the vector meson rest frame) angular distributions of the form

$$\sin^2 \theta_{\pi\pi} \quad \text{or} \quad \cos^2 \theta_{\pi\pi} .$$

This is true independently of the parent decaying particle state of polarization or alignment.

A $\cos^2 \theta_{\pi\pi}$ term allows for the occurrence of events with the 3 mesons along the same line in the parent decaying particles rest frame and would show that the relative parity to the vector meson is $(-1)^j$. Taking into account the negative parity of the ρ meson yields a parity $(-1)^{j+1}$ for the parent particle decaying into an intermediate $\rho\pi$ state.

This simple 2 body approach neglects any interference effects between the third particle. Nevertheless it can be confidently applied when there is no doubt that two of the mesons form a vector meson the third one being unaffected. The ρ bands for instance, in a 3π decay, (outside of any overlapping region) can be selected for this purpose.

To complete this discussion we give in (35) the angular distribution obtained from (34). The method for arriving at this expression follows the derivation of Equation (38) given below.

The angular distribution of the vector meson in the parent meson rest frame is then

$$I(\theta,\varphi) = \frac{1}{4} \sum_{m\,m'} \left\{ \cos(m-m')\varphi \operatorname{Re}\left[\rho_{mm'} + (-1)^{m-m'} \rho_{-m-m'}\right] - \sin(m-m')\varphi \operatorname{Im}\left[\rho_{mm'} - (-1)^{m-m'} \rho_{-m-m'}\right]\right\} \times \left[|F_1|^2 Z^{j1+}_{mm'}(\theta) + 2|F_o|^2 Z^{jo+}_{mm'}(\theta)\right]$$

IV. ISOBAR TWO AND THREE BODY DECAYS

We consider next the decay of a particle of arbitrary half integer spin j into a spin 1/2 hyperon and two spinless mesons. Parity is assumed to be conserved in the decay and hence the decay state corresponding to a pure spin state $J_z = m$ is written according to (17), as

$$\sum_M F_M(|j,m,M \tfrac{1}{2}> + \epsilon\ (-1)^M\ |j,m,M, -\tfrac{1}{2}>) \tag{36}$$

where ϵ stands for the parity of the decaying particle, relative to the decay baryon. M takes all half integer values $-j \leq M \leq j$.

Since all M values may appear in the expression obtained for the angular distribution of the normal to the decay plane this distribution will appear slightly more complicated than the one obtained in the 3π case. Nevertheless, the a priori unknown parameters, the $2j + 1$ decay amplitudes and the density matrix elements which describe the polarization and alignment of the decaying particle, also predict the polarization state of the daughter hyperon. Its density matrix can in turn be fully determined from the knowledge of the decay asymmetries.

Since our approach using the helicity formalism, generalizes the derivation of well known relations for two body decays, to three body decays, we first briefly introduce our method for the two body case. Many of these results are already known[20] but have not been given in the same concise and simple form presented here. Furthermore, in many practical cases 2-body and 3-body decays occur with similar branching ratios ($Y_1^* \to \Lambda\pi$, $Y_1^* \to \Lambda\pi\pi$ and $Y_0^* \to \Sigma + \pi$, $Y_0^* \to \Lambda\pi\pi$ ...) and it may be useful to have the various decay distributions compiled together as they both refer to the same set of density matrices.

Consider now the parity conserving two body decay of a particle into a hyperon and a spin zero meson. From Equation (33) we find that parity conservation assumes that the decay state corresponding to a pure spin state ($J_z = m$) takes the form[21]

$$\frac{1}{\sqrt{2}}\left(\left|j\ m, \tfrac{1}{2}\right> + \epsilon\ (-1)^{j+\frac{1}{2}}\left|j\ m, -\tfrac{1}{2}\right>\right) \tag{37}$$

Hence there is only one amplitude associated with a parity conserving decay. It follows from (37) that the angular distribution and the longitudinal polarization of the decay hyperon, depend on the angular momentum, and polarization state of the decaying particle but not on the relative parity ϵ. The transverse polarization, which is an interference term between the two helicity states, however changes sign with ϵ which is a well known result.[22] If we take the z axis and the hyperon momentum (in the isobar rest frame) to define a decay plane, then the polarization vector of the final hyperon is in this decay plane (Figure 4).

From (37) one readily finds angular distribution of the decay spin 1/2 hyperon as

$$\begin{aligned} I(\theta,\varphi) &= |F|^2 \times \frac{1}{2} \sum_{mm'} \rho_{mm'} \left\{ D^{j}_{m\frac{1}{2}}(\varphi,\theta,0)\, D^{j*}_{m\frac{1}{2}}(\varphi,\theta,0) + D^{j}_{m'-\frac{1}{2}}(\varphi,\theta,0)\, D^{j*}_{m-\frac{1}{2}}(\varphi,\theta,0) \right\} \\ &= \frac{1}{2} \sum_{mm'} \left(\mathrm{Re}\ \rho_{mm'} \cos(m-m')\ \varphi - \mathrm{Im}\ \rho_{mm'} \sin(m-m')\ \varphi \right) Z^{j\frac{1}{2}+}_{m'm}(\theta) \\ &= \frac{1}{4} \sum_{mm'} \Big\{ \cos(m-m')\ \varphi\ \mathrm{Re}\left(\rho_{mm'} + (-1)^{m-m'} \rho_{-m-m'}\right), \\ &\qquad - \sin(m-m')\ \varphi\ \mathrm{Im}\left(\rho_{mm'} + (-1)^{m-m'} \rho_{-m-m'}\right) \Big\}\ Z^{j\frac{1}{2}+}_{m'm}(\theta) \end{aligned} \tag{38}$$

For the longitudinal polarization, i.e., the expectation value of the helicity, we have merely to replace $Z^+(\theta)$ by $Z^-(\theta)$ and thus we obtain

$$p_L \times I(\theta,\varphi) = \frac{|F|^2}{4} \sum_{mm'} \Big\{ \cos(m-m')\varphi \,\mathrm{Re}\,(\rho_{mm'} - (-1)^{m-m'} \rho_{-m-m'}) - \sin(m-m')\varphi \,\mathrm{Im}\,(\rho_{mm'} + (-1)^{m-m'} \rho_{-m-m'}) \Big\} Z^{j\frac{1}{2}-}_{m'm}(\theta) \tag{39}$$

The longitudinal polarization given by (39) vanishes if the isobar is not polarized. If the isobar is polarized it is zero, as expected, when averaged over the angular distribution since $Z^{\frac{1}{2}-}(\pi-\theta) = -Z^{\frac{1}{2}-}(\theta)$.

The transverse polarization, directly obtained from (37) reads

$$p_T I(\theta,\varphi) = \epsilon (-1)^{j+\frac{1}{2}} \frac{|F|^2}{2} \sum_{mm'} \rho_{mm'}$$

$$\times \Big\{ D^j_{m'\frac{1}{2}}(\varphi,\theta,0)\, D^{j*}_{m-\frac{1}{2}}(\varphi,\theta,0) + D^j_{m'-\frac{1}{2}}(\varphi,\theta,0)\, D^{j*}_{m\frac{1}{2}}(\varphi,\theta,0) \Big\}$$

$$= \epsilon (-1)^{j+\frac{1}{2}} \frac{|F|^2}{2} \sum_{mm'} \Big(\mathrm{Re}\,\rho_{mm'} \cos(m-m')\varphi - \mathrm{Im}\,\rho_{mm'} \sin(m-m')\varphi \Big) X^j_{m'm}(\theta)$$

where

$$X^j_{m'm}(\theta) = d^j_{m'\frac{1}{2}}(\theta)\, d^j_{m-\frac{1}{2}}(\theta) + d^j_{m'-\frac{1}{2}}(\theta)\, d^j_{m\frac{1}{2}}(\theta)$$

With the relation $X^j_{-m'-m}(\beta) = (-1)^{m'+m} X^j_{m'm}(\beta)$ we rewrite the transverse polarization as

$$p_T \times I(\theta,\varphi) = \epsilon(-1)^{j+\frac{1}{2}} \frac{|F|^2}{4} \sum_{mm'} \Big\{ \cos(m-m')\varphi \,\mathrm{Re}\,(\rho_{mm'} + (-1)^{m+m'} \rho_{-m-m'}) - \sin(m-m')\varphi \,\mathrm{Im}\,(\rho_{mm'} - (-1)^{m+m'} \rho_{-m-m'}) \Big\} X^j_{m'm}(\theta) \tag{40}$$

Examination of Equation (40) shows that the transverse polarization also vanishes if the isobar is not polarized, and furthermore that if the azimuthal angle φ is not observed, only diagonal terms of the density matrix contribute.

The simplicity of the method is related to the fact that the ratio of the helicity amplitudes does not change when transformed from the isobar rest system to the hyperon rest frame.

For a specific illustration, we give the above decay distributions obtained for the decay of a spin 1/2 and spin 3/2 isobar. The $Z^{\pm}$ and X functions are obtained from the values of the d function given in the Appendix. In order to give relatively simple expressions we average over the φ angle. The effect of any other density matrix elements whose contributions have been averaged out can be obtained in a straightforward way if this azimuthal average is not performed.

For $j = \frac{1}{2}$ we have the well known results

$$I(\theta) = \int_0^{2\pi} I(\theta,\varphi)\, d\varphi = 2\pi \, |F|^2 \, \frac{1}{4}$$

$$p_L \times I(\theta) = 2\pi \, |F|^2 \, \frac{1}{4} \left(\rho_{\frac{1}{2}\frac{1}{2}} - \rho_{-\frac{1}{2}-\frac{1}{2}}\right) \cos\theta \tag{41}$$

$$p_T \times I(\theta) = 2\pi \, |F|^2 \, \frac{1}{4} \, \epsilon \left(\rho_{\frac{1}{2}\frac{1}{2}} - \rho_{-\frac{1}{2}-\frac{1}{2}}\right) \sin\theta$$

In the $j = 3/2$ case, it reads

$$I(\theta) = 2\pi\,|F|^2\,\frac{1}{16}\left\{\left(\rho_{\frac{1}{2}\frac{1}{2}} + \rho_{-\frac{1}{2}-\frac{1}{2}}\right)\left(1 + 3\cos^2\theta\right) + \left(\rho_{\frac{3}{2}\frac{3}{2}} + \rho_{-\frac{3}{2}-\frac{3}{2}}\right) 3\sin^2\theta\right\}$$

$$p_L\, I(\theta) = \frac{2\pi}{16}\,|F|^2\left\{\left(\rho_{\frac{1}{2}\frac{1}{2}} - \rho_{-\frac{1}{2}-\frac{1}{2}}\right)\left(9\cos^2\theta - 5\right) + 3\left(\rho_{\frac{3}{2}\frac{3}{2}} - \rho_{-\frac{3}{2}-\frac{3}{2}}\right)\sin^2\theta\right\}\cos\theta \tag{42}$$

$$p_T\, I(\theta) = -\frac{2\pi}{16}\,|F|^2\,\epsilon\left\{\left(\rho_{\frac{1}{2}\frac{1}{2}} - \rho_{-\frac{1}{2}-\frac{1}{2}}\right)\left(9\cos^2\theta - 1\right) + 3\left(\rho_{\frac{3}{2}\frac{3}{2}} - \rho_{-\frac{3}{2}-\frac{3}{2}}\right)\sin^2\theta\right\}\sin\theta$$

At this point we may easily derive a useful result. From (39) and (40) we get the ratio of the expectation values of $< p_L(\theta,\varphi)\, I\,(\theta,\varphi)\,\cos\theta >$ and $< p_T(\theta,\varphi)\, I\,(\theta,\varphi)\,\sin\theta >$ where the bracket symbol means average over all directions. We use the Clebsch-Gordan series expansion (12) together with the orthogonality property of the $\mathbb{D}$ functions. We find

$$< p_L(\theta,\varphi)\, I\,(\theta,\varphi)\,\cos\theta > =$$

$$\frac{1}{3}\,\frac{|F|^2}{2}\,\sum_m (-1)^{m-\frac{1}{2}}\,\mathrm{Re}\left(\rho_{mm} - \rho_{-m-m}\right) C(j\;j\;1|m,-m)\; C(j\;j\;1|\tfrac{1}{2},-\tfrac{1}{2})$$

and

$$< p_T(\theta,\varphi)\, I\,(\theta,\varphi)\,\sin\theta > =$$

$$\frac{|F|^2}{2}\,\frac{1}{3}\sqrt{2}\,\epsilon(-1)^{j+\frac{1}{2}}\,\sum_m (-1)^{m+\frac{1}{2}}\,\mathrm{Re}\left(\rho_{mm} - \rho_{-m-m}\right) C(j\;j\;1\,|\,m,-m)\; C(j\;j\;1|\tfrac{1}{2}\,\tfrac{1}{2})$$

It follows that

$$R_1 = \frac{< p_L(\theta,\varphi)\ I(\theta,\varphi)\ \cos\theta >}{< p_T(\theta,\varphi)\ I(\theta,\varphi)\ \sin\theta >} = -\,\epsilon(-1)^{j+\frac{1}{2}} \frac{C(j\ j\ 1\,|\,\frac{1}{2}\ -\frac{1}{2})}{\sqrt{2}\ C(j\ j\ 1\,|\,\frac{1}{2}\ \frac{1}{2})}$$

The ratio of the two Clebsch-Gordan coefficients is readily obtained and we find

$$R_1 = \epsilon(-1)^{j-\frac{1}{2}} \frac{1}{2j+1}$$

It should be stressed, however, that the two quantities which appear in this ratio are both proportional to the parent particle polarization. This result can be generalized to higher moments of the type illustrated below with the restriction of ℓ being odd.[20] For example, we can calculate the ratios

$$R_\ell = \frac{< p_L(\theta,\varphi)\ I(\theta,\varphi)\, P_\ell(\cos\theta) >}{< p_T(\theta,\varphi)\ I(\theta,\varphi) \mathcal{P}^1_\ell\ (\theta)\ \ >}$$

where $\mathcal{P}^m_\ell\ (\theta) = e^{-im\varphi}\ Y^m_\ell\ (\theta,\varphi)\sqrt{\frac{4\pi}{2\ell+1}}$

In a similar way we find for the average longitudinal polarization

$$< p_L(\theta,\varphi)\ I(\theta,\varphi)\ P_\ell(\cos\theta) > =$$

$$\frac{1}{2\ell+1}\ \frac{|F|^2}{2}\ \sum_m (-1)^{m-\frac{1}{2}}\ \mathrm{Re}(\rho_{mm} - \rho_{-m-m})\ C(j\ j\ \ell\,|\,m,-m)\ C(j\ j\ \ell\,|\,\tfrac{1}{2},-\tfrac{1}{2})$$

which vanishes for even ℓ and for the average transverse polarization

$$< P_T(\theta,\varphi)\; I(\theta,\varphi) \mathcal{P}^1_\ell\;(\theta) > =$$

$$\epsilon\, \frac{(-1)^{j+\frac{1}{2}}}{2\ell+1}\; \frac{|F|^2}{2}\; \sum_m\; (-1)^{m+\frac{1}{2}}\, \mathrm{Re}(\rho_{mm} - \rho_{-m-m})\; C(j\; j\; \ell\,|\,m,-m)\; C(j\; j\; \ell\,|\,\tfrac{1}{2},\tfrac{1}{2})$$

It then follows that[20] R_ℓ can be expressed as the ratio of Clebsch-Gordan coefficients

$$R_\ell = \epsilon(-1)^{j-\frac{1}{2}}\; \frac{C(j\; j\; \ell\,|\,\frac{1}{2},-\frac{1}{2})}{C(j\; j\; \ell\,|\,\frac{1}{2},\,\frac{1}{2})} = \epsilon(-1)^{j-\frac{1}{2}}\; \frac{\sqrt{\ell(\ell+1)}}{2j+1}$$

Similar relations can also be obtained in the same simple way when the Legendre polynomials and Legendre functions are replaced by D functions. (One always obtains the ratio of two Clebsch-Gordan coefficients but off diagonal density matrix elements are introduced.)

We now turn to the three body decay into a spin 1/2 hyperon and two spinless mesons. The angular distribution of the normal to the decay plane is given by (13) and (14). This is a simple generalization of (38) where the normal to the decay plane replaces the momentum as an analyzer of the decaying particle polarization. However for a three body decay into two spin 0 mesons and a spin 1/2 hyperon, there are in general $2j + 1$ independent amplitudes instead of one as in (38). The $2j + 1$ decay amplitudes F_M are in general unknown functions of the invariant scalars s, t and u. However, the kind of angular functions which arise in the normal angular distribution do not depend on the explicit form of the F_M but only on the parameter M. Just as in the case of 3π decays if some of the decay products are in a fixed isospin state, then there can be some additional relations among the amplitudes F_M . For example the 2π mesons will be in a state of well-defined isotopic spin for the decay Y_1^* (1660) $\rightarrow \Lambda\, 2\pi$ (branching ratio 0.23), and for the decay Y_0^* (1520) $\rightarrow \Lambda 2\pi$ (branching ratio 0.16) ... The decay amplitudes F_M with opposite values of M are then related by (19) and just as in the case treated above for the three pion decay the R_M^- amplitudes will vanish when summed over all energy configurations.

As an illustration of the general formula (14), we give the angular distribution of the normal obtained when the parent decaying particle has angular momentum 3/2. In order to make the resultant expression more compact we define the 12 quantities

$$C_1 = \rho_{\frac{3}{2}\frac{3}{2}} + \rho_{-\frac{3}{2}-\frac{3}{2}}$$

$$C_2 = \rho_{\frac{1}{2}\frac{1}{2}} + \rho_{-\frac{1}{2}-\frac{1}{2}}$$

$$C_3(\alpha) = \cos\alpha \,\mathrm{Re}\left(\rho_{\frac{3}{2}\frac{1}{2}} - \rho_{-\frac{3}{2}-\frac{1}{2}}\right) - \sin\alpha \,\mathrm{Im}\left(\rho_{\frac{3}{2}\frac{1}{2}} + \rho_{-\frac{3}{2}-\frac{1}{2}}\right)$$

$$C_4(\alpha) = \cos 2\alpha \,\mathrm{Re}\left(\rho_{\frac{3}{2}-\frac{1}{2}} + \rho_{-\frac{3}{2}\frac{1}{2}}\right) - \sin 2\alpha \,\mathrm{Im}\left(\rho_{\frac{3}{2}-\frac{1}{2}} - \rho_{-\frac{3}{2}\frac{1}{2}}\right)$$

$$C_5(\alpha) = \cos 3\alpha \,\mathrm{Re}\left(\rho_{\frac{3}{2}-\frac{3}{2}} - \rho_{-\frac{3}{2}\frac{3}{2}}\right) - \sin 3\alpha \,\mathrm{Im}\left(\rho_{\frac{3}{2}-\frac{3}{2}} + \rho_{-\frac{3}{2}\frac{3}{2}}\right)$$

$$C_6(\alpha) = \cos\alpha \,\mathrm{Re}\left(\rho_{\frac{1}{2}-\frac{1}{2}} - \rho_{-\frac{1}{2}\frac{1}{2}}\right) - \sin\alpha \,\mathrm{Im}\left(\rho_{\frac{1}{2}-\frac{1}{2}} + \rho_{-\frac{1}{2}\frac{1}{2}}\right)$$

and

$$C_1' = \rho_{\frac{3}{2}\frac{3}{2}} - \rho_{-\frac{3}{2}-\frac{3}{2}}$$

$$C_2' = \rho_{\frac{1}{2}\frac{1}{2}} - \rho_{-\frac{1}{2}-\frac{1}{2}}$$

$$C_3'(\alpha) = \cos\alpha \,\mathrm{Re}\left(\rho_{\frac{3}{2}\frac{1}{2}} + \rho_{-\frac{3}{2}-\frac{1}{2}}\right) - \sin\alpha \,\mathrm{Im}\left(\rho_{\frac{3}{2}\frac{1}{2}} - \rho_{-\frac{3}{2}-\frac{1}{2}}\right)$$

$$C_4'(\alpha) = \cos 2\alpha \,\mathrm{Re}\left(\rho_{\frac{3}{2}-\frac{1}{2}} - \rho_{-\frac{3}{2}\frac{1}{2}}\right) - \sin 2\alpha \,\mathrm{Im}\left(\rho_{\frac{3}{2}-\frac{1}{2}} + \rho_{-\frac{3}{2}\frac{1}{2}}\right)$$

$$C_5'(\alpha) = \cos 3\alpha \,\mathrm{Re}\left(\rho_{\frac{3}{2}-\frac{3}{2}} + \rho_{-\frac{3}{2}\frac{3}{2}}\right) - \sin 3\alpha \,\mathrm{Im}\left(\rho_{\frac{3}{2}-\frac{3}{2}} - \rho_{-\frac{3}{2}\frac{3}{2}}\right)$$

$$C_6'(\alpha) = \cos\alpha \,\mathrm{Re}\left(\rho_{\frac{1}{2}-\frac{1}{2}} + \rho_{-\frac{1}{2}\frac{1}{2}}\right) - \sin\alpha \,\mathrm{Im}\left(\rho_{\frac{1}{2}-\frac{1}{2}} - \rho_{-\frac{1}{2}\frac{1}{2}}\right)$$

In terms of these quantities the angular distribution of the normal may be expressed as

$$\begin{aligned}
\frac{dN}{d\Omega} = \; & \Bigg\{ C_1 \frac{1}{4}\left((1 + 3\cos^2\beta)\, R^+_{\frac{3}{2}} + 3\sin^2\beta\, R^+_{\frac{1}{2}}\right) + C_2 \frac{1}{4}\left(3\sin^2\beta\, R^+_{\frac{3}{2}} + (1 + 3\cos^2\beta)\, R^+_{\frac{1}{2}}\right) \\
& + \frac{\sqrt{3}}{2} C_3(\alpha)\, \sin 2\beta \left(R^+_{\frac{3}{2}} - R^+_{\frac{1}{2}}\right) + \frac{\sqrt{3}}{2} C_4(\alpha)\, \sin^2\beta \left(R^+_{\frac{3}{2}} - R^+_{\frac{1}{2}}\right)\Bigg\} \\
& + \Bigg\{ C_1' \frac{1}{4} \cos\beta \left((\cos^2\beta + 3)\, R^-_{\frac{3}{2}} + 3\sin^2\beta\, R^-_{\frac{1}{2}}\right) + C_2' \frac{3}{4}\cos\beta\left(\sin^2\beta\, R^-_{\frac{3}{2}} + (3\cos^2\beta - \tfrac{5}{3}) R^-_{\frac{1}{2}}\right) \\
& + \frac{\sqrt{3}}{2} C_3'(\alpha)\, \sin\beta \left((1 + \cos^2\beta)\, R^-_{\frac{3}{2}} + (1 - 3\cos^2\beta)\, R^-_{\frac{1}{2}}\right) + \frac{\sqrt{3}}{2} C_4'(\alpha) \cos\beta\, \sin^2\beta \left(R^-_{\frac{3}{2}} - 3R^-_{\frac{1}{2}}\right) \\
& + \frac{1}{4} C_5'(\alpha)\, \sin\beta\, \sin^2\beta \left(R^-_{\frac{3}{2}} - 3R^-_{\frac{1}{2}}\right) + \frac{1}{4} C_6'(\alpha)\, \sin\beta \left((9\cos^2\beta - 1)\, R^-_{\frac{1}{2}} + 3\sin^2\beta\, R^-_{\frac{3}{2}}\right)\Bigg\}
\end{aligned} \tag{43}$$

Analysis of the three body decay in terms of Equation (43) would provide 16 different functions of α and β which can in principle fully determine the decaying particles density matrix.

We now turn to the polarization of the decay spin 1/2 hyperon. As follows from the way we decomposed the parity operation where the z and y axis were defined to be along the hyperon momentum and along the normal to the decay plane respectively, the state

$$\frac{1}{\sqrt{2}}\left(\,|\, j,m,M,\, \tfrac{1}{2} > + \,\epsilon\, (-1)^{M} \,|\, j,m,M,\, -\tfrac{1}{2} > \right)$$

is an eigenstate of the spin component of the hyperon normal to the decay plane, with eigenvalue $\epsilon(-1)^{M-\frac{1}{2}}$. As usual this polarization is defined in the hyperon rest system.

It follows from (36) that the expectation value of the polarization of the hyperon, normal to the decay plane, can be easily expressed in terms of the decaying particle density matrix. The polarization is defined as the expectation value of $\sigma\cdot\hat{n}$ where $\hat{n}$ is a unit vector along the normal to the decay plane. In terms of the parent decaying particles density matrix $\rho_{mm'}$ the distribution of transverse polarization along the normal can be expressed as

$$P_T \frac{dN}{d\Omega} = \epsilon \sum_{M} (-1)^{M-\frac{1}{2}} \left| F_M \right|^2 \sum_{mm'} \rho_{mm'} \left\{ D^{j}_{m'M}(\alpha\ \beta\ 0)\ D^{j*}_{mM}\ (\alpha\ \beta\ 0) \right\} \qquad (44)$$

Just as for the angular distribution of the normal, we regroup terms with opposite values of M and obtain

$$p_T \frac{dN}{d\Omega} = \epsilon \sum_{M>0} (-1)^{M-\frac{1}{2}} \Bigg\{ R_M^+ \sum_{mm'} \Big(\mathrm{Re}\ \rho_{mm'} \cos (m-m')\alpha$$

$$- \mathrm{Im}\ \rho_{mm'} \sin (m-m')\alpha \Big) Z_{m'm}^{jM-} (\beta)$$

$$+ R_M^- \sum_{mm'} \Big(\mathrm{Re}\ \rho_{mm'} \cos (m-m')\alpha + \mathrm{Im}\ \rho_{mm'} \sin (m-m')\alpha \Big) Z_{m'm}^{jM+} (\beta) \Bigg\}$$

$$= \epsilon \sum_{M>0} (-1)^{M-\frac{1}{2}} \frac{1}{2} \Bigg\{ \sum_{mm'} \Big(\mathrm{Re} \left\{ \rho_{mm'} + (-1)^{m+m'} \rho_{-m-m'} \right\} \cos (m-m')\alpha$$

$$- \mathrm{Im} \left\{ \rho_{mm'} - (-1)^{m+m'} \rho_{-m-m'} \right\} \sin (m-m')\alpha \Big) Z_{m'm}^{jM-} (\beta) R_M^+$$

$$+ \Big(\mathrm{Re} \left\{ \rho_{mm'} - (-1)^{m+m'} \rho_{-m-m'} \right\} \cos (m-m')\alpha - \mathrm{Im}$$

$$\left\{ \rho_{mm'} + (-1)^{m-m'} \rho_{-m-m'} \right\} \sin (m-m')\alpha \Big) Z_{m'm}^{jM+} (\beta) R_M^- \Bigg\} \tag{45}$$

In order to illustrate this general relation we consider the case of the parent decaying particle to have angular momentum $\frac{3}{2}$. Applying the $Z^{\pm}$ functions already obtained for the normal angular distribution we find

$$
\begin{aligned}
p_T \frac{dN}{d\Omega} = -\epsilon \Bigg\{ & \frac{1}{4} C_1' \cos\beta \left[(\cos^2\beta + 3) R^+_{\frac{3}{2}} - 3\sin^2\beta \, R^+_{\frac{1}{2}} \right] \\
& + \frac{3}{4} C_2' \cos\beta \left(\sin^2\beta \, R^+_{\frac{3}{2}} - \left(3\cos^2\beta - \frac{5}{3}\right) R^+_{\frac{1}{2}} \right) \\
& + \frac{\sqrt{3}}{2} C_3' (\alpha) \sin\beta \left((1 + \cos^2\beta) R^+_{\frac{3}{2}} + (3\cos^2\beta - 1) R^+_{\frac{1}{2}} \right) \\
& + \frac{\sqrt{3}}{2} C_4' (\alpha) \cos\beta \sin^2\beta \left(R^+_{\frac{3}{2}} + 3 R^+_{\frac{1}{2}} \right) + \frac{1}{4} C_5'(\alpha) \sin\beta \sin^2\beta \left(R^+_{\frac{3}{2}} + 3 R^+_{\frac{1}{2}} \right) \\
& + \frac{1}{4} C_6' (\alpha) \sin\beta \left(3\sin^2\beta \, R^+_{\frac{3}{2}} - (9\cos^2\beta - 1) R^+_{\frac{1}{2}} \right) \\
& + C_1 \frac{1}{4} (1 + 3\cos^2\beta) \left(R^-_{\frac{3}{2}} - 3\sin^2\beta \, R^-_{\frac{1}{2}} \right) \\
& + C_2 \frac{1}{4} \left(3\sin^2\beta \, R^-_{\frac{3}{2}} - (1 + 3\cos^2\beta) R^-_{\frac{1}{2}} \right) \\
& + \frac{\sqrt{3}}{2} C_3(\alpha) \sin 2\beta \left(R^-_{\frac{3}{2}} + R^-_{\frac{1}{2}} \right) \\
& + \frac{\sqrt{3}}{2} C_4(\alpha) \sin^2\beta \left(R^-_{\frac{3}{2}} + R^-_{\frac{1}{2}} \right) \Bigg\}
\end{aligned}
\tag{46}
$$

Equations (43) and (46) can be used to determine the spin and parity of the decaying isobar by fitting to the three body data, or at least can be used to impose further consistency requirements when the two body decay data are simultaneously analyzed in terms of (38, 39, and 40).

For example (46) when applied to the Y_1^* (1660) data should yield polarizations of the same sign for the Λ and Σ, when averaged over both the Dalitz plot and the azimuthal angle of the normal if the Λ and Σ particles have the same parity. This comparison could be considered as an independent determination of the $\Sigma\Lambda$ relative parity and generalizes to three body decays a result already known for two-body decays.[23]

If desired, the expectation value of the hyperon polarization along any other direction is readily obtained from (36). However, the polarization normal to the decay plane is the only component of polarization which does not vanish when an average is performed over γ . This is because the normal component of the polarization does not depend on any interference terms between the decay amplitudes. The observation of γ, or of the decay hyperon polarization component in the decay plane (as a function of γ), would yield further information on the decay amplitudes. However, the relations of the type illustrated by (43) and (46), obtained by averaging over γ, provide enough constraints to fully determine the decaying particles density matrix and further provide an independent means of determining its spin[24] and parity.

Isobar-Pion Decay

Since a three body decay of a high mass isobar may proceed through an intermediate isobar-pion decay we now consider, as in the case of the three pion decay, two successive parity conserving two-body decays eventually producing a final three-body state of 1/2 baryon and two spinless mesons. We restrict the arguments below to exclude any possible overlapping isobar bands, thus eliminating any possible ambiguities as to the kind of two-body decay. The $N^*\ (1688) \rightarrow N^*\ (1238) + \pi$ and $\Xi^*\ (1810) \rightarrow \Xi^*\ (1530) + \pi$ provide two such examples.[5] In both cases, one of the daughter particles is a decuplet member with angular momentum $3/2^+$. For the first step of this two step process parity conservation implies two independent decay amplitudes. Assuming that the intermediate particles are a spin 3/2 particle and a spin zero particle we have that the intermediate decay state corresponding to a pure spin state of the initial particle can be expressed as

$$\frac{1}{\sqrt{2}}\left\{F_{\frac{3}{2}}\left(\left|j,m,\tfrac{3}{2}\right> + \epsilon\ (-1)^{j-\frac{1}{2}}\left|j,m,-\tfrac{3}{2}\right>\right) + F_{\frac{1}{2}}\left(\left|j,m,\tfrac{1}{2}\right> + \epsilon\ (-1)^{j-\frac{1}{2}}\left|j,m,-\tfrac{1}{2}\right>\right)\right\} \tag{47}$$

where j is the parent isobar angular momentum and ϵ stands for the relative parity of the parent and daughter isobars. For the special case of the parent isobar having spin 1/2 there is only one decay amplitude and $F_{\frac{3}{2}}$ would not appear in (47).

The density matrix ρ' of the daughter isobar can be expressed in terms of the parent density matrix ρ as

$$\rho'_{\mu\nu} = F_\mu F^*_\nu \sum_{mm'} e^{i(m-m')\varphi}\, d^j_{m\mu}(\theta)\, d^j_{m'\nu}(\theta)\, \rho_{mm'} \tag{48}$$

The density matrix ρ' is defined in terms of a coordinate system derived from the initial coordinate system in the parent isobar rest frame by a rotation of angles θ and φ where θ and φ are polar and azimuthal angles of the momentum of the daughter isobar in the parent isobar system. (Figure 5). Parity conservation as expressed by equations of the form (47) then implies that for an unpolarized parent particle

$$\rho'_{-\mu-\nu} = \rho'_{\mu\nu} \tag{49}$$

As follows from the transformation property of the helicity amplitudes under Lorentz transformation the density matrix ρ' is the same in either the parent isobar rest frame or the daughter isobar rest frame. We note also that Equations (48) and (49) are valid for all spin of the daughter isobar.

If the daughter isobar subsequently has a two-body decay her density matrix given by (48) may now be used directly in (38), (39) and (40) to express the resultant angular distributions. In particular for the case of the daughter isobar having spin 3/2 the density matrix (48) can be substituted directly in (42). The results obtained in the beginning of Section IV pertaining to two body decays can now be applied directly to the daughter isobar decay, especially the theorem on the ratio of transverse to longitudinal polarization.

The succession of reference frames used in the analysis of such a two step process, followed by the eventual isobar decay into $Y + \pi$, are shown on Figure 5.

It is perhaps by this last example of the two stage decay that the simplicity of a method using helicity state is clearly demonstrated. The more traditional treatment would require recoupling coefficients to describe the second stage of the decay in terms of the parameters describing the first stage, a complication avoided by our presentation.

APPENDIX

We list together the d functions which are useful for the analysis of the decay of particles of spin less than or equal to 3. Not all the d functions are given. The missing ones are easily obtained using the simple symmetry relations

$$d^j_{m'm}(\beta) = (-1)^{m-m'} d^j_{-m'-m}(\beta)$$

$$d^j_{m'm}(\beta) = (-1)^{m-m'} d^j_{m\,m'}(\beta)$$

Several recurrent relations useful for the calculation of the d functions are given in the appendix of Reference 7. More relations are given in References 12 and 25.

The relevant d are now listed below.

Spin $\frac{1}{2}$

$$d_{\frac{1}{2}\frac{1}{2}}(\beta) = \cos\frac{\beta}{2} \qquad d_{-\frac{1}{2}\frac{1}{2}}(\beta) = \sin\frac{\beta}{2}$$

Spin 1

$$d_{11}(\beta) = \frac{1+\cos\beta}{2} \qquad d_{01}(\beta) = \frac{\sin\beta}{\sqrt{2}}$$

$$d_{1-1}(\beta) = \frac{1-\cos\beta}{2} \qquad d_{00}(\beta) = \cos\beta$$

Spin $\frac{3}{2}$

$$d_{\frac{3}{2}\frac{3}{2}}(\beta) = \frac{1 + \cos\beta}{2}\cos\frac{\beta}{2} \qquad d_{\frac{3}{2}\frac{1}{2}}(\beta) = -\sqrt{3}\,\frac{1 + \cos\beta}{2}\sin\frac{\beta}{2}$$

$$d_{\frac{3}{2}-\frac{1}{2}}(\beta) = \sqrt{3}\,\frac{1 - \cos\beta}{2}\cos\frac{\beta}{2} \qquad d_{\frac{3}{2}-\frac{3}{2}}(\beta) = -\frac{1 - \cos\beta}{2}\sin\frac{\beta}{2}$$

$$d_{\frac{1}{2}\frac{1}{2}}(\beta) = \frac{3\cos\beta - 1}{2}\cos\frac{\beta}{2} \qquad d_{\frac{1}{2}-\frac{1}{2}}(\beta) = -\frac{1 + 3\cos\beta}{2}\sin\frac{\beta}{2}$$

Spin 2

$$d_{22}(\beta) = \left(\frac{1 + \cos\beta}{2}\right)^2 \qquad d_{21}(\beta) = -\frac{1 + \cos\beta}{2}\sin\beta$$

$$d_{20}(\beta) = \frac{\sqrt{6}}{4}\sin^2\beta \qquad d_{2-1}(\beta) = -\frac{1 - \cos\beta}{2}\sin\beta$$

$$d_{2-2}(\beta) = \left(\frac{1 - \cos\beta}{2}\right)^2 \qquad d_{11}(\beta) = \frac{1 + \cos\beta}{2}(2\cos\beta - 1)$$

$$d_{10}(\beta) = -\sqrt{\frac{3}{2}}\sin\beta\cos\beta \qquad d_{1-1}(\beta) = \frac{1 - \cos\beta}{2}(2\cos\beta + 1)$$

$$d_{00}(\beta) = \frac{3\cos^2\beta - 1}{2}$$

Spin $\frac{5}{2}$

$$d_{\frac{5}{2}\frac{5}{2}}(\beta) = \left(\frac{1+\cos\beta}{2}\right)^2 \cos\frac{\beta}{2}$$

$$d_{\frac{5}{2}\frac{3}{2}}(\beta) = -\sqrt{5}\left(\frac{1+\cos\beta}{2}\right)^2 \sin\frac{\beta}{2}$$

$$d_{\frac{5}{2}\frac{1}{2}}(\beta) = \frac{\sqrt{10}}{4}\sin^2\beta\cos\frac{\beta}{2}$$

$$d_{\frac{5}{2}-\frac{1}{2}}(\beta) = -\frac{\sqrt{10}}{4}\sin^2\beta\sin\frac{\beta}{2}$$

$$d_{\frac{5}{2}-\frac{3}{2}}(\beta) = \sqrt{5}\left(\frac{1-\cos\beta}{2}\right)^2 \cos\frac{\beta}{2}$$

$$d_{\frac{5}{2}-\frac{5}{2}}(\beta) = -\left(\frac{1+\cos\beta}{2}\right)^2 \sin\frac{\beta}{2}$$

$$d_{\frac{3}{2}\frac{3}{2}}(\beta) = \frac{5\cos\beta - 3}{2}\cos^3\frac{\beta}{2}$$

$$d_{\frac{3}{2}\frac{1}{2}}(\beta) = \frac{-(5\cos\beta-1)}{\sqrt{2}}\cos^2\frac{\beta}{2}\sin\frac{\beta}{2}$$

$$d_{\frac{3}{2}-\frac{1}{2}}(\beta) = \frac{1+5\cos\beta}{\sqrt{2}}\sin^2\frac{\beta}{2}\cos\frac{\beta}{2}$$

$$d_{\frac{3}{2}-\frac{3}{2}}(\beta) = -\frac{5\cos\beta+3}{2}\sin^3\frac{\beta}{2}$$

$$d_{\frac{1}{2}\frac{1}{2}}(\beta) = \frac{5\cos^2\beta-2\cos\beta-1}{2}\cos\frac{\beta}{2}$$

$$d_{\frac{1}{2}-\frac{1}{2}}(\beta) = -\frac{5\cos^2\beta+2\cos\beta-1}{2}\sin\frac{\beta}{2}$$

Spin 3

$$d_{33}(\beta) = \left(\frac{1+\cos\beta}{2}\right)^3$$

$$d_{32}(\beta) = -\frac{\sqrt{6}}{8}\sin\beta\,(1+\cos\beta)^2$$

$$d_{31}(\beta) = \frac{\sqrt{15}}{8}\sin^2\beta\,(1+\cos\beta)$$

$$d_{30}(\beta) = -\frac{\sqrt{5}}{4}\sin^3\beta$$

$$d_{3-1}(\beta) = \frac{\sqrt{15}}{8}\sin^2\beta\,(1-\cos\beta)$$

$$d_{3-2}(\beta) = -\frac{\sqrt{6}}{8}\sin\beta\,(1-\cos\beta)^2$$

$$d_{3-3}(\beta) = \left(\frac{1-\cos\beta}{2}\right)^3$$

$$d_{22}(\beta) = \left(\frac{1+\cos\beta}{2}\right)^2(3\cos\beta-2)$$

$$d_{21}(\beta) = -\frac{\sqrt{5}}{4\sqrt{2}} \sin\beta(3\cos^2\beta + 2\cos\beta - 1) \qquad d_{20}(\beta) = \frac{\sqrt{15}}{2\sqrt{2}} \cos\beta \sin^2\beta$$

$$d_{2-1}(\beta) = \frac{\sqrt{5}}{4\sqrt{2}} \sin\beta(3\cos^2\beta - 2\cos\beta - 1) \qquad d_{2-2}(\beta) = \left(\frac{1-\cos\beta}{2}\right)^2 (3\cos\beta + 2)$$

$$d_{11}(\beta) = \frac{1+\cos\beta}{8} (15\cos^2\beta - 10\cos\beta - 1) \qquad d_{10}(\beta) = -\frac{\sqrt{3}}{4} \sin\beta\,(5\cos^2\beta - 1)$$

$$d_{1-1}(\beta) = \frac{1-\cos\beta}{8} (15\cos^2\beta + 10\cos\beta - 1) \qquad d_{00}(\beta) = \frac{5\cos^3\beta - 3\cos\beta}{2}$$

REFERENCES AND FOOTNOTES

1. G. C. Wick, Annals of Physics 18, 65, (1961).

2. R. Omnès, UCRL (1964) unpublished.

3. See for instance J. D. Jackson, Rev. Mod. Phys. (To be published); S. M. Berman and S. D. Drell, Phys. Rev. 133, B 791, (1964).

4. For a detailed discussion on the energy distribution of three pions see C. Zemach, Phys. Rev. 133, B 1201 (1964).

5. For a recent compilation of resonance quantum numbers see for example A. H. Rosenfeld *et al*., Rev. Mod. Phys. 36, 977 (1964).

6. E. P. Wigner, Gruppentheorie und ihre Anwendung auf die Quantunmechanik der Atomspektren. Edwards Bros., Ann Arbor, Michigan, 1954 Chapter 12.

7. M. Jacob and G. C. Wick, Annals of Physics 7, 404 (1959).

8. The separation of angular variables by means of the Wigner projection method has been used in connection with the three particle scattering problem by Omnès[2] and also by Bhatia and Temkin[9] in the study of the two electron and fixed nucleon problem. Even though the general procedure is perhaps well known the practical application to three body decays as a means for spin and parity determination appears to carry enough interest for the specialized and detailed discussion given here.

9. A. K. Bhatia and A. Temkin, Rev. Mod. Phys. 36, 1050 (1964).

10. We do not follow here the convention chosen in Ref. 7 for the construction of two particle states (see Eq. 13 of Ref. 7) but treat all particles on the same footing according to (1).

11. This result may be seen as follows. The integration indicated by (6) is over all possible directions of two vectors whose relative angle is fixed. But this integration may as well be considered as ranging over all possible notations of a rigid body. In this case we may apply the well known result that the differential element may be written as $dR = d\,\alpha\, d\,\gamma\, \sin\beta\, d\,\beta$ where α, β and γ are the usual Euler angles. For a detailed derivation see Ref. 4.

12. We follow the notations of M. E. Rose "Elementary Theory of Angular Momentum" John Wiley and Sons, Inc., 1957. We refer the reader to this book for the various relations among rotation matrix elements used throughout this paper.

13. See Eq. (9') of Ref. 7. All helicity states in this paper are defined with the "conventional" z axis along the particle's momentum and the "conventional" y axis normal to the decay plane.

14. For example see M. Jacob and A. Morel, Phys. Letters 7, 350 (1963).

15. We use a metric such that $a \cdot b = a_o b_o - \vec{a} \cdot \vec{b}$

16. Ph. Dennery and A. Krzywicki, Phys. Rev. 136 B 839 (1964).

17. We use the $(\varphi, \theta, 0)$ representation of Ref. 7.

18. This follows from the property that the generator of a Lorentz transformation along a particular axis M_{i4} commutes with the component of the angular momentum along that axis, i.e., $[M_{jK}, M_{i4}] = 0$.

19. See for example M. Gell-Mann, D. Sharp, W. Wagner, Phys. Rev. Letters 8, 71 (1962).

20. N. Byers and S. Fenster, Phys. Rev. Letters 11, 52 (1963).

21. When we consider two particle-states one of the particles is assumed to be spin zero. Footnote 10 is then irrelevant.

22. R. Gatto and H. Stapp, Phys. Rev. 121, 1553 (1961).

23. Ph. Meyer, J. Prentki, Y. Yarnaganchi, Phys. Rev. Letters 5, 442 (1960).

24. In effect this gives a lower limit for the spin since a higher spin particle, with a particular polarization, could give the same angular distribution.

25. A. R. Edmonds, Angular Momentum in Quantum Mechanics, Princeton University Press (1957).

FIGURE CAPTIONS

Figure 1 The angles of rotation of the one particle helicity state.

Figure 2 The decay plane configuration - Triangle 1 represents the decay plane in the standard position. Triangle 2 shows the plane after rotation of angle γ. Triangle 3 shows the decay plane in its actual position with its normal indicated by $\hat{n}$.

Figure 3 ρ-π decay. The ρ momentum is taken along z' - the relative momentum of the decay pions is taken along z''.

Figure 4 Hyperon-pion decay. The decay hyperon momentum is taken along z'. p_T, p_L and p are the transverse, longitudinal and total polarizations of the decay hyperon respectively.

Figure 5 The two stage decay $Y^{**} \to Y^* + \pi$, $Y^* \to Y + \pi$, the coordinate system (x,y,z) is the rest frame associated with the Y^{**}. The Y^* momentum is along z' and (x',y',z') is the rest frame associated with the Y^*. The Y momentum is along z''. In addition the coordinate system (x'',y'',z'') in the Y rest frame, used for the analysis of the final hyperon polarization is indicated on the figure where z''' is along the nucleon in the analyzing decay $Y \to N + \pi$. Note that the direction of the y axis remains invariant between any two successive frames of reference.

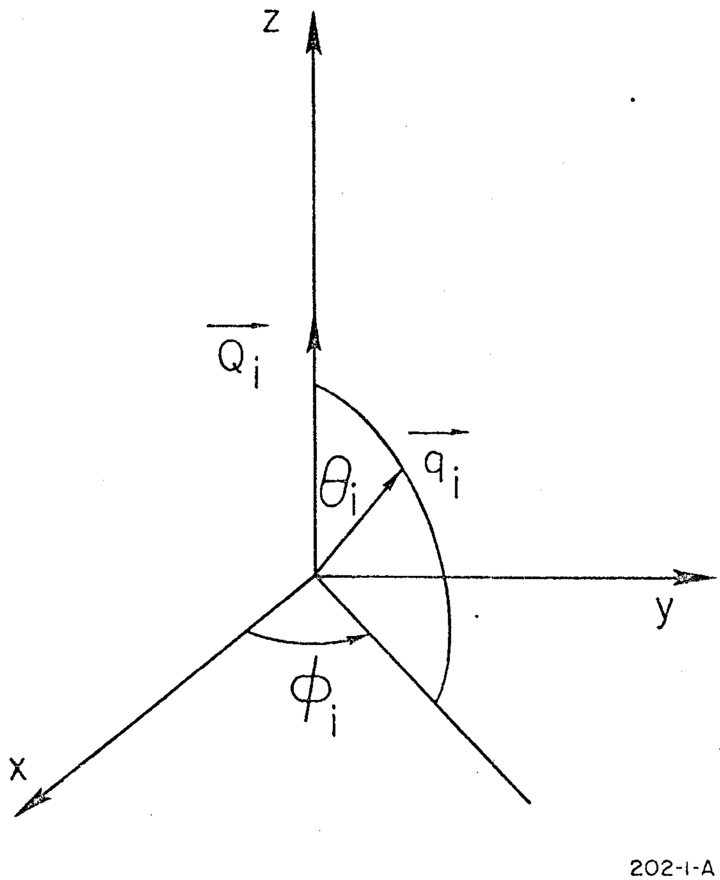

FIGURE 1

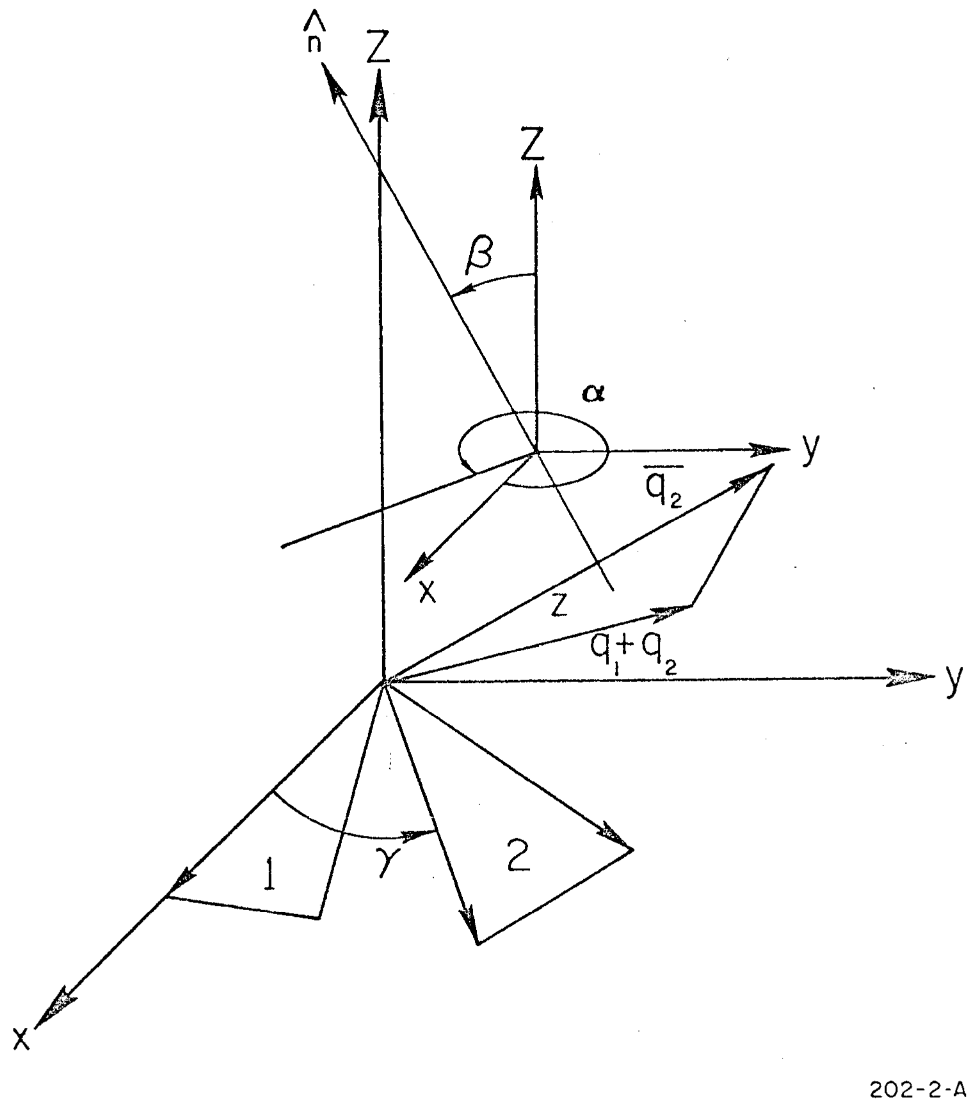

FIGURE 2

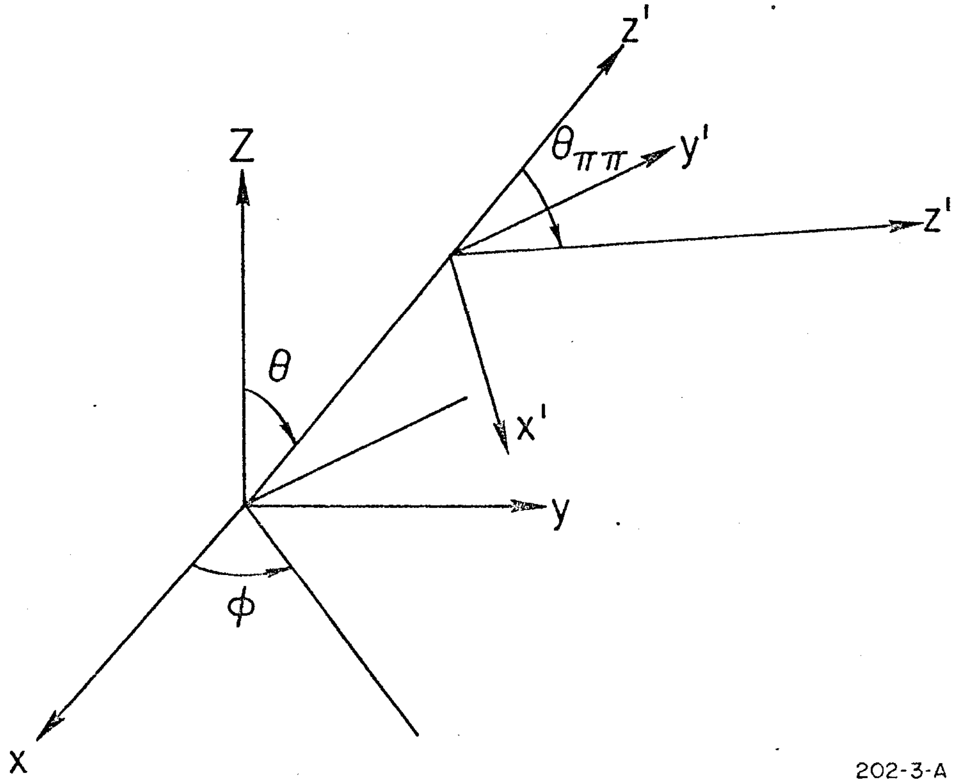

FIGURE 3

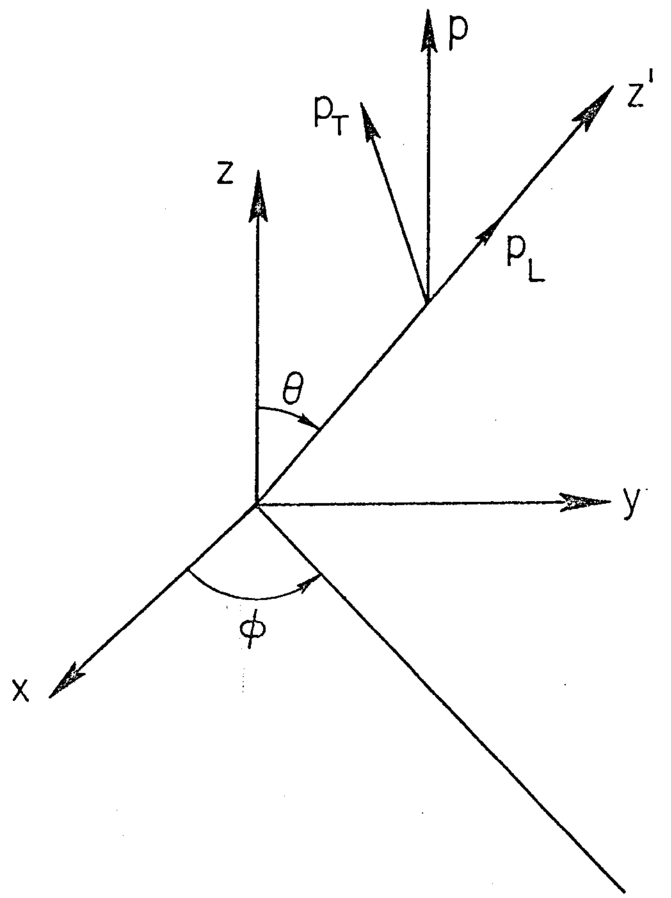

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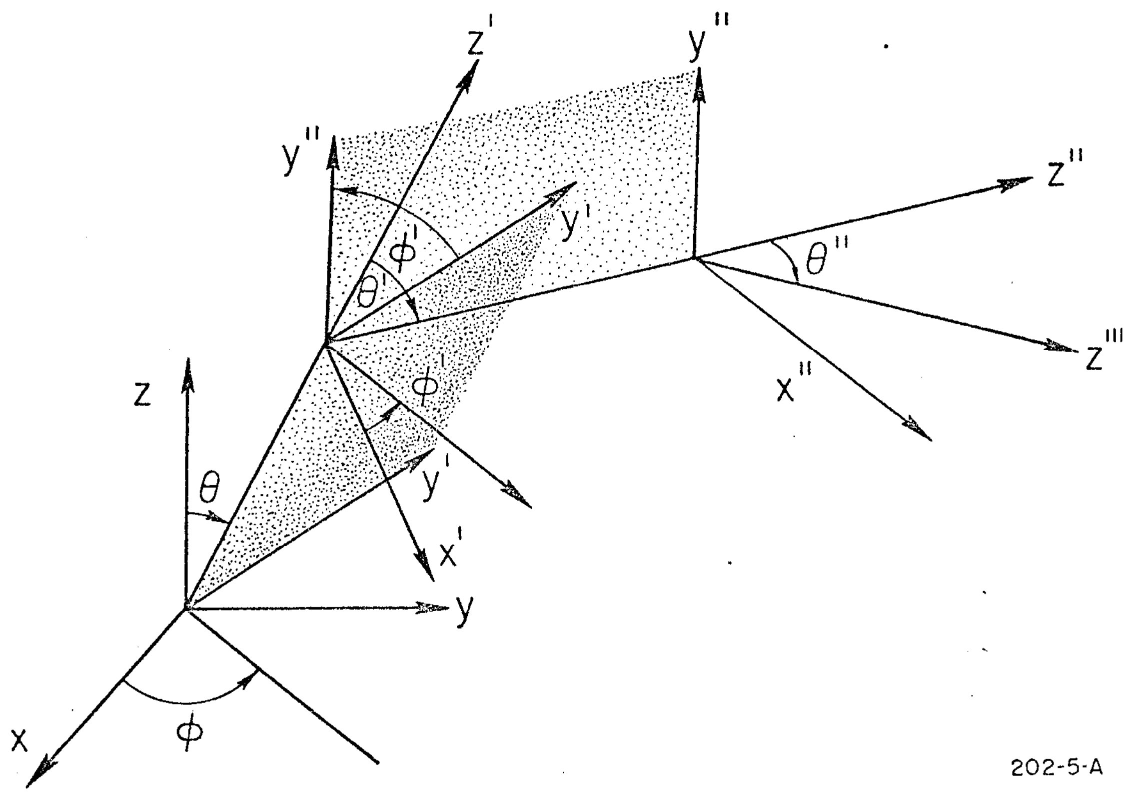
FIGURE 4



x, y, z	Y**	REST FRAME	$Y^{**} \rightarrow Y^{*} + \pi$
x', y', z'	Y*	REST FRAME	$Y^{*} \rightarrow Y + \pi$
x'', y'', z''	Y	REST FRAME	$Y \rightarrow N + \pi$

FIGURE 5

TABLE I

	j even	j odd
parity even	j	j+1
parity odd	j+1	j

Number of independent amplitudes describing the angular distribution of the three pion decay of a spin j particle. The columns refer to the angular momentum j and the rows to the parity of the decaying particle.