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NONEQUIVALENCE OF THE ONE CHANNEL N/D EQUATIONS WITH INELASTIC UNITARITY AND THE MULTICHANNEL ND^{-1} EQUATIONS

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Consider a partial wave elastic scattering amplitude for two spinless equal mass, M, particles as a function of $s(= 4(k^2 + M^2))$[1]

$$A = \frac{1}{2i\rho}(S-1) = \frac{1}{2i\rho}(\eta e^{2i\delta}-1) = B + {}^{R}A \tag{1}$$

where ρ is a kinematical factor and the "generalized potential" B is regular in the physical region, whereas ${}^{R}A$ has cuts for $s > 4M^2 \equiv s_E$. The inelastic partial wave cross section σ_r^{ℓ} is determined by η alone:

$$\sigma_r^{\ell} = \pi k^2(2\ell+1)(1-\eta^2) \quad . \tag{2}$$

Given B and η, we can determine $A \equiv N/D$ using the Frye-Warnock equations:[2,3]

$$\frac{2\eta(s)}{1+\eta(s)} \operatorname{Re} N(s) = \bar{B}(s) + \frac{1}{\pi}\int_{s_E}^{\infty} \frac{(\bar{B}(s') - \bar{B}(s))\, 2\rho(s') \operatorname{Re} N(s')ds'}{(s'-s)(1+\eta(s'))}$$

$$\bar{B}(s) = B(s) + \frac{P}{\pi}\int_{s_I}^{\infty} \frac{(1-\eta(s'))\, ds'}{2\rho(s)(s'-s)} \quad , \tag{3}$$

$$D(s) = 1 - \frac{P}{\pi}\int_{s_E}^{\infty} \frac{2\rho(s') \operatorname{Re} N(s')\, ds'}{(s'-s)(1+\eta(s'))} - i\,\frac{2\rho(s)}{1+\eta(s)} \operatorname{Re} N(s)\, \theta(s-s_E),$$

$$\operatorname{Im} N(s) = \frac{1-\eta(s)}{2\rho(s)} \operatorname{Re} D(s)\, \theta(s-s_I) \quad ,$$

where s_I is the lowest inelastic threshold. On the other hand consider a set of coupled 2 body channels with potentials B_{ij}. The amplitudes

$$A_{ij} \left(= (S_{ij} - \delta_{ij}) \frac{1}{2i(\rho_i \ \rho_j)^{\frac{1}{2}}} \right)$$

may be determined by the multichannel ND^{-1} formalism from the B_{ij}. Now take B_{11} and η determined from the $|A_{ij}|^2$ and calculate A from (3).[4]

The purpose of this note is to demonstrate by a simple example that the solution A is not in general equal to A_{11}.[5] We generalize from the results of calculations described below that a sufficient condition for the two amplitudes A (calculated from (3)) and A_{11} (calculated by the multichannel ND^{-1} equations) to be identical is that the diagonal forces $B_{ii}(i \neq 1)$ are not strong enough to produce bound states in channel i in the absence of coupling between the channels. As one increases the strengths for the B_{ii} $(i \neq 1)$ beyond these values (necessary to produce binding), complex conjugate pairs of zeros in S_{11} move onto the physical sheet through the inelastic cut $(s > s_I)$. The two calculations then disagree. Thus the physical situation in which we have a B_{ii} strong enough to produce a bound state in channel i and then weakly couple it to the open channel 1 to produce a narrow resonance in A_{11} cannot be reproduced in the one channel calculation (3). In addition, we demonstrate that, in our simple example, there are no poles of the S matrix on the physical sheet for complex values of s.

In order to carry out a substantial amount of the calculations analytically, we consider a two channel non-relativistic s wave ($\rho_i = (s-s_i)^{\frac{1}{2}}$) system with the input (symmetric) B given by a single pole

$$B_{ij} = g_{ij}/(s+m). \tag{4}$$

Then

$$A_{ij} = \frac{g_{ik}(D^{-1})_{kj}}{s+m} \quad ,$$

$$\frac{1-\eta^2}{4} = \rho_1 \rho_2 |A_{12}|^2 \, \theta(s-s_2) \ , \tag{5}$$

$$D_{ij} = \delta_{ij} - g_{ij} \, \phi_i$$

$$\phi_i = -\frac{1}{2(s_i+m)^{\frac{1}{2}}} + \frac{(s_i+m)^{\frac{1}{2}}}{s+m} - \frac{(s_i-s)^{\frac{1}{2}}}{s+m} \quad .$$

The procedure is for given g_{ij} and m calculate A_{11} and η from (5). Then using B_{11} and η as input we calculate A from (3) and compare it with A_{11}. (The integral equation (3) for Re N(s) is solved numerically by the matrix inversion technique.) The next step in the program is to locate the zeros and poles of $S_{11}(= 2i\rho_1 A_{11} + 1)$. This problem is easily reduced to solving a quartic equation in the variable $(s-s_2)^{\frac{1}{2}}$; the same equation gives both zeros and poles of S_{11} as a function of the 3 g_{ij}'s for a given input pole position m. After solving for the roots, we determine whether they correspond to poles or zeros of S_{11} on the physical sheet (where $\mathrm{Im}(s-s_2)^{\frac{1}{2}} \geq 0$ and $\mathrm{Im}(s-s_1)^{\frac{1}{2}} \geq 0$) by putting these values back into the expressions for A_{11} and S_{11}. We find that there are no poles in S_{11} on the physical sheet for complex values of s.

Now for given g_{11} and g_{12}, take g_{22} small; then A_{11} agrees with A as calculated from (3). Increase g_{22} : for all $g_{22} >$ some value $\bar{g}_{22}(g_{11}, g_{12}, m) > 2(s_2+m)^{\frac{1}{2}}$ (the value for which channel 2 in the absence of coupling to channel 1 developes a bound state) the 2 amplitudes A_{11} and A disagree. Returning to the location of the zeros in S_{11}, we find that $\bar{g}_{22}$ corresponds to the value for which a (double) zero in S_{11} occurs along the real axis above the inelastic threshold, i.e., η for some $s > s_2$ is equal to zero. We see for this situation that the integral equation (3) for Re N is no longer Fredholm. For $g_{22} > \bar{g}_{22}(g_{11}, g_{12}, m)$ a pair of zeros in S_{11} (at complex conjugate points) move from the real axis onto the physical sheet.

We investigated in great detail the case $g_{11} = 0$, i.e., no left hand cut in channel 1. In this case the Ball-Frazer[6] representation is applicable: We write a dispersion relation for the phase shift in channel 1:

$$\delta = -(s-s_1)^{\frac{1}{2}} \frac{P}{2\pi} \int_{s_2}^{\infty} \frac{\ln \eta(s')ds'}{(s'-s_1)^{\frac{1}{2}}(s'-s)} \quad . \qquad (6)$$

In addition, we note that the quartic equation for the zeros in S_{11} reduces to a cubic. We find that in all cases ($g_{11} = 0$) that both one channel calculations (3) and (6) for A agree. They both break down and disagree[7] with the two channel A_{11} when zeros in S_{11} appear on the physical sheet, coming through the inelastic cut. It is clear that A as calculated from (6) will disagree with A_{11} then since zeros in S_{11} amount to cuts in δ which are not taken into account by (6).

The appearance of zeros (at α and α^*) of S_{11} on the physical sheet through the inelastic cut will also cause the Froissart[8] one channel N/D formalism to disagree with A_{11}. He introduces

$$R = \exp\left(-\frac{i(s-s_1)^{\frac{1}{2}}}{\pi}\int_{s_I}^{\infty}\frac{ds' \, \ln \eta(s')}{(s'-s_1)^{\frac{1}{2}}(s'-s)}\right)$$

and notes that $R^{-1}S$ satisfies elastic unitarity. However R is not unique since we could multiply it by the factor

$$G = \frac{(\alpha - i(s-s_1)^{\frac{1}{2}})(\alpha^* - i(s-s_i)^{\frac{1}{2}})}{(\alpha + i(s-s_1)^{\frac{1}{2}})(\alpha^* + i(s-s_1)^{\frac{1}{2}})} \quad .$$

This would presumably bring the one channel calculation in agreement with the multichannel one. The G factor is clearly related to specifying the CDD ambiguity.[9]

In summary, we speculate that a sufficient condition for one channel calculation (3) to agree with the multichannel amplitude A_{11} is that the diagonal forces in the channels not explicitly considered should not be strong enough to produce bound states in the absence of coupling to channel 1.

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REFERENCES

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4. We consider the situation in which all the channels i have thresholds $s_i >$ than that of channel 1.

5. A similar problem has been considered from a different approach by G. Chew (private communication) and E. Squires (preprint).

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7. Note that a numerical comparison between A_{11} and A as calculated by (6) can be made to more significant figures than the calculation (3) since (3) involves the solution of an integral equation in addition to numerical integration.

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