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Analyses of Muon Electrodynamic Tests*

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ABSTRACT

A simple model for a possible modification of the fermion propagator in quantum electrodynamics is presented along with the accompanying change in the vertex function dictated by the Ward-Takahashi identity in order to maintain current conservation. This model is used to compute corrections to the calculations of the muon g-2 value, to the μ-pair production and to the muon-proton scattering cross sections. The purpose is to show by explicit calculation, and thereby to emphasize, that these are independent and complementary probes of the theory. Therefore, each measurement should be carried out to a maximum obtainable accuracy independent of the results found in the other two.

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INTRODUCTION

Recent experiments on the electrodynamic interactions of μ mesons have been in agreement with the predictions of the theory of quantum electrodynamics. The muon appears to be just a heavy electron. The principal results of interest to us here are the comparison of muon with electron scattering from hydrogen,[1] and measurements of the g-2 value of the muon[2] and of the cross section for large angle photoproduction of μ-pairs.[3]

These experiments have been analyzed in terms of limits they put on possible deviations from the standard theory of quantum electrodynamics. Since different ingredients of the theory enter into the different processes listed above, different features of the theory are probed by these experiments. In particular the Feynman diagrams in Fig. 1 show that the comparison of muon with electron scattering gives the ratio of muon to electron electromagnetic form factors as functions of momentum transfer q^2. In Fig. 2 the lowest order g-2 calculation involves photon and muon propagators as well as the electromagnetic vertex when two lines, one photon and one muon, are off their mass shells. In Fig. 3 for the Bethe-Heitler pair-production amplitude there appear a muon propagator and vertices with the muon propagator off the mass shell and with the photon real or virtual. In contrast with the g-2 diagram in which one integrates over all virtual four momenta flowing through the lines and vertices in the closed loop, for the pair production the muon has a fixed (large) virtual four momentum. Also the virtual photon "carrying" the Coulomb interaction is almost on its mass shell for the symmetric arrangement of equal energies and equal angles left and right for the μ^- and μ^+.

From these diagrams it is clear that the close agreement of the different experiments with theory limits the possible deviations that can be introduced from quantum electrodynamic predictions. For example the equality of muon with electron scattering at identical energies and momentum transfers constrains the structure of the muon vertex for real muon lines as a function of the photon four momentum but does not limit (to this order in $\alpha = 1/137$) the behavior of the muon propagator or the vertex contribution for a virtual muon line entering or leaving. This latter behavior is constrained by the g-2 and large angle pair experiments which do not, however, constrain vertices with photons of large q^2.

As a convenient mnemonic device it has become customary to regulate propagators appearing in the cross section calculations and to refer to the lower limit of the regulator mass Λ, or correspondingly the upper limit of the cutoff length $1/\Lambda$, consistent with the experimental results as the limit to which quantum electrodynamics has been probed within the errors of the experiments.[4] For example a regulated photon propagator

$$\frac{1}{q^2} \rightarrow \frac{1}{q^2} - \frac{1}{q^2 - \Lambda_\gamma^2} \quad ; \quad q^2 = q_o^2 - \vec{q}^{\,2} \tag{1}$$

in the g-2 calculation, Fig. 2 corrects the magnetic moment as first computed by Berestetsky, Krokhin, and Khlebnikov[5] to

$$g\text{-}2 = \frac{\alpha}{\pi}\left(1 - \frac{2}{3}\, m^2/\Lambda_\gamma^2\right) \qquad \Lambda_\gamma \gg m \ . \tag{2}$$

Similarly a regulated muon propagator in the large angle pair production experiment

$$\frac{1}{p^2 - m^2} \rightarrow \frac{1}{p^2 - m^2} - \frac{1}{p^2 - m^2 - \Lambda_\mu^2} \tag{3}$$

gives a correction proportional to

$$\frac{p^2}{\Lambda_\mu^2} \approx -\frac{k^2\theta^2}{\Lambda_\mu^2}$$

where k is the incident photon energy and θ the angle of a muon emerging with energy $\sim k/2$.

It is an old remark that neither of these regulator modifications (1) and (3) are in a form that is attractive from a purely theoretical viewpoint. Their charm is solely that of simplicity. The regulated photon propagator is designed to improve convergence at $q^2 \to \infty$ when carrying out the renormalization program. However, it takes a change in sign of the second term to $\frac{1}{q^2} + \frac{1}{q^2 - \Lambda^2}$ to yield a simple and special model not in conflict with the general spectral form

$$\frac{1}{q^2} + \int_0^\infty \frac{\rho(\sigma^2)\ d\sigma^2}{q^2 - \sigma^2} \quad ; \quad \rho(\sigma^2) \geq 0 \tag{4}$$

based on local, relativistic field theory. The only effect of this sign change is to change the sign of the correction term in (2) from minus to plus.

The regulation (3) of the muon propagator wreaks greater havoc with the theory, however, because we are now dealing with a charge bearing line. In order to preserve a differential current conservation law the alteration (3) must be accompanied by a change in the vertex function as dictated by the Ward-Takahashi identity.[6]

The purpose of this note is to discuss the matter of modifying the muon (fermion) propagator in a more systematic way and to compute in a simple model the resulting corrections to the g-2 and μ-pair production calculations. There is no contribution to the muon-proton scattering since no virtual muon line appears in Fig. 1. We then make a comparison with experiment in order to emphasize the main point motivating this note: The g-2 and μ-pair measurements are complementary and independent probes of the theory of quantum electrodynamics and a precise measurement of one of these numbers in no way pre-empts the urgency of measuring the other to the highest accuracy attainable.

GENERAL FORMULAS

We wish to consider only the effects on these processes which arise from a modification of the fermion propagator. From invariance considerations this propagator may in general be written in the form (up to irrelevant gauge terms)

$$S_F'(p) = \frac{B_1 \not{p} + B_2 m}{p^2 - m^2} \tag{5}$$

where the B_i are arbitrary functions of p^2. The fermion electromagnetic vertex with all three particles off the mass shell may similarly be written in terms of 12 arbitrary functions[7]

$$\Gamma_\mu(p', p, q) = \sum_{j,k=0,1} (m)^{-(j+k)} (\not{p}' - m)^j \left[C_1^{jk} \gamma_\mu + C_2^{jk} \sigma_{\mu\nu} q^\nu + C_3^{jk} q_\mu \right] (\not{p} - m)^k \tag{6}$$

where the C_i^{jk} depend on the scalars p'^2, p^2 and $q^2 = (p' - p)^2$, and by charge conjugation invariance,

$$C^{jk}(p'^2, p^2, q^2) = C^{kj}(p^2, p'^2, q^2) \tag{7}$$

It is our purpose to generalize the vertex only as much as is necessary to satisfy the Ward identity

$$S_F'(p')\ q^\mu\ \Gamma_\mu(p',\ p,\ q)\ S_F'(p) = S_F'(p) - S_F'(p') \tag{8}$$

where $S_F'(p)$ is given in (5) and Γ_μ is the complete Dyson irreducible vertex. Also because we are only interested in deviations associated with internal fermion lines we do not consider any q^2 dependence of the form factors. We can accomplish these aims by keeping only the terms

$$\Gamma_\mu(p',\ p) = \sum_{j,k=0,1} (m)^{-(j+k)} (\not{p}' - m)^j \left[C^{jk}(p'^2,\ p^2)\ \gamma_\mu \right] (\not{p} - m)^k \tag{9}$$

which introduce 4 arbitrary functions.

If one of the fermions is on the mass shell there are only two functions and the vertex may be written in the form

$$\bar{u}(p')\ \Gamma_\mu(p',\ p) = \bar{u}(p')\ \gamma_\mu \left[C_1(p^2)\ \frac{\not{p} - m}{m} + C_2(p^2) \right] \tag{10}$$

Application of Ward's identity now determines the C_i in terms of the B_i. Furthermore the relationship is such that a cancellation between the vertex and the propagator occurs in this model; specifically[8]

$$\bar{u}(p')\ \Gamma_\mu(p',\ p)\ S_F'(p) = \bar{u}(p')\ \gamma_\mu \frac{1}{\not{p} - m} \tag{11}$$

At a vertex with both fermions off the mass shell the Ward identity plus the symmetry requirement (7) determine the C^{jk} in terms of the B_i and hence in terms of the C_i

$$C^{00}(p'^2, p^2) = \frac{1}{p'^2 - p^2}\left[(p'^2 - m^2)\, C_2(p'^2) - (p^2 - m^2)\, C_2(p^2)\right]$$

$$C^{01}(p'^2, p^2) = C^{10}(p'^2, p^2) = \frac{1}{p'^2 - p^2}\left[(p'^2 - m^2)\, C_1(p'^2) - (p^2 - m^2)\, C_1(p^2)\right]$$

$$C^{11}(p'^2, p^2) = - \frac{m^2}{p'^2 - p^2}\left\{2\left[C_1(p'^2) - C_1(p^2)\right] + \left[C_2(p'^2) - C_2(p^2)\right]\right\} \qquad (12)$$

With these alterations we can write down the expressions pertinent to the anomalous moment and pair production. Using the cancellation between propagator and vertex at the outside vertices the first order vertex modification (Fig. 2) is

$$\Lambda_\mu(p', p) = - e^2 \frac{i}{(2\pi)^4} \int d^4k \frac{1}{k^2} \gamma_\nu \frac{1}{(\not{p}' - \not{k}) - m} \Gamma_\mu(p' - k, p - k) \frac{1}{(\not{p} - \not{k}) - m} \gamma^\nu \qquad (13)$$

where Γ_μ is given by (9) and (12).

Similarly the amplitude for pair production in an external Coulomb field, Fig. 3, is of the form

$$\mathcal{M} = \bar{u}(p_-)\left\{\gamma_\mu\left[\frac{1}{m} C_1(q'^2) + \frac{1}{\not{q}' - m} C_2(q'^2)\right]\not{\epsilon} + \not{\epsilon}\left[\frac{1}{m} C_1(q''^2) + \frac{1}{\not{q}'' - m} C_2(q''^2)\right]\gamma_\mu\right\} v(p_+)\, A^\mu_{Coul} \qquad (14)$$

where $q' = k - p_+$, $q'' = p_- - k$.

As they stand, neither amplitude (13) nor (14) is gauge invariant. The situation is analogous to the electrodynamics of spin zero bosons in which there appear two photon as well as single photon vertices as in Fig. 4 which arise from the derivative coupling. The form factors here play the role in momentum space of derivative interactions and we must include an amplitude from diagrams as in Fig. 4. This cannot be constructed uniquely in the absence of a theory but it will serve our present purposes to write an amplitude which when added suitably to (13) and (14) restores gauge invariance. Such a two photon vertex is given by

$$V_{\mu\nu} = \left\{ - 2C_1 \frac{k^\mu q^\nu + k^\nu q^\mu}{k \cdot q} + (\not{p}_2 - m) \, L_{\mu\nu}(k,q,p_2) + R_{\mu\nu}(k,q,p_1)(\not{p}_1 - m) \right\} \quad (15)$$

The first term of (15) when added to Eq. (14) gives a gauge invariant amplitude for symmetric pairs and the last two terms vanish in virtue of the inverse propagators appearing there sandwiched by $\bar{u}(p_2) \; .... \; u(p_1)$. In particular it is now clear that the term proportional to C_1 does not contribute to the Bethe-Heitler cross section for symmetric pairs since in transverse gauge $(\gamma^\mu \not{\epsilon} + \not{\epsilon}\gamma^\mu) \, g_{\mu o} = 2\epsilon_o = 0$.

In the anomalous moment calculation the one particle matrix element of the fermion current is of interest and as found from (13) is of the form

$$\bar{u}(p') \, \Lambda_\mu(p', p) \, u(p) = \bar{u}(p') \left[A\gamma_\mu + iB\sigma_{\mu\nu} + Dq_\mu \right] u(p) \quad (16)$$

where A, B, and D are functions of q^2 resulting from carrying out the integrals over d^4k involving the C_i. The q_μ term is the only one which is not conserved and on general invariance grounds must be removed from (16). When graphs containing the two photon vertex are considered all terms of $V_{\mu\nu}$ in (15) contribute since one of the fermion lines emerging from this vertex is off the mass shell. We simply choose $L_{\mu\nu}$ and $R_{\mu\nu}$ so as to cancel both the contributions of the first term in $V_{\mu\nu}$ and also the q_μ term in (16). The amplitude is now gauge invariant.

By standard methods we can project from $\Lambda_\mu(p', p)$ the coefficient of the $\sigma_{\mu\nu}q^\nu$ term. This is conveniently done in the limit of small momentum transfer to the external field, $q = (p' - p) \rightarrow 0$. The coefficient will in general depend on p and the moment is obtained from its static limit. In particular we find that the Schwinger correction $\alpha/2\pi$ becomes $\alpha/2\pi\, A(0)$ where $A(0)$ is the small p value of

$$A(p) = -\frac{i}{\pi^2}\int d^4k\,\frac{1}{k^2}\left[(p-k)^2-m^2\right]^{-2}\left\{\frac{2}{m}K_2[(p-k)^2](p\cdot k - m^2)\right.$$
$$\left. + K_1[(p-k)^2]\left[4m^2 + \frac{3}{m^2}(k\cdot p)^2 - 6p\cdot k - k^2 + \lim_{q^2\to 0}\frac{(k\cdot q)}{q^2}\right]\right\} \qquad (17)$$

with

$$K_1(\ell^2) = \frac{d}{d\ell^2}\left[(\ell^2 - m^2)\, C_2(\ell^2)\right]$$
$$K_2(\ell^2) = \frac{2}{m^2}\left\{m^2\frac{d}{d\ell^2}\left[(\ell^2 - m^2)\, C_2(\ell^2)\right] + (\ell^2 - m^2)\frac{d}{d\ell^2}\left[(\ell^2 - m^2)\, C_1(\ell^2)\right]\right\}$$

For the photoproduction of symmetric pairs, the amplitude (14) gives simply a multiple of the Bethe-Heitler cross section

$$d\sigma = \left[C_2(q'^2)\right]^2 d\sigma_{BH} \qquad (18)$$

COMPARISON WITH EXPERIMENT

As a first application of these results consider a propagator modification of the type mentioned in the introduction. This is of the form

$$S(p) = f(p^2) \frac{1}{\not p - m}$$

Calculating in the usual way from (3) without correcting the vertices for Ward's identity, this propagator leads to a symmetric pair cross section

$$d\sigma = f^2(p^2)\ d\sigma_{BH}$$

If however the Ward identity (8) is imposed with $B_1 = B_2 = f$ we find $C_1 = 0$ and $C_2 = f^{-1}$ and from (18) obtain

$$d\sigma = f^{-2}(p^2)\ d\sigma_{BH}$$

along with appropriate modifications of the anomalous moment. For the particular function given in the introduction and $d\sigma/d\sigma_{BH} \approx 1$ the effect of this inversion on the estimated value of $1/\Lambda$ is small.

To make further comparisons with experiment it is necessary to choose some particular functions $C_i(p^2)$ and evaluate the integral (17). A suitable pair are

$$C_1 = \frac{m^2}{p^2 - m^2 - m^2\eta_1}, \qquad C_2 = 1 - \frac{p^2 - m^2}{p^2 - m^2 - m^2\eta_2}$$

where η_1 and η_2 are parameters such that the usual theory appears in the limit $\eta_1, \eta_2 \to \infty$.

With these C_i, $A(p)$ is independent of p, and for both η_1, η_2 large gives a Schwinger correction

$$\frac{g-2}{2} = \frac{\alpha}{2\pi}\left[1 + 2\,\frac{\log\,\eta_1}{\eta_1} - \frac{2}{3}\,\frac{\log\,\eta_2}{\eta_2}\right] \tag{19}$$

Because it is the difference of the log terms that enters, comparison with experiment puts no definite bounds on η_1 and η_2. To within the validity of the approximation that η_1 and η_2 are large there is a considerable range of η's for which the moment can be made arbitrarily close to $\frac{\alpha}{2\pi}$. This is a feature, of course, of the particular C_i's we have chosen. Others can be readily contrived so that the moment measurement does bound both terms. But the result (19) illustrates the point here, namely, in these lowest order calculations the fermion propagator can be considerably different from the usual form and still give an anomalous moment in close agreement with experiment.

We may obtain estimates on the regulator masses in C_1 and C_2 by considering the anomalous moment and pair production results together. Since C_1 does not appear in the symmetric pair cross section this may be used to bound η_2. From the recent measurements of de Pagter, et al,[3] we find ($\hbar = c = 1$)

$$\left(\frac{1}{m_\mu^2\eta_2}\right) < (0.16f)^2$$

This limit together with the most accurate g-2 measurements for the muon[2] gives from (19)

$$\left(\frac{1}{m_\mu^2\eta_1}\right) < (0.09f)^2 \ .$$

The usual comparison of experiments by insertion of regulating functions uses only one function. We can make such a comparison by setting $C_1 = 0$. This gives an idea of the effect of the Ward identity on the limit imposed by the g-2 experiment. From (19) this yields

$$\left(\frac{1}{m_\mu^2 \eta_2}\right)_{\eta_1 = \infty} < (0.06f)^2 .$$

This value is somewhat smaller than found in previous estimates.[3]

In summary we note that the g-2 expression depends on both the parameters which our generalization introduces while the symmetric pair cross section depends on only one. Thus a comparison with experiment of either of these results by itself does not completely probe the muon propagator. For the particular functions we chose the limits on the regulator masses are consistent between the two experiments and are roughly in agreement with previous estimates.

Finally we remark that the generalization we have made is computationally simpler than might be expected because of the propagator-vertex cancellation implied by the Ward relation.

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FIGURE CAPTIONS

1. Diagrams for muon and electron scattering from protons.
2. Electromagnetic vertex for g-2 calculation.
3. Diagrams for electromagnetic pair production.
4. Two photon interaction vertex.

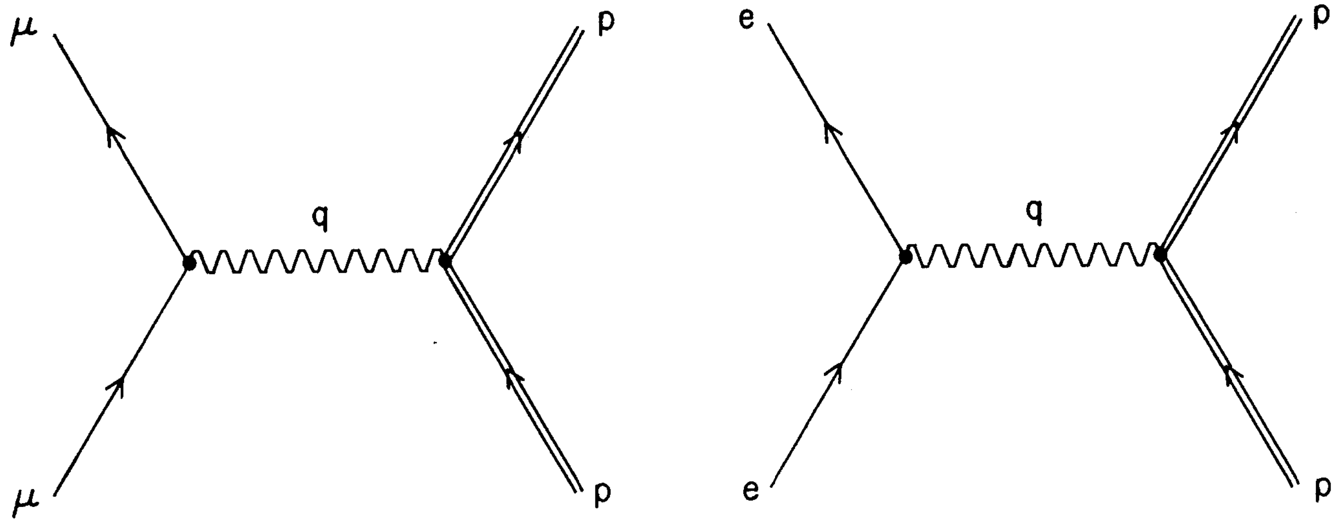

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FIG. 1

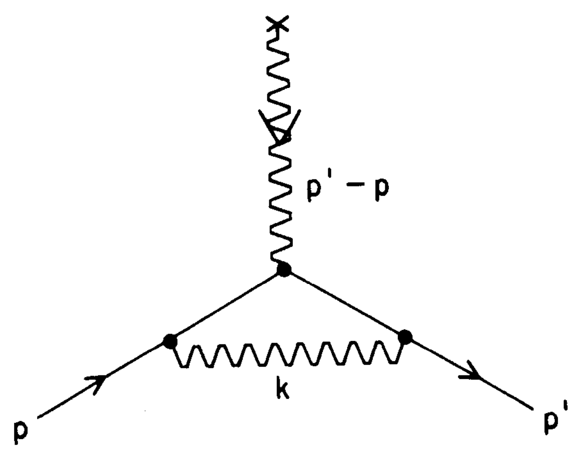

FIG. 2 192-2-A

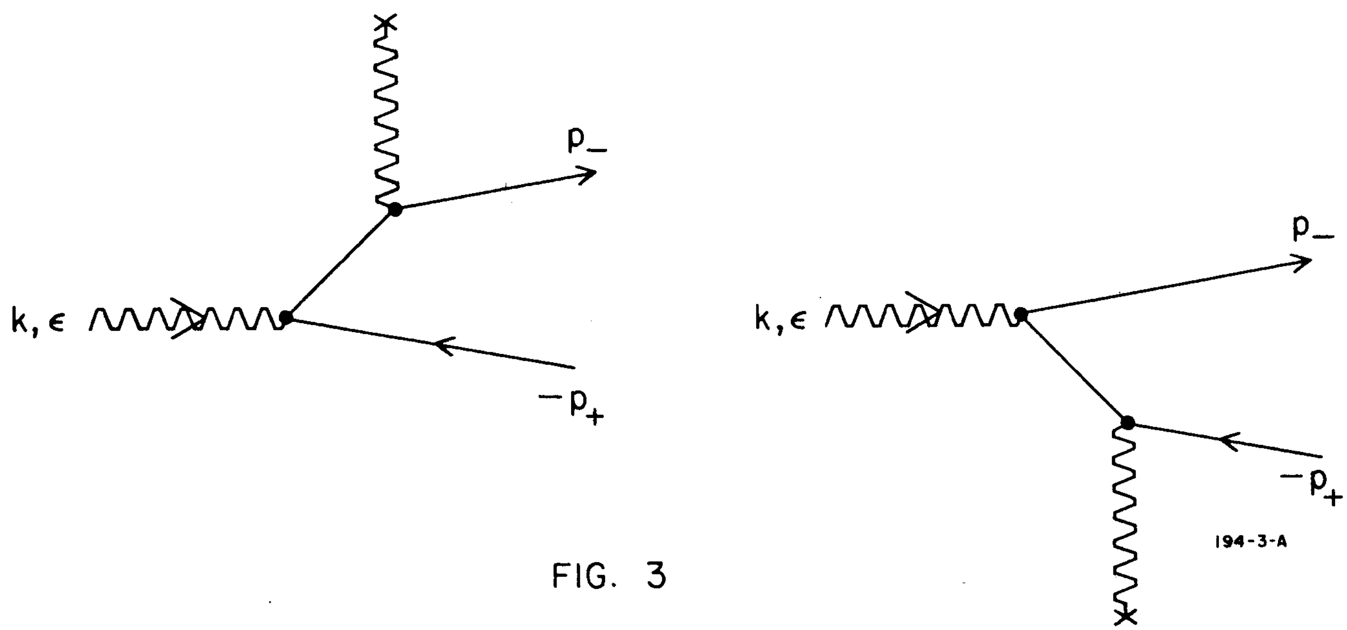

FIG. 3

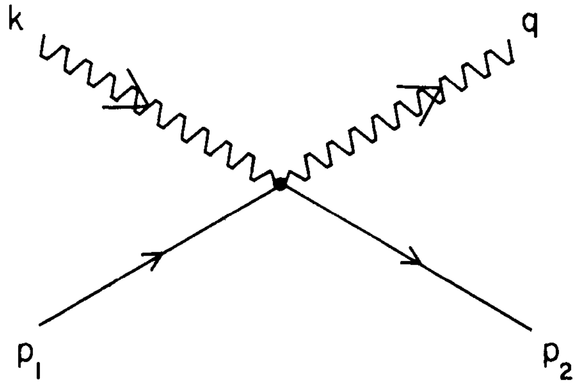

FIG. 4

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