ON THE POSSIBILITY OF RELATING INTERNAL SYMMETRIES AND LORENTZ INVARIANCE[*]

by

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ABSTRACT

In this note we investigate the possibility that
the inhomogeneous Lorentz group is only a subgroup of
a larger Lie group G of symmetries for strong inter-
action physics. The discussion is restricted to the
Lie algebra $\mathcal{L}$ of G. We make the assumption that the
remaining generators A_i of G commute with the gen-
erators of translations P^λ which build the ideal $\mathcal{T}$
in $\mathcal{L}$. It is then shown that the A_i generate an
ideal in $\mathcal{L}$ modulo $\mathcal{T}$. If this ideal is semi-simple
then $\mathcal{L}$ breaks up in a direct sum $\mathcal{L} = \mathcal{P} \oplus \mathcal{A}$
where $\mathcal{P}$ is isomorphic with the Lie algebra of the
inhomogeneous Lorentz group and $\mathcal{A}$ is semi-simple.

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1. INTRODUCTION

It is interesting to discuss the various possibilities of mixing "internal symmetries" in strong interaction physics with Lorentz invariance. However this cannot be done in an arbitrary way as was recently shown by Mac-Glinn.[1] We take here a point of view closely related to Mac-Glinn's, by assuming that the inhomogeneous Lorentz group is only a subgroup of a larger Lie group G. This is not the only possibility as pointed out by Louis Michel[2] who assumes the inhomogeneous Lorentz group to be only a factor group of G. However, we do not discuss here these other ways of inter-relating internal symmetries and Lorentz invariance. In fact we shall restrict our discussion to the Lie algebra of G, call it   . Our main hypothesis will be that it is possible to choose a basis of this algebra in such a way that

(i) The generators of the usual translations P^{λ}, $0 \leq \lambda \leq 3$ and homogeneous Lorentz transformations $M_{\mu\nu} = - M_{\nu\mu}$, $\mu, \nu = 0,1,2,3$ have their usual commutation relations (this being another way of describing the fact that the inhomogeneous Lorentz group is a subgroup of G), and that

(ii) The other generators A_i $1 \leq i \leq n$ commute with P^{λ}. This assures that any irreducible representation of our group G will be characterized by a common mass so that we do not expect the relation of internal symmetries to Lorentz invariance to account for the actual mass splitting within the same super-multiplet. Had we only assumed $\left[A_i, P^0\right] = 0$, then Lorentz co-variance would require it to be true in any frame

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so that $\left[A_i, P^\lambda\right] = 0$ whatever the index λ. Now we will show the following:

Theorem

Under assumptions (i) and (ii), the P^λ generate an ideal $\mathcal{T}$ in $\mathcal{L}$. The images of the A_i under the application $\mathcal{L} \to \mathcal{L}/\mathcal{T} = \mathcal{L}'$ generate an ideal $\mathcal{A}'$ in $\mathcal{L}'$. If we assume that this algebra $\mathcal{A}'$ is semi-simple then $\mathcal{L}$ is equal to a direct sum of Lie algebras $\mathcal{L} = \mathcal{A} \oplus \mathcal{P}$ where $\mathcal{A}$ is isomorphic to $\mathcal{A}'$ and $\mathcal{P}$ is isomorphic to the usual Lie algebra of the inhomogeneous Lorentz group.

This result is in some sense parallel to the one of Mac-Glinn's. The essential difference being here that in order to split $\mathcal{L}$ into this direct sum one may have to change the labels of the operators. Before proving the theorem in section 4 we recall in section 2 some definitions about Lie algebras and devote section 3 to the proof of two lemmas which will clear the way. Section 5 gives some comments on our result.

For convenience we choose all our Lie algebras to be on the real numbers. In this way P^λ and $M_{\mu\nu}$ have the commutation relations of $\frac{\partial}{\partial x_\lambda}$ and $x_\mu \frac{\partial}{\partial x^\nu} - x_\nu \frac{\partial}{\partial x^\nu}$ respectively. The greek indices will always run from 0 to 3; $g_{\mu\nu} = g^{\mu\nu}$ is the usual diagonal real Lorentz metric. We use freely of $g_{\mu\nu}$ to raise or lower indices.[*]

2. DEFINITIONS

To make our language precise, we recall briefly some facts about Lie algebras. For details one can refer for instance to S. Helgasson's

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[*] Whenever two indices, one upper and one lower, appear in the same formula and are equal, they are meant to be summed over; the summing for greek indices to be from 0 to 3, the summing for latin indices to be from 1 to n.

book.[3] By Lie algebra $\mathcal{L}$ over R, the real numbers, we mean a finite

dimensional vector space over R with an internal bilinear product

satisfying

$$[x, x] = 0$$

and the Jacobi identity

$$\Big[x, [y,z]\Big] + \Big[y, [z,x]\Big] + \Big[z, [x,y]\Big] = 0$$

A subalgebra of $\mathcal{L}$ is a vector subspace closed under the product. An

ideal is a true vector subspace $\mathcal{I} \subset \mathcal{L}$ such that for any $l \in \mathcal{L}$ and

$i \in \mathcal{I}$, $[l,i]$ belongs to $\mathcal{I}$. The factor algebra $\mathcal{L}/j$ is the factor

vector space with the inner product inherited from $\mathcal{L}$. We shall say

that a Lie algebra $\mathcal{L}$ is a direct sum of two ideals $\mathcal{L}_1$ and $\mathcal{L}_2$ and

write $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$ if $[l_1, l_2] = 0$, $[l_1', l_1''] = \epsilon$, $\mathcal{L}_1 [l_2', l_2''] \in \mathcal{L}_2$,

$l_1, l_1', l_1'' \in \mathcal{L}_1$, $l_2, l_2', l_2'' \in \mathcal{L}_2$, and if any element l in $\mathcal{L}$ can be

uniquely written as $l = l_1 + l_2$. Mathematicians often call $\mathcal{L}$ the

direct product of $\mathcal{L}_1$ and $\mathcal{L}_2$.

Given $l \in \mathcal{L}$ the linear correspondence $x \to [l,x] = \text{adj}\,(l) \cdot x$

has the property $\text{adj}\,l'\,\text{adj}\,l'' - \text{adj}\,l''\,\text{adj}\,l' = \text{adj}\,[l',l'']$ and defines

the adjoint representation of $\mathcal{L}$: $\text{adj}\,(\mathcal{L})$. If the symmetric bi-

linear from $K(l_1', l'') = \text{Trace adj}\,l'\,\text{adj}\,l''$ is non-degenerate, the Lie

algebra is called semi-simple. In this case $\mathcal{L}$ has no abelian ideal

except zero and its center is also zero. Finally by derivation on a Lie

algebra one means a linear correspondence:

$$x \to Dx$$

with the property

$$D[x,y] = [Dx,y] + [x,Dy]$$

These derivations themselves build up a Lie algebra $\mathcal{D}$ ($\mathcal{L}$), with $[D',D''] = D'\,D'' - D''\,D'$. It is easy to see that the adj ($\mathcal{L}$) is an ideal in $\mathcal{D}$ ($\mathcal{L}$) (in this respect adj ℓ is also called an inner derivation of $\mathcal{L}$) with

$$[D,\ \text{adj}\ \ell] = \text{adj}\ D\,(\ell)$$

Now the main result we shall use is the following.

Theorem

For a semi-simple Lie algebra every derivation is an inner derivation. In other words $\mathcal{D}$ ($\mathcal{L}$) = adj ($\mathcal{L}$). The proof of this theorem is given on page 122 of reference 3.

3.   TWO LEMMAS

In this section we prove the following two lemmas:

Lemma 1

Suppose that we have a Lie algebra over R and that, in terms of a basis P^λ, $N_{\mu\nu}$, the following commutation relations hold:

$$\left[P^\lambda, P^\rho\right] = 0, \quad \left[P^\lambda, N_{\mu\nu}\right] = \delta^\lambda_\mu P_\nu - \delta^\lambda_\nu P_\mu$$

$$\left[N_{\mu_1\nu_1}, N_{\mu_2\nu_2}\right] = g_{\mu_1\nu_2}\,N_{\nu_1\mu_2} + g_{\nu_1\mu_2}\,N_{\mu_1\nu_2} - g_{\mu_1\mu_2}\,N_{\nu_1\nu_2} - g_{\nu_1\nu_2}\,N_{\mu_1\mu_2} \quad (1)$$

$$+ a_{\mu_1\nu_1,\mu_2\nu_2,\lambda}\,P^\lambda$$

$$N_{\mu\nu} = -N_{\nu\mu}$$

and

$$a_{\mu_1\nu_1,\mu_2\nu_2,\lambda} = -a_{\nu_1\mu_1,\mu_2\nu_2,\lambda} = -a_{\mu_1\nu_1,\nu_2\mu_2,\lambda} = -a_{\mu_2\nu_2,\mu_1\nu_1,\lambda} \qquad (2)$$

Then the Lie algebra just described is necessarily isomorphic to the one of the inhomogeneous Lorentz group.

Proof of Lemma 1 [*]

The statement of the lemma can be rephrased in the following way. There exists a set of (real) coefficients $f_{\mu\nu,\lambda}$ such that putting

$$L_{\mu\nu} = N_{\mu\nu} - f_{\mu\nu,\lambda} \, P^\lambda$$

the $L_{\mu\nu}$ and P^λ generate the same Lie algebra as before; however these new operators have among them the usual commutation relations of the generators of homogeneous Lorentz transformations and translations, respectively, that is, this change of basis has the virtue of supressing the "a" coefficients in the commutation relations (1) when they are written in terms of P^λ and $L_{\mu\nu}$.

We have to make sure that our Lie algebra satisfies the Jacobi identities:

$$\sum_{\text{circular permutation of } 1,2,3} \left[N_{\mu_1\nu_1}, \left[N_{\mu_2\nu_2}, N_{\mu_3\nu_3} \right] \right] = 0$$

This gives us a set of linear relations among the coefficients $a_{\mu_1\nu_1,\mu_2\nu_2,\lambda}$:

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[*] In spite of the simplicity of the lemma the author was unable to find a simpler proof than this rather lengthy one.

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$$\sum_{\substack{\text{circular permutation} \\ \text{of } 1,2,3}} \left(g_{\mu_2 \nu_3} \, a_{\mu_1 \nu_1, \nu_2 \mu_3, \lambda} + g_{\nu_2 \mu_3} \, a_{\mu_1 \nu_1, \mu_2 \nu_3, \lambda} - g_{\nu_2 \nu_3} \, a_{\mu_1 \nu_1, \mu_2 \mu_3, \lambda} \right.$$

$$(3)$$

$$\left. - g_{\mu_2 \mu_3} \, a_{\mu_1 \nu_1, \nu_2 \nu_3, \lambda} + g_{\mu_1 \lambda} \, a_{\mu_2 \nu_2, \mu_3 \nu_3, \nu_1} - g_{\nu_1 \lambda} \, a_{\mu_2 \nu_2, \mu_3 \nu_3, \mu_1} \right) = 0$$

Our task now is to find the most general solution of these equations. Let $\alpha, \beta, \gamma, \delta$ stand for $0,1,2,3$ in any order. Because of the symmetry properties of the a's we have only to compute the following quantities.

$$a_{\alpha\beta, \alpha\gamma, \delta}; \quad a_{\alpha\beta, \gamma\delta, \delta}; \quad a_{\alpha\beta, \alpha\gamma, \alpha}; \quad a_{\alpha\beta, \beta\gamma, \gamma}.$$

The value of all $a_{\mu_1 \nu_1, \mu_2 \nu_2, \lambda}$ is then related to the preceding ones using (2). Let us first write (3) for $\mu_1 = \alpha$, $\nu_1 = \gamma$, $\mu_2 = \beta$, $\nu_2 = \delta$, $\mu_3 = \alpha$, $\nu_3 = \beta$ and $\lambda = \alpha$:

$$g_{\alpha\alpha} \left(a_{\beta\delta, \alpha\beta, \gamma} + a_{\alpha\gamma, \beta\delta, \beta} - a_{\beta\delta, \beta\gamma, \alpha} \right) + g_{\beta\beta} \, a_{\alpha\gamma, \delta\alpha, \alpha} = 0 \; . \qquad (4)$$

Now we put in (3) $\mu_1 = \alpha$, $\nu_1 = \beta$, $\mu_2 = \alpha$, $\nu_2 = \gamma$, $\mu_3 = \alpha$, $\nu_3 = \delta$, $\lambda = \beta$

$$\left(g_{\alpha\alpha} \, a_{\alpha\beta, \gamma\delta, \beta} + a_{\alpha\gamma, \delta\beta, \beta} + a_{\alpha\delta, \beta\gamma, \beta} \right) + g_{\beta\beta} \, a_{\alpha\gamma, \alpha\delta, \alpha} = 0 \qquad (5)$$

Let us add (4) and (5)

$$a_{\beta\delta, \alpha\beta, \gamma} + a_{\delta\alpha, \gamma\beta, \beta} = a_{\beta\gamma, \delta\beta, \alpha} + a_{\gamma\delta, \alpha\beta, \beta} \qquad (6)$$

Call

$$\varphi^{\beta}_{\delta\alpha\gamma} = a_{\beta\delta, \alpha\beta,} + a_{\delta\alpha, \gamma\beta, \beta}$$

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Obviously

$$\varphi^{\beta}{}_{\delta\alpha\gamma} = - \varphi^{\beta}{}_{\alpha\delta\gamma} \tag{7'}$$

Equation (6) when translated in terms of φ reads

$$\varphi^{\beta}{}_{\delta\alpha\gamma} = \varphi^{\beta}{}_{\gamma\delta\alpha} \tag{7''}$$

Hence $\varphi^{\alpha}{}_{\beta\gamma\delta}$ is invariant by a circular permutation of $\beta\gamma\delta$ and changes sign under an odd permutation. We will get a new equation for φ if we write (3) with $\mu_1 = \alpha, \nu_1 = \beta, \mu_2 = \alpha, \nu_2 = \gamma, \mu_3 = \alpha, \nu_3 = \delta,$ and $\lambda = \alpha$

$$a_{\alpha\gamma,\alpha\delta,\beta} + a_{\alpha\delta,\alpha\beta,\gamma} + a_{\alpha\beta,\alpha\gamma,\delta} - a_{\alpha\beta,\gamma\delta,\alpha} - a_{\alpha\gamma,\delta\beta,\alpha} - a_{\alpha\delta,\beta\gamma,\alpha} = 0 \tag{8}$$

Using φ just defined we rewrite (8) as

$$\varphi^{\alpha}{}_{\delta\gamma\beta} + \varphi^{\alpha}{}_{\beta\delta\gamma} + \varphi^{\alpha}{}_{\gamma\beta\delta} = 0 \tag{9}$$

Comparing equations $(7)''$ and (9) we conclude that all three terms in equation (9) are equal and thus $\varphi^{\alpha}{}_{\beta\gamma\delta} = 0$ that means

$$a_{\alpha\beta,\gamma\alpha,\delta} + a_{\beta\gamma,\delta\alpha,\alpha} = 0 \tag{10}$$

We now define $f_{\alpha\beta,\gamma}$ through the following equation (δ being known as soon as we have made a definite choice for α β and γ):

$$a_{\alpha\beta,\gamma\delta,\delta} = g_{\delta\delta}\, f_{\alpha\beta,\gamma} \qquad \text{with } f_{\alpha\beta,\gamma} = -f_{\beta\alpha,\gamma} \tag{11}$$

With the help of (10) we find

$$a_{\alpha\beta,\alpha\gamma,\delta} = g_{\alpha\alpha} \, f_{\beta\gamma,\delta} \qquad\qquad (12)$$

Using (4) we also get

$$a_{\alpha\beta,\alpha\gamma,\alpha} = g_{\alpha\alpha} \, (f_{\beta\gamma,\alpha} + f_{\alpha\gamma,\beta} - f_{\alpha\beta,\gamma}) \qquad\qquad (13)$$

So far we have only introduced $f_{\mu\nu,\rho}$ for $\mu \neq \nu \neq \rho$ with the property of being antisymmetric with respect to its first two indices. We were able to express in terms of these quantities our first three sets of unknowns. We turn now to the last set. For that purpose we write again (3) with $\mu_1 = \alpha, \nu_1 = \beta, \mu_2 = \alpha, \nu_2 = \gamma, \mu_3 = \beta, \nu_3 = \gamma,$ and $\lambda = \alpha$

Using also (2) we get

$$a_{\alpha\gamma,\beta\gamma,\beta} = -a_{\alpha\beta,\gamma\beta,\gamma} \qquad\qquad (14)$$

We make a final use of (3) by setting $\mu_1 = \alpha, \nu_1 = \beta, \mu_2 = \gamma, \nu_2 = \delta, \mu_3 = \alpha, \nu_3 = \gamma$ and $\lambda = \beta$.

$$-g_{\beta\beta} \, a_{\gamma\delta,\alpha\gamma,\alpha} + g_{\gamma\gamma} a_{\alpha\beta,\delta\alpha,\beta} - g_{\alpha\alpha} \, a_{\gamma\delta,\gamma\beta,\beta} = 0 \qquad\qquad (15)$$

We want to combine (14) and (15). Let us call $\psi_{\alpha,\gamma\beta} = a_{\alpha\gamma,\beta\gamma,\beta}$

The last two equations read:

$$\psi_{\delta,\alpha\beta} = -\psi_{\delta,\beta\alpha} \qquad\qquad (14')$$

$$g_{\beta\beta} \, \psi_{\delta,\gamma\alpha} + g_{\gamma\gamma} \, \psi_{\delta,\alpha\beta} + g_{\alpha\alpha} \, \psi_{\delta,\beta\gamma} = 0 \qquad\qquad (15')$$

Because of the antisymmetry of $\psi_{\delta,\alpha\beta}$ in its last two indices, as soon as we choose δ we are left with only three quantities linked by the last equation. The most general solution is of the following form

$$\psi_{\delta,\alpha\beta} = g_{\alpha\alpha} h_{\delta,\beta} - g_{\beta\beta} h_{\delta,\alpha}$$

with arbitrary $h_{\delta,\beta}$. Indeed (14) and (15) are now identities. The three $h_{\delta,\alpha}, h_{\delta,\beta}, h_{\delta,\gamma}$ only appear in the expression of the six $\psi_{\delta,\alpha\beta}$ $= -\psi_{\delta,\beta\alpha}$, $\psi_{\delta,\alpha\gamma} = -\psi_{\delta,\gamma\alpha}$, $\psi_{\delta,\beta\gamma} = -\psi_{\delta,\gamma\beta}$ and the conditions for solving back for the h_{δ}, with one degree of arbitrariness, given the $\psi_{\delta,\cdot\cdot}$ are precisely equations (14) and (15).

At last we set

$$f_{\delta\alpha,\alpha} = -f_{\alpha\delta,\alpha} = h_{\delta,\alpha}$$

These new f's are obviously independent of the ones previously defined. Together they build up $f_{\mu\nu,\rho} = f_{\nu\mu,\rho}$ and we now have:

$$a_{\alpha\beta,\beta\gamma,\gamma} = \psi_{\alpha,\gamma\beta} = g_{\gamma\gamma} f_{\alpha\beta,\beta} - g_{\beta\beta} f_{\alpha\gamma,\gamma} \tag{16}$$

Using (2) we can unite equations (11), (12), (13) and (16) in a single expression:

$$a_{\mu_1\nu_1,\mu_2\nu_2,\lambda} = f_{\mu_1\nu_1,\mu_2} g_{\nu_2\lambda} - f_{\mu_1\nu_1,\nu_2} g_{\mu_2\lambda} - f_{\mu_2\nu_2,\mu_1} g_{\nu_1\lambda} + f_{\mu_2\nu_2,\nu_1} g_{\mu_1\lambda}$$

$$\tag{17}$$

$$-g_{\mu_1\nu_2} f_{\nu_1\mu_2,\lambda} - g_{\nu_1\mu_2} f_{\mu_1\nu_2,\lambda} + g_{\mu_1\mu_2} f_{\nu_1\nu_2,\lambda} + g_{\nu_1\nu_2} f_{\mu_1\mu_2,\lambda}$$

Equations (2) and (3) are now identities.

We are now in position to return to our Lie algebra. We set

$$L_{\mu\nu} = N_{\mu\nu} - f_{\mu\nu,\lambda} \; P^{\lambda} \tag{18}$$

with the same f's as the one's which appear in (17).

It is clear that $L_{\mu\nu} = - L_{\mu\nu}$, and that $L_{\mu\nu}$ and P^{λ} generate the same Lie algebra as the $N_{\mu\nu}$ and P^{λ} do. But in terms of these new operators the commutation relations now read (using (17)):

$$\left[P^{\lambda}, P^{\rho} \right] = 0 \quad \left[P^{\lambda}, L_{\mu\nu} \right] = \delta^{\lambda}_{\mu} \; P_{\nu} - \delta^{\lambda}_{\nu} \; P_{\mu}$$

$$\left[L_{\mu_1 \nu_1}, L_{\mu_2 \nu_2} \right] = g_{\mu_1 \nu_2} \; L_{\nu_1 \mu_2} + g_{\nu_1 \mu_2} \; L_{\mu_1 \nu_2} - g_{\mu_1 \mu_2} \; L_{\nu_1 \nu_2} - g_{\nu_1 \nu_2} \; L_{\mu_1 \mu_2}$$

which are the usual commutation relations for the generators of the inhomogeneous Lorentz group. Hence our lemma is proved.

Lemma 2

Let $\mathcal{L}$ be a Lie algebra over R which as vector space is the direct sum of an ideal A and a subalgebra M; if A is a semi-simple Lie algebra then there exists an ideal N in $\mathcal{L}$ isomorphic to M and such that $\mathcal{L}$ is the direct sum of the Lie algebras N and A:

$$\mathcal{L} = N \; \oplus \; A$$

Proof of Lemma 2

Since A is an ideal in $\mathcal{L}$ if we choose a definite m in M, [m,a]

belongs to A and the correspondence

$$a \rightarrow [m,a]$$

defined for every a in A is linear. Because of Jacobi identity it is
even a derivation on A:

$$\Big[b\ [m,a] \Big] + \Big[m\ [a,b] \Big] + \Big[a\ [b,m] \Big] = 0$$

can be read

$$\Big[m\ [a,b] \Big] = \Big[[m,a],\ b \Big] + \Big[a,\ [m,b] \Big] \tag{1}$$

We now combine this fact, our hypothesis of the semi-simplicity of A
with the theorem quoted at the end of section 2 to conclude that the
derivation given by m is in fact an inner derivation of A; that means
there exists an element $\varphi(m)$ belonging to A such that for any a ϵ A.

$$[m,a] = [\varphi(m),a] \tag{2}$$

This element is unique for the equality $[\varphi_1(m),a] = [\varphi_2(m),a]$ which
if valid for any a ϵ A means that $\varphi_1(m) - \varphi_2(m)$ belongs to the center
of A which for a semi-simple Lie algebra is reduced to zero. We now
have a correspondence

$$\varphi : M \rightarrow A \quad m \rightarrow \varphi(m)$$

We prove that φ is an homomorphism. Because of the unicity of $\varphi(m)$, it is

clear that φ is linear.

We use Jacobi identity and the fact that M is a subalgebra of L.

$$\Big[\ [m_1,m_2]\quad,a\Big] = \Big[\ \varphi(\ [m_1 m_2]\ ),\ a\Big] = \Big[\ m_1\ [m_2,a]\ \Big] - \Big[\ m_2\ [m_1,a]\ \Big]$$

$$= \Big[\varphi(m_1)\ [\varphi(m_2),a]\ \Big] - \Big[\varphi(m_2),\ [\varphi(m_1),a]\ \Big]$$

$$= \Big[[\varphi(m_1),\ \varphi(m_2)]\quad,a\ \Big]$$

The unicity of φ gives

$$\varphi(\ [m_1,m_2]\ ) = [\varphi(m_1),\ \varphi(m_2)] \qquad\qquad (3)$$

This proves our statement that φ is indeed an homomorphism. Let us further note that

$$\Big[m_1,\ \varphi(m_2)\Big] = \Big[\varphi(m_1),\ \varphi(m_2)\Big] \qquad\qquad (4)$$

We define now the subset N of $\mathcal{L}$. It is the image of the following application:

$$\psi:\quad M \to \mathcal{L} \qquad m \to \psi(m) = m - \varphi(m)$$

ψ is linear and moreover $\psi(\ [m_1,m_2]\quad) = \Big[\psi(m_1),\ \psi(m_2)\Big]$

which turns N into a subalgebra of $\mathcal{L}$ homomorphic to M. This last equality is a direct consequence of (3) and (4). It is also clear that $\psi(m) = 0$ implies m = 0. Indeed $\psi(m) = 0$ means m = $\varphi(m)$ but m $\in$ M and $\varphi(m) \in$ A means that   m = $\varphi(m) = 0$ in virtue of our hypothesis that as a vector

space, $\mathcal{L}$ is the direct sum of M and A. Hence N is isomorphic to M.

Further, any n in N is an image under ψ of an element in M: n = ψ(m) so

that [n,a] = [m-φ(m),a] = 0 because of (2).

It remains to show that any ℓ belonging to $\mathcal{L}$ can be written in a

unique way as ℓ = n + a with n $\in$ N and a $\in$ A.

By hypothesis we know that ℓ = m + a = m - φ(m) + a = n + a and this

n is obviously unique. This concludes the proof of lemma 2.

4. PROOF OF THE THEOREM

We turn to the proof of the theorem. The assumptions made imply

that the commutation relations between our operators have the following form

$$\left[P^\lambda, P^\mu \right] = 0, \quad \left[P^\lambda, M_{\mu\nu} \right] = \delta^\lambda{}_\mu P_\nu - \delta^\lambda{}_\nu P_\mu$$

$$\left[M_{\mu_1\nu_1}, M_{\mu_2\nu_2} \right] = g_{\mu_1\nu_2} M_{\nu_1\mu_2} + g_{\nu_1 u_2} M_{\mu_1\nu_2} - g_{\mu_1\mu_2} M_{\nu_1\nu_2} - g_{\nu_1\nu_2} M_{\mu_1\mu_2}$$

$$\left[A_i, P^\lambda \right] = 0 \tag{1}$$

$$\left[A_i, M_{\mu\nu} \right] = a_{i,\mu\nu}{}^{\rho\sigma} M_{\rho\sigma} + b_{i,\mu\nu}{}^\ell A_\ell + C_{i,\mu\nu,\lambda} P^\lambda$$

$$\left[A_i, A_j \right] = d_{ij}{}^{,\rho\sigma} M_{\rho\sigma} + e_{ij}{}^\ell A_\ell + f_{ij,\lambda} P^\lambda$$

with obvious symmetry properties for the coefficients a, ..., f. First let

us prove that $a_{i,\mu\nu}{}^{\rho\sigma}$ = 0 and $d_{ij}{}^{\rho\sigma}$ = 0. We use the Jacobi identity

$$\left[P^\lambda, \left[A_i, A_j \right] \right] + \left[A_i, \left[A_j, P^\lambda \right] \right] + \left[A_j, \left[P^\lambda, A_i \right] \right] = 0$$

Because of (1) it reads

$$d_{ij}{}^{\rho\sigma} \left[P^\lambda, M_{\rho\sigma} \right] = 0$$

hence

$$d_{ij}^{\rho\sigma} = 0$$

Again

$$\left[A_i, \left[M_{\mu\nu}, P^\lambda\right]\right] + \left[M_{\mu\nu}, \left[P^\lambda, A_i\right]\right] + \left[P^\lambda, \left[A_i, M_{\mu\nu}\right]\right] = 0$$

$$a_{i,\mu\nu}^{\rho\sigma} \left[P^\lambda, M_{\rho\sigma}\right] = 0$$

So also $a_{i,\mu\nu}^{\rho\sigma} = 0$

It is clear by inspection of (1) that the P^λ generate an ideal $\mathcal{T}$ in $\mathcal{L}$

and so we can go to the factor Lie algebra $\mathcal{L}' = \mathcal{L}/\mathcal{T}$. We denote by A_i'

and $M_{\mu\nu}'$ the images of A_i and $M_{\mu\nu}$ under the mapping $\mathcal{L} \to \mathcal{L}/\mathcal{T} = \mathcal{L}'$.

Using the fact that the a's and d's are zero we get for the commutation

relations

$$\left[M_{\mu_1\nu_1}', M_{\mu_2\nu_2}'\right] = g_{\mu_1\nu_2} M_{\nu_1\mu_2}' + g_{\nu_1\mu_2} M_{\mu_1\nu_2}' - g_{\mu_1\mu_2} M_{\nu_1\nu_2}' - g_{\nu_1\nu_2} M_{\mu_1\mu_2}'$$

$$\left[A_i', M_{\mu\nu}'\right] = b_{i,\mu\nu}^\ell A_\ell' \tag{2}$$

$$\left[A_i', A_j'\right] = e_{ij}^\ell A_\ell'$$

Eq. (2) expresses the fact that the A_i' generate an ideal $\mathcal{A}' \subset \mathcal{L}$. We

thus have proved the first two statements of the theorem.

Not only do the A_i' build up the ideal $\mathcal{A}'$, but also the $M_{\mu\nu}'$ generate a

subalgebra $\mathcal{M}'$ and, as a vector space, $\mathcal{L}'$ is the direct sum of $\mathcal{M}'$ and $\mathcal{A}'$.

If we suppose that the algebra $\mathcal{A}'$ is semi-simple we can make use of lemma 2

which asserts that one can find real coefficients $k_{\mu\nu}^i = - k_{\nu\mu}^i$ such that

defining:

$N'_{\mu\nu} = M'_{\mu\nu} - k^i_{\mu\nu} A'_i$, the $N'_{\mu\nu}$ also generate an ideal in $\mathcal{L}'$ and the commutation relations in $\mathcal{L}'$ are the same as in (2) with $M'_{\mu\nu}$ replaced by $N'_{\mu\nu}$ and the coefficients $b^\ell_{i,\mu\nu}$ put equal to zero.

This means, going back to the initial Lie algebra, that by replacing the $M_{\mu\nu}$ by $N_{\mu\nu} = M_{\mu\nu} - k^i_{\mu\nu} A_i$ we can manage to make the b coefficients disappear. Since $\left[A_i, P^\lambda\right] = 0$, $\left[N_{\mu\nu}, P^\lambda\right] = \left[M_{\mu\nu}, P^\lambda\right]$; on the other hand

$$\left[N_{\mu_1\nu_1}, N_{\mu_2\nu_2}\right] = \left[M_{\mu_1\nu_1}, M_{\mu_2\nu_2}\right] \text{ modulo } \mathcal{J} \,.$$

This is the same statement as

$$\left[N'_{\mu_1\nu_1}, N'_{\mu_2\nu_2}\right] = \left[M'_{\mu_1\nu_1}, M'_{\mu_2\nu_2}\right] \text{ in } \mathcal{L}' \,.$$

In other words, the Lie algebra $\mathcal{L}$ is also generated by P^λ, $N_{\mu\nu}$, A_i with the commutation relations now reading

$$\left[P^\lambda, P^\rho\right] = 0$$

$$\left[P^\rho, N_{\mu\nu}\right] = \delta^\rho_{\ \mu} P_\nu - \delta^\rho_{\ \nu} P_\mu$$

$$\left[N_{\mu_1\nu_1}, N_{\mu_2\nu_2}\right] = g_{\mu_1\nu_2} N_{\nu_1\mu_2} + g_{\nu_1\mu_2} N_{\mu_1\nu_2} - g_{\mu_1\mu_2} N_{\nu_1\nu_2} - g_{\nu_1\nu_2} N_{\mu_1\mu_2}$$

$$+ \, m_{\mu_1\nu_1,\mu_2\nu_2,\rho} P^\rho \tag{3}$$

$$\left[A_i, N_{\mu\nu}\right] = c_{i,\mu\nu,\lambda} P^\lambda$$

$$\left[A_i, A_j\right] = e^\ell_{ij} A_\ell + f_{ij,\lambda} P^\lambda$$

We now observe that P^λ and $N_{\mu\nu}$ generate a subalgebra in $\mathcal{L}$ and the commutation relations are just the ones discussed in lemma 1 where it was proved that there exist linear combinations

$$L_{\mu\nu} = N_{\mu\nu} - \ell_{\mu\nu,\lambda} \, P^\lambda = M_{\mu\nu} - k^i_{\mu\nu} A_i - \ell_{\mu\nu,\lambda} \, P^\lambda$$

such that P^λ and $L_{\mu\nu}$ have the same commutation relations as the previous P^λ and $M_{\mu\nu}$ had. Again P^λ commutes with A_i so that

$$\left[A_i, L_{\mu\nu}\right] = \left[A_i, N_{\mu\nu}\right] \quad \text{and also} \quad \left[P^\lambda, L_{\mu\nu}\right] = \left[P^\lambda, N_{\mu\nu}\right]$$

Hence our algebra $\mathcal{L}$ is still generated by P^λ, $L_{\mu\nu}$ and A_i but we now have the commutation rules:

$$P^\lambda P^\rho = 0$$

$$\left[P^\rho, L_{\mu\nu}\right] = \delta^\rho_\mu P_\nu - \delta^\rho_\nu P_\mu$$

$$\left[L_{\mu_1\nu_1}, L_{\mu_2\nu_2}\right] = g_{\mu_1\nu_2} L_{\nu_1\mu_2} + g_{\nu_1\mu_2} L_{\mu_1\nu_2} - g_{\mu_1\mu_2} L_{\nu_1\nu_2} - g_{\nu_1\nu_2} L_{\mu_1\mu_2} \qquad (4)$$

$$\left[A_i, L_{\mu\nu}\right] = c_{i,\mu\nu,\lambda} \, P^\lambda$$

$$\left[A_i, A_j\right] = e^\ell_{ij} A_\ell + f_{ij,\lambda} \, P^\lambda$$

We have not yet made full use of the Jacobi identities in $\mathcal{L}$. Indeed we have to make sure that

$$\left[\left[A_i, L_{\mu\nu}\right], L_{\rho\sigma}\right] + \left[\left[L_{\mu\nu}, L_{\rho\sigma}\right], A_i\right] + \left[\left[L_{\rho\sigma}, A_i\right], L_{\mu\nu}\right] = 0$$

Using (4) we get

$$c_{i,\mu\nu,\rho}\, g_{\sigma\lambda} - c_{i,\mu\nu,\sigma}\, g_{\rho\lambda} - c_{i,\rho\sigma,\mu}\, g_{\nu\lambda} + c_{i,\rho\sigma,\nu}\, g_{\mu\lambda}$$
$$= c_{i,\nu\rho,\lambda}\, g_{\mu\sigma} + c_{i,\mu\sigma,\lambda}\, g_{\nu\rho} - c_{i,\mu\rho,\lambda}\, g_{\nu\sigma} - c_{i,\nu\sigma,\lambda}\, g_{\mu\rho} \qquad (5)$$

We now try to find the most general solution of (5). Let us put in (5) $\mu = \rho \neq \nu \neq \sigma$ and $\lambda = \nu$. The result is

$$g_{\nu\nu}\, c_{i,\mu\sigma,\mu} = g_{\mu\mu}\, c_{i,\nu\sigma,\nu}$$

Hence

$$c_{i,\mu\sigma,\mu} = -g_{\mu\mu} p_{i,\sigma}$$

where the $p_{i,\sigma}$ are some real numbers.

Again let us write (5) with $\mu = \rho \neq \nu \neq \sigma \neq \lambda$ which is possible since we have four values at our disposal. The result is

$$c_{i,\nu\sigma,\lambda} = 0 \quad \text{if } \lambda \text{ is different from } \nu \text{ and } \sigma.$$ Combining this result with the previous one we get

$$c_{i,\mu\nu,\sigma} = g_{\nu\sigma} p_{i,\mu} - g_{\mu\sigma} p_{i,\nu} \tag{6}$$

In writing (6) we have also made use of the fact that $c_{i,\mu\nu,\sigma} = -c_{i,\nu\mu,\sigma}$.

Using (6) one can now check that (5) is an identity. We set $B_i = A_i - p_{i,\mu} P^\mu$

Now $\left[B_i, L_{\mu\nu} \right] = 0$ and since $\left[P^\lambda, A_i \right] = 0$ and $\left[P^\lambda, P^\rho \right] = 0$ we also have

$$\left[B_i, B_j \right] = \left[A_i, A_j \right] = e_{ij}{}^\ell A_\ell + f_{ij,\lambda} P^\lambda$$

$$= e_{ij}{}^\ell B_\ell + \bar{f}_{ij,\lambda} P^\lambda$$

with $\bar{f}_{ij,\lambda} = f_{ij,\lambda} + e_{ij}{}^\ell p_{\ell,\lambda}$

The last Jacobi identity, namely

$$\left[\left[B_i B_j\right], L_{\mu\nu}\right] + \left[\left[B_j, L_{\mu\nu}\right] B_i\right] + \left[\left[L_{\mu\nu}, B_i\right], B_j\right] = 0$$

tells us

$$f_{ij,\lambda}\left[P^\lambda, L_{\mu\nu}\right] = 0$$

that is $\qquad f_{ij,\lambda} = 0$

Our proof is now complete since using the assumptions of the theorem
we were able to show that starting with a set of P^λ, $M_{\mu\nu}$ and A_i
satisfying (1) there exists a set of numbers $k_{\mu\nu}{}^i$, $\ell_{\mu\nu,\lambda}$, $p_{i,\mu}$ such that
setting

$$L_{\mu\nu} = M_{\mu\nu} - k_{\mu\nu}{}^i A_i - \ell_{\mu\nu,\lambda} P^\lambda$$

$$\text{(7)}$$

$$B_i = A_i - p_{i,\mu} P^\mu$$

the P^λ $L_{\mu\nu}$, B_i generate the same Lie algebra as before but their commutation
relations now read

$$\left[P^\lambda, P^\rho\right] = 0 \qquad \left[P^\lambda, L_{\mu\nu}\right] = \delta^\lambda{}_\mu P_\nu - \delta^\lambda{}_\nu P_\mu$$

$$\left[L_{\mu_1\nu_1}, L_{\mu_2\nu_2}\right] = g_{\mu_1\nu_2} L_{\nu_1\mu_2} + g_{\nu_1\mu_2} L_{\mu_1\nu_2} - g_{\mu_1\mu_2} L_{\nu_1\nu_2} - g_{\nu_1\nu_2} L_{\mu_1\mu_2}$$

$$\left[B_i, P^\lambda\right] = \left[B_i, L_{\mu\nu}\right] = 0 \qquad\qquad\qquad \text{(8)}$$

$$\left[B_i, B_j\right] = e_{ij}{}^\ell B_\ell$$

Hence we have succeeded in splitting our Lie algebra into a direct sum of two algebras, one generated by the P^λ and $L_{\mu\nu}$, call it $\mathscr{P}$, isomorphic to the one of the generators of the inhomogeneous Lorentz group, the second generated by the B_i, call it $\mathscr{A}$ with $\mathscr{A}$ semi-simple. $\mathscr{A}$ is clearly isomorphic with $\mathscr{A}'$ introduced earlier. The theorem is proved.

5. REMARKS

The crucial point in section 4 was in assuming that $\mathscr{A}'$ is semi-simple. This of course has not much to do with any kind of physical assumption. However, as our discussion shows, if we keep the hypothesis that A_i , $P^\lambda = 0$, the only way to get a final answer different from the one given in the theorem is to find a Lie algebra $\mathscr{A}$ such that it should be impossible to interpret the derivations $A' \to M_{\mu\nu}, A_i'$ as inner derivations. A typical case would be the following. Call $\mathscr{D}$ the Lie algebra of derivations on $\mathscr{A}'$ and $\mathscr{D}'$ the Lie algebra of inner derivations on $\mathscr{A}'$. If there exists a nontrivial homomorphism of the Lie algebra $\mathscr{M}$ of the homogeneous Lorentz group in $\mathscr{D}/\mathscr{D}'$, then the last conclusion of the theorem would be false. By non-trivial homomorphism, we mean any homomorphism except the one which sends $\mathscr{M}$ on $0 \in \mathscr{D}/\mathscr{D}'$. In our previous considerations we made the assumption of semi-simplicity in order to assure $\mathscr{D}/\mathscr{D}' = 0$ which forced the homomorphism $\mathscr{M} \to \mathscr{D}/\mathscr{D}' = 0$ to be the trivial one. We see that the theorem extends to the case where $\mathscr{A}$ has zero center and $\mathscr{M} \to \mathscr{D}/\mathscr{D}'$ has to be the trivial homomorphism.

In conclusion one can point out the parallelism of this result with Mac-Glinn's theorem which assumed that $[A_i, M_{\mu\nu}] = 0$. The outcome was then that the A_i generate a sub-algebra $\mathscr{A}$ of $\mathscr{L}$; if this subalgebra

was supposed to be semi-simple (or more generally to have no abelian factor algebra[4]) then $A_i, P^\lambda = 0$ and $\mathcal{L}$ splits into $\mathcal{A} \oplus \mathcal{P}$.

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