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DOUBLE POLES AND NON-EXPONENTIAL DECAYS*

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ABSTRACT

Models are constructed with double poles in partial wave scattering amplitudes. The associated unstable particles have non-exponential decay laws which contain parameters dependent on the production and detection arrangements.

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INTRODUCTION

Attention has been called recently[1] to the possibility of multiple complex poles in partial wave scattering amplitudes and non-exponential decays of the associated unstable particles. Examples of this situation are presented. The first is a potential well with two regions of trapping; the second is a kind of Lee model with two unstable V particles. In both cases it is shown that double poles can be contrived by suitably adjusting the parameters. The time dependence is considered, for the Lee model in some detail. It is found that the decay law is a continuously variable function of production and detection arrangements. This is in contrast with the conclusion of Ref. 1 that alternatives to the usual exponential law form a discrete set.

FIRST EXAMPLE: DOUBLE POTENTIAL WELL

One would expect that the required degeneracy can be contrived with a potential well that has two possible regions of trapping. To make the working as explicit as possible, we take the potential barriers to be delta functions. The S-wave radial Schrodinger equation is

$$\frac{d^2\varphi}{dr^2} + k^2\varphi = \left\{ \left(\frac{\pi}{\alpha}\right) \delta(r-a) + \left(\frac{\pi}{\beta}\right) \delta(r-a-b) \right\} \varphi \qquad (1)$$

In the various regions the wave function, normalized to unit slope at the origin, is as follows:

$$
\left.
\begin{aligned}
&\text{for } o < r < a \\
&\qquad \varphi = k^{-1}\sin kr \\
&\text{for } a < r < a+b \\
&\qquad \varphi = k^{-1}\sin kr + k^{-1}\sin ka\ \pi\alpha^{-1}k^{-1}\sin k(r-a) \\
&\text{for } b < r \\
&\qquad \varphi = k^{-1}\sin kr + k^{-1}\sin ka\ \pi\alpha^{-1}k^{-1}\sin k(r-a) + (k^{-1}\sin k(a+b) \\
&\qquad\quad + k^{-1}\sin ka\ \pi\alpha^{-1}k^{-1}\sin kb)\ \pi\beta^{-1}k^{-1}\sin k(r-a-b)
\end{aligned}
\right\} \tag{2}
$$

The last expression can be written as a combination of an outgoing wave $\exp(ikr)$ and an incoming wave $\exp(-ikr)$. The coefficient of the latter is $-f(-k)/2ik$ with

$$
\left.
\begin{aligned}
f(-k) &= \pi^2\,\alpha^{-1}\,\beta^{-1}\,k^{-2}\,e^{ik(a+b)}\,F(k) \\
F(k) &= \sin ka \sin kb + \pi^{-1}\alpha k \sin k(a+b) \\
&\qquad + \pi^{-1}\beta k \sin ka\ e^{-ikb} + \pi^{-2}k^2\alpha\beta\ e^{-ik(a+b)}
\end{aligned}
\right\} \tag{3}
$$

The zeros of f, or F, give poles in the S-matrix. For a double pole we require that with some k

$$
F = \partial F/\partial k = 0 .
$$

It will be shown that this is possible for small β, with α and $\delta = (k-\pi)$ of order β^2, $\epsilon = \pi(b-1)$ of order β, and $a = 1$. From Eq. (3)

$$
\begin{aligned}
F &= \delta(\delta+\epsilon) + \alpha(\epsilon+2\delta) + \beta\delta(1-i\epsilon) + \alpha\beta(1-i\epsilon) + O(\beta^5) \\
&= \left(\delta + \alpha + \tfrac{1}{2}\beta + \tfrac{1}{2}\epsilon - \tfrac{1}{2}i\epsilon\beta\right)^2 - \left(\tfrac{1}{2}\epsilon + \tfrac{1}{2}\beta\right)^2 + \tfrac{1}{4}\epsilon^2\beta^2 - \alpha^2 \\
&\qquad + i\epsilon\beta\left(\tfrac{1}{2}\epsilon + \tfrac{1}{2}\beta\right) + O(\beta^5)
\end{aligned}
$$

The vanishing of F' is given by

$$\delta = -\left(\alpha + \frac{1}{2}\beta + \frac{1}{2}\epsilon - \frac{1}{2}i\epsilon\beta\right) + O(\beta^3)$$

and then the vanishing of imaginary and real parts of F by

$$\frac{1}{2}\epsilon + \frac{1}{2}\beta = O(\beta^3)$$

$$\alpha = \frac{1}{2}\beta^2 + O(\beta^3)$$

So we can indeed arrange for a double pole.

SECOND EXAMPLE: MODIFIED LEE MODEL

The second example is particularly simple for the study of time dependence. It is a Lee-type model[2] in which there are two unstable V particles, a stable N particle, and a stable non-relativistic meson θ, with allowed transitions

$$\begin{aligned} V_1 &\rightleftharpoons V_2 \\ V_2 &\rightleftharpoons N+\theta \end{aligned} \tag{4}$$

Let ψ_1 and ψ_2 be the probability amplitudes for V_1 and V_2, and $\psi(\underset{\sim}{x})$ the probability amplitude for a meson at $\underset{\sim}{x}$ and the N particle at the origin. The Schrodinger equation is

$$\frac{d\psi_1}{dt} = -i(M_1\psi_1 + \alpha\psi_2) \tag{5}$$

$$\frac{d\psi_2}{dt} = -i\left(M_2^o\psi_2 + \alpha\psi_1 + g\psi(o)\right) \tag{6}$$

$$\frac{\partial \psi(\underset{\sim}{x})}{\partial t} = - i \left(-\nabla^2 \psi(x) + g\delta(\underset{\sim}{x})\psi_2 \right) \tag{7}$$

where α and g are coupling constants for the above two processes. It is formally convenient to suppose that ψ_1 and ψ_2 are zero at large negative times; so to permit consideration of any desired initial conditions we add source terms $f_1(t)$ and $f_2(t)$ to the right-hand sides of Eqs. (5) and (6). Then taking Fourier transforms

$$\psi_1(t) = \int \frac{d\omega}{2\pi} \; e^{-i\omega t} \; \tilde{\psi}_1(\omega),$$

etc., we have

$$\omega\tilde{\psi}_1 = M_1 \tilde{\psi}_1 + \alpha\tilde{\psi}_2 + i\tilde{f}_1$$

$$\omega\tilde{\psi}_2 = M_2^o \tilde{\psi}_2 + \alpha\tilde{\psi}_1 + i\tilde{f}_2 + g\tilde{\psi}(o)$$

$$\omega\tilde{\psi}(\underset{\sim}{x}) = - \nabla^2 \tilde{\psi}(\underset{\sim}{x}) + g\delta(\underset{\sim}{x}) \tilde{\psi}_2$$

The retarded solution of the last equation is

$$\tilde{\psi}(\underset{\sim}{x}) = - \frac{g}{4\pi} \frac{e^{ik|\underset{\sim}{x}|}}{|\underset{\sim}{x}|} \tilde{\psi}_2 + \tilde{\varphi}(\underset{\sim}{x}) \tag{8}$$

where φ is an incident meson wave, and $k = \sqrt{\omega}$ (the positive root for positive ω). Eliminating $\tilde{\psi}(o)$ from the other equations then

$$(\omega - M_1)\, \tilde{\psi}_1 - \alpha\tilde{\psi}_2 = i\tilde{f}_1$$

$$\left(\omega - M_2 + ikg^2/4\pi\right) \tilde{\psi}_2 - \alpha\tilde{\psi}_1 = i\tilde{f}_2 + g\tilde{\varphi}(o)$$

where a real (divergent) self-energy has been absorbed into M_2. This yields

$$\left.\begin{aligned}
\tilde{\psi}_1 &= \left[\left(\omega - M_2 + ikg^2/4\pi\right) i\tilde{f}_1 + \alpha\left(i\tilde{f}_2 + g\tilde{\varphi}(o)\right)\right] /\Delta(\omega) \\
\tilde{\psi}_2 &= \left[\left(\omega - M_1\right)\left(i\tilde{f}_2 + g\tilde{\varphi}(o)\right) + \alpha\, i\tilde{f}_1\right] /\Delta(\omega) \\
\Delta(\omega) &= \left(\omega - M_1\right)\left(\omega - M_2 + ikg^2/4\pi\right) - \alpha^2
\end{aligned}\right\} \qquad (9)$$

To obtain a double pole $\Delta(\omega)$ and its derivative with respect to ω, $\Delta'(\omega)$, must vanish together. From Eq. (9)

$$\Delta'(\omega) = \left(\omega - M_2 + ikg^2/4\pi\right) + \left(\omega - M_1\right)\left(1 + \frac{ig^2}{4\pi}\frac{dk}{d\omega}\right)$$

This vanishes at $\omega = \omega_r$, with

$$\left(\omega_r - M_1\right) = \left(1 + \frac{ig^2}{8\pi}\frac{dk}{d\omega}\right)^{-1}\left(\frac{M_2 - M_1}{2} - \frac{ikg^2}{8\pi}\right)$$

where k and $dk/d\omega$ are evaluated at $\omega = \omega_r$. The requirement that $\Delta(\omega)$ vanishes at the same time gives

$$\left(1 + \frac{ig^2}{8\pi}\frac{dk}{d\omega}\right)^{-2}\left(1 + \frac{ig^2}{4\pi}\frac{dk}{d\omega}\right)\left(\frac{M_2 - M_1}{2g^2} - \frac{ik}{8\pi}\right)^2 + \frac{\alpha^2}{g^4} = 0$$

The imaginary part of this can be made to vanish by choice of $M_2 - M_1$, and then the real part by adjustment of α^2. For small g^2

$$\left.\begin{aligned} (M_2 - M_1)/g^2 &= 0 \\ \alpha^2 &= (kr\ g^2/8\pi)^2 \\ \omega_r &= M_1 - \frac{1}{2} i\Gamma \\ \Gamma &= k_r g^2/4\pi = 2\alpha \end{aligned}\right\} \qquad (10)$$

and in the neighborhood of the pole, ignoring an overall factor of order g^2 different from unity,

$$\left[\Delta(\omega)\right]^{-1} = (\omega - \omega_r)^{-2} \qquad (11)$$

Inverting the Fourier transform, and retaining only the pole contribution as in the study[3] of almost exponential decays,

$$\begin{aligned} \psi_1(t) &= \left[\frac{d}{d\omega} e^{-i\omega t} \left\{\left(\omega - \omega_r + \frac{1}{2} i\Gamma\right) \tilde{f}_1 + \frac{1}{2}\Gamma\left(\tilde{f}_2 - ig\tilde{\varphi}(o)\right)\right\}\right]_{\omega_r} \\ \psi_2(t) &= \left[\frac{d}{d\omega} e^{-i\omega t} \left\{\left(\omega - \omega_r - \frac{1}{2} i\Gamma\right)\left(\tilde{f}_2 - ig\tilde{\varphi}(o)\right) + \frac{1}{2}\Gamma \tilde{f}_1\right\}\right]_{\omega_r} \end{aligned} \qquad (12)$$

These are correct up to terms of order g^2; kg^2 has been replaced by $k_r g^2$, because differentiation of k would only give a term in g^2 not multiplied by t. Any desired initial values of ψ_1 and ψ_2 may be obtained by taking for $f_1(t)$ and $f_2(t)$ suitable multiples of $\delta(t)$. Then $\tilde{f}_1$ and $\tilde{f}_2$ in Eq. (12) are constants. It is clear that there is a continuous variety of "decay laws" depending on these initial conditions and on whether ψ_1 or ψ_2 or some combination thereof is observed. Consider some special cases:

a) Suppose the system is started in the V_1 state, i.e., $\tilde{f}_2(o) = \tilde{\varphi}(o) = 0$, $\tilde{f}_1 = 1$. Then from Eq. (10)

$$\left|\psi_1(t)\right| = (1 + \frac{1}{2}\,\Gamma t)\; e^{-\frac{1}{2}\Gamma t} \tag{13}$$

$$\left|\psi_2(t)\right| = \frac{1}{2}\,\Gamma t\; e^{-\frac{1}{2}\Gamma t} \tag{14}$$

The first of these is the decay law of Goldberger and Watson.[1] Note that $\psi(x)$, the probability amplitude for the decay products, is closely related by Eq. (8) to ψ_2. It is in fact proportional to it if

$$\Gamma\,\frac{dk}{dE}\,\left|\underset{\sim}{x}\right| \ll 1$$

i.e., if retardation effects are negligible. So for close-in observation of the decay products Eq. (14) rather than the Goldberger-Watson (Eq. (13)) is appropriate.

b) Suppose the system to be started in the state V_2. Then $\tilde{f}_2 = 1$, $\tilde{f}_1 = 0 = \tilde{\varphi}(o)$. One finds

$$\left|\psi_1(t)\right| = \frac{1}{2}\,\Gamma t\; e^{-\frac{1}{2}\Gamma t} \tag{15}$$

$$\left|\psi_2(t)\right| = (1 - \frac{1}{2}\,\Gamma t)\; e^{-\frac{1}{2}\Gamma t} \tag{16}$$

c) If ψ_1 and ψ_2 are initially equal but in quadrature, $\tilde{f}_1 = i\tilde{f}_2 = 1$, $\tilde{\varphi}(o) = 0$, the decay is exponential:

$$\left|\psi_1(t)\right| = \left|\psi_2(t)\right| = e^{-\frac{1}{2}\Gamma t}$$

d) Suppose the resonance is excited by the incidence of a wave packet in the decay channel. Then $\tilde{f}_1 = \tilde{f}_2 = 0$, but $\tilde{\varphi}(o) \neq 0$. In working out Eq. (12) in this case one meets the derivative

$$\left[\frac{d}{d\omega} \tilde{\varphi}(o) \right]_{\omega=\omega_r} \tag{17}$$

This could be made large by an inappropriate choice of time origin, for a shift of origin by τ introduces into $\tilde{\varphi}$ a factor $e^{i\omega\tau}$. However a sensible choice of origin makes (17) zero or small. Then case d) reduces to b) with $\tilde{\varphi}(o)$ for $\omega = \omega_r$ replacing $\tilde{f}_2$. Thus the decay laws for excitation from the decay channel are Eqs. (15) and (16), and in particular Eq. (16) is appropriate if the observations also are made in the decay channel close to the source.

EXCITATION FROM THE DECAY CHANNEL

The conclusion that Eq. (16) is the decay law for observation in the decay channel and excitation from the decay channel is of course more general than the above model. We can write in general for the S-wave

$$\psi(r) = \int_0^\infty d\omega \frac{dk}{d\omega} c(k) \psi_k^+(r) e^{-i\omega t} \tag{18}$$

where the scattering states have the form, outside the interaction region,

$$r\psi_k^+(r) = e^{-ikr} - S(\omega) e^{ikr} \tag{19}$$

The quantity $c(k)$ is essentially the Fourier transform of the incoming wave packet[3] and can be supposed to vary smoothly near the assumed double pole of the S-matrix. Near the singularity, because of unitarity, S can be represented by

$$S = \left(\frac{\omega - \omega_r^*}{\omega - \omega_r}\right)^2 F(\omega) \tag{20}$$

where F is a smooth function, unimodular for real ω. The pole contribution to Eq. (18), ignoring small terms of relative order Γ not multiplied by t or r, is proportional to

$$\left[\frac{d}{d\omega}\left(\omega-\omega_r^*\right)^2 e^{ikr-i\omega t}\right]_{\omega=\omega_r}$$

whence

$$\left|\psi(r)r\right| \propto \left(1 + \frac{1}{2}\Gamma\left(v^{-1}r - t\right)\right) \times e^{-\frac{1}{2}\Gamma(t-rv^{-1})} \tag{21}$$

where $v^{-1} = dk/d\omega$. This reduces to Eq. (16) for small r.

THE GOLDBERGER-WATSON APPROACH

The approach of Goldberger and Watson, as applied to S-wave potential scattering, is the following. The wave function is written as a superposition of outgoing wave scattering states

$$\psi(r,t) = \int_0^\infty \left(\frac{dk}{2\pi}\right) a(k)\, e^{-i\omega t}\, \psi_k^+(r) \tag{22}$$

where ω is the energy corresponding to wave number k. The coefficients are determined by the initial wave function ψ_a:

$$a(k) = \left(\psi_k^+, \psi_a\right) = \int_0^\infty dr\ r^2 \left[\psi_k^+(r)\right]^* \psi_a(r) \tag{23}$$

The projection of $\psi(r,t)$ on a state ψ_b is

$$A(t) = (\psi_b, \psi) = \int_0^\infty \left(\frac{dk}{2\pi}\right) a(k)\ b^*(k)\ e^{-i\omega t} \tag{24}$$

where

$$\left(\psi_k^+, \psi_b\right) = b(k)$$

Actually the above authors take $\psi_a = \psi_b$; so we make a slight generalization. They introduce a real solution φ of the radial Schrodinger equation normalized to unit slope at the origin:

$$\lim r \to o \left\{ r^{-1}\ \varphi(k,r) \right\} = 1$$

and outside the interaction

$$\varphi(k,r) = \frac{1}{2ik} \left\{ f(k)\ e^{ikr} - f(-k)\ e^{-ikr} \right\}$$

Thus

$$r\psi^+(k,r) = i\varphi(k,r)\ \ 2k/f(-k)$$

and

$$a(k) = \alpha(k)/\left(f(-k)\right)^*,\ b(k) = \beta(k)/\left(f(-k)\right)^* \tag{25}$$

$$\left.\begin{aligned} \alpha(k) &= \left(\varphi_k/r,\ \psi_a\right) \quad (+2ik) \\ \beta(k) &= \left(\varphi_k/r,\ \psi_b\right) \quad (+2ik) \end{aligned}\right\} \tag{26}$$

The S-matrix is $f(k)/f(-k)$, so that we are concerned with double zeros of $f(-k)$. In the neighborhood of such a zero

$$f(-k) = (\omega-\omega_r)^2\, N$$

where N is a smooth function. The contribution of the double pole to the integral (24) is

$$A(t) = -i\left[\frac{d}{d\omega}\frac{dk}{d\omega}\,(NN^*)^{-1}\ \frac{\alpha(k)\,\beta^*(k)}{(\omega-\omega_r{}^*)^2}\,e^{-i\omega t}\right]_{\omega=\omega_r} \tag{27}$$

In evaluating this, terms of relative order Γ not multiplied by t are dropped. Then

$$A(t) \propto \frac{d}{d\omega}\ \frac{\alpha(k)\,\beta^*(k)}{(\omega-\omega_r{}^*)^2}\,e^{-i\omega t} \tag{28}$$

Goldberger and Watson at this point ignore the derivative of $\alpha\beta^*$, on the ground that $\varphi(k,r)$ is an integral function of k, and that the states ψ_a and ψ_b are well localized, so that singularities of α and β given by Eq. (26) are remote. Then

$$A(t) \propto \left(-it - \frac{2}{\omega-\omega_r{}^*}\right) e^{-i\omega_r t}$$

$$\propto \left(-it + \frac{2}{i\Gamma}\right) e^{-i\omega_r t}$$

Normalizing at $t = o$, they have the unique decay law

$$\left|A(t)\right| = \left(1 + \frac{1}{2}\,\Gamma t\right) e^{-\frac{1}{2}\Gamma t} \tag{29}$$

Now in fact one cannot in general ignore the variation with ω of $\alpha\beta^*$ in Eq. (25). Although $\varphi(k,r)$ is for given r an integral function of k, it is one which, in the neighborhood of the resonance, varies increasingly violently with k as Γ is decreased. Consider for example a resonant state confined by a large potential barrier, as in example 1 above. If the radial wave equation is integrated outwards from a prescribed slope at the origin it will in general be very large in and beyond the barrier. Only near the resonant energy is a given wave inside compatible with a small one outside. There is, however, one case in which $\alpha\beta^*$ may be regarded as slowly varying; that is when the wave packets ψ_a and ψ_b are confined to the inner part of the potential well where φ is normalized. Thus the decay law Eq. (29), of Goldberger and Watson corresponds to creating the particle near the center of the well and observing the probability that it remains there. It is no surprise therefore that it agrees with Eq. (11), obtained under analogous circumstances, since V_1 is the "innermost state" of the model.

Consider now, in this approach, the excitation of the resonance by a wavepacket incident in the decay channel. Then $a(k)$ is essentially the Fourier transform of the incident wavepacket and has nothing to do with the singularities of the S-matrix. To cancel the vanishing $f(-k)$ in Eq. (25) we must therefore take

$$\alpha(k) \propto (\omega-\omega_r^{\,*})^2$$

If we observe at a point r_b outside the interaction region

$$\psi_b = \delta(r-r_b) \ ,$$

so that

$$\left(\beta(k)\right)^* \propto f(k)\ e^{ikr_b} - f(-k)\ e^{-ikr_b} \ .$$

We are then led via Eq. (25) to

$$A(t) \propto \frac{d}{d\omega} \ (f(k)) \ \ e^{ikr_b - i\omega t}$$

Since on the real axis

$$f(k) = \left[f(-k)\right]^*$$

we have $f(k) \propto (\omega-\omega_r^*)^2$, and then Eq. (21) as before.

CONCLUSION

We have found that double poles can indeed occur in simple models. However they give rise to a continuously variable rather than to a unique decay law.[5] Indeed the situation is quite analogous to that arising in the classical theory of small vibrations of non-conservative systems.[6] There it can happen that two (or more) of the characteristic frequencies are degenerate and associated with a single normal mode. Another independent solution is required, and is obtained by a limiting process of subtracting one from another the almost identical normal modes before the degeneracy becomes complete with the variation of a parameter; the resulting "abnormal" mode may have as well as the exponential factor a factor linear in time - i.e., a non-exponential decay. However, the non-exponential decay law is not unique. The double root is associated with an arbitrary combination of normal and "abnormal" modes, determined by initial conditions. In fact it is associated with two degrees of freedom, and not just one, of the system.

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