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The Interaction Effect in n-p Capture[†]

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[†]Work performed under the auspices of the U. S. Atomic Energy Commission

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Abstract: The non-relativistic prediction of the *n-p* capture cross section at a neutron laboratory velocity of 2200 m/sec is evaluated as 302.5 ± 4.0 mb assuming that $\underline{r}_s^{np}$ = 2.73 ± 0.03 fm as predicted by charge independence, or 308.5 ± 5.0 mb if $\underline{r}_s^{np}$ = 2.44 ± 0.11 fm. The smaller value for $\underline{r}_s^{np}$, derived from the two most recent *n-p* total cross section measurements, raises serious difficulties with the hypothesis of charge independence, which are discussed. The prediction includes corrections for the shape dependence of the triplet effective range, the deuteron D-state probability, and the intermediate range behaviour of the wave functions, which contribute half the quoted uncertainty; these corrections are model independent to within that uncertainty if the longest range part of the interaction is due to one pion exchange. The experimental value of 334.2 ± 0.5 mb obtained by Cox, Wynchank, and Collie therefore shows an interaction effect of 9.5 ± 1.2% (or 7.7 ±1.5%). A recently proposed approximation based on dispersion theory which would, if accepted, explain two-thirds of the effect, is shown to be quantitatively inadequate. Hence covariant calculations which reduce to this approximation must be rejected, except for those parts of the calculation which give meson current corrections. G. Stranahan predicts

a 2.9% correction coming from the π-N 33 resonance, while the calculation of M. H. Skolnick, interpreted as indicated, gives a non-resonant contribution of 2.2%. Even if these effects could be added, a substantial part of the meson current contribution remains unexplained. It is suggested that at least part of this residual effect might be due to the large π-γ-ρ coupling recently invoked by Adler and Drell to account for the static magnetic moment of the deuteron and the large forward photopion production cross section in the multi-GeV region.

1. Introduction

Although the n-p capture cross section near threshold can be mainly accounted for by an approximate evaluation of the magnetic dipole capture matrix element, using only the static magnetic moments of neutron and proton and the n-p effective range expansion parameters[1)], there is a residual disagreement with experiment of about 10%. As has been emphasized by Austern[2)], this effect provides a significant measure of the distortion of the electromagnetic properties of the free neutron and proton by the nuclear interaction, which he calls the "interaction effect." Using data available up to 1959, Austern and Rost[3)] found an interaction effect of 28 ± 12 mb, or 8.4 ± 3.6% of the experimental cross section of 331.5 ± 1.7 mb. The high precision of the new measurement by Cox, Wynchank, and Collie[4)], who report a value of 334.2 ± 0.5 mb, calls for a review of relevant experimental and theoretical developments which affect the comparison calculation. A critical re-examination of these developments is presented in this paper.

The parameter which has the largest uncertainty in the calculation of Austern and Rost[3] is the n-p $^1S_0$ effective range, for which they adopt the value $r_s^{np} = 2.68 \pm 0.30$ fm obtained from p-p scattering and the hypothesis of charge independence. At first sight this uncertainty can be reduced by adopting the value $r_s^{np} = 2.51 \pm 0.11$ fm obtained in a recent analysis[5] of low energy n-p experiments. But this value raises difficulties with the hypothesis of charge independence[6]. These are discussed in detail in sec. 2, making use of the singlet model given by Signell, Yoder and Heller[7], and a new analysis of the low energy n-p data is presented. In sec. 3 we make use of these results to evaluate the non-relativistic prediction of the matrix element both in the shape-independent approximation and with the shape-dependent corrections. Detailed examination of the latter shows that they are reasonably well-determined by empirically established restrictions on the two-nucleon interaction. In sec. 4 we examine an alternative approach to the problem via non-relativistic dispersion theory, and covariant calculations which reduce to the same result for the problem at hand. We also briefly review calculations of the interaction effect in terms of meson currents. The conclusions reached are summarized in sec. 5.

2. The $^1S_0$ effective range and charge independence

The hypothesis of charge independence can be given verifiable content only to the extent deviations from the exact symmetry law are calculable. These effects fall into two classes: (a) direct electromagnetic effects, which include the electrostatic interaction between two protons, the finite size of the charge and magnetic moment distributions of the neutron and proton, and vacuum polarization, and (b) charge-dependent differences in the strong interactions which may be indirectly due to electromagnetic effects, but which also could conceivably arise from other causes. The most important of the direct effects is obviously the electrostatic potential e^2/r. Schwinger[8] showed that a point magnetic dipole interaction has a significant effect on the singlet scattering length if the nuclear interaction is due to a simple Yukawa potential, but this effect is considerably reduced if the nuclear interaction has a hard core, as shown by Salpeter[9], or if the point magnetic dipole interaction is replaced by the actual extended magnetic moment distribution[9,10]. Because of the latter effect, this correction is quantitatively much less significant than effects of class (b). Unfortunately only a few of the latter are calculable. One known effect which will result in charge dependence of the strong interactions[11] is the $\pi^{\pm}$-π^{0} mass difference. Even if the "bare" coupling constant is charge independent, this splitting can be expected to result in a splitting of the "renormalized" coupling constants. Estimates based on field theory show[12,13] that this effect will be quantitatively more important than the (finite radius) magnetic moment interactions. From the point of view of dispersion theory, this effect can be calculated, so far as the one-pion-exchange part of the nuclear force is concerned, by using the physical pion masses, and the four physical coupling constants $G_{\pm p}, G_{\pm n}, G_{0p}, G_{0n}$. Unfortunately, if this splitting is found to be significant, it can also be expected to have effects on the strong interaction at shorter range which are at present beyond the reach of reliable calculation. Our procedure will therefore be to investigate what freedom these uncertainties allow in models constrained to fit the empirical nucleon-nucleon scattering lengths, and then investigate the effect of these uncertainties on the prediction of the n-p $^1S_0$ effective range.

Our problem can be considerably simplified if we separate the question of charge symmetry (i.e. the equality of n-n and p-p nuclear forces) from that of full charge independence (equality of these two forces with the corresponding states in the n-p system). We note that charge symmetry is preserved (in a pion theory) even in the presence of $\pi^{\pm}$-π^{0} mass splitting, if we require in addition $G^2_{\pm p} = G^2_{\pm n}$ and $G^2_{0p} = G^2_{0n}$. As is well known, there is impressive evidence for exact charge symmetry from the coulomb energies of light mirror nuclei, but two recent measurements[14,15] of the final state interaction in the reaction $\pi^- + d \rightarrow 2n + \gamma$ which determine a_s^{nn} to be about -17 fm, provide useful direct evidence. This value is not in agreement with the Born approximation analysis of the breakup reaction $n + d \rightarrow 2n + p$, which gives[16,17] values of -22 to -24 fm, but this could easily be the result of distortion by three-body final state effects. Since Bander[18] has shown that the theoretical uncertainty in the analysis of the π^- capture reaction is only ±1 fm, we accept -17 fm as a reliable determination of the value of a_s^{nn}.

Signell, Yoder, and Heller[7] have recently studied this question using an "updated Hamada-Johnston[19] model" for p-p singlet scattering of the form

$$V(r) = \frac{e^2}{r} - G^2\left(\frac{\mu_0}{2M}\right)^2 \frac{e^{-\mu_0 r}}{r} + \sum_{n=2}^{6} A_n \left(\frac{e^{-\mu r}}{\mu r}\right)^n, \quad r > 0.48\,\text{fm} \qquad (1)$$

$$= +\infty, \quad r < 0.48\,\text{fm}$$

$$\mu_0 = 135\ \text{MeV}/c^2, \quad \mu = 137.35\ \text{MeV}/c^2, \quad G^2 = 14.4$$

This model is fitted to ${}^1\underline{S}_0$, ${}^1\underline{D}_2$, and ${}^1\underline{G}_4$ $\underline{p}$-$\underline{p}$ phase shifts up to 320 MeV and to two $\underline{p}$-$\underline{p}$ phase shifts below 3 MeV by adjusting the parameters $A_2 - A_6$ to the values given in table 1. If we take $\underline{e}^2 = 0$, $\underline{M} = \underline{M}_n = 939.550$ MeV/c^2, we find $\underline{a}_s^{nn} = -16.96$ fm, $\underline{r}_s^{nn} = 2.8455$ fm, in agreement with ref. 7 and earlier results summarized there. We therefore interpret the measured value of $\underline{a}_s^{nn}$ as strong evidence for exact charge symmetry, and in particular will assume that $\underline{G}^2_{\pm p} = \underline{G}^2_{\pm n}$, and $\underline{G}^2_{0p} = \underline{G}^2_{0n}$ in what follows.

Since this result is in disagreement with the prediction of $a_s^{nn} \approx -28$ fm given by Wong and Noyes[20], we must investigate what went wrong with that calculation. The assumption made there was that if the p-p amplitude $\exp(i\delta_{pp}) \sin \delta_{pp}/\mathcal{C}^2 q$ has the discontinuity $(2K/\mu-1)^{-Me^2/K} D(K^2)$ for $q^2 < -\mu^2/4$ (with $q = iK$), then the n-n amplitude $\exp(i\delta_{nn}) \sin \delta_{nn}/q$ has the discontinuity $D(K^2)$. The specific model presented used the OPE cut for $D(K^2)$ plus a single pole whose position and residue were adjusted to fit two low energy p-p phase shifts. As has been noted by Heller[†], in

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[†] private communication

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changing variables to carry out the numerical work, the sign of the exponent $-Me^2/K$ was inadvertently taken as positive. It was also assumed that the same factor modified the residue of the pole at $q = i\beta$. Since this pole represents the interaction due to physical processes distinct from OPE, it would be more consistent to assume that this pole is an approximation to a cut starting at $q = i\beta/\sqrt{2}$, and hence carries the factor $(\sqrt{2} - 1)^{-Me^2/\beta}$. Since it has been shown[6] that the effect of the OPE cut can be accurately reproduced by a second pole of residue $\frac{1}{2}f^2M$ at $q = i\mu/\sqrt{2}$, we have recomputed the prediction for this two pole model using the correct prescription for the coulomb modification of both residues, and find an n-n scattering length of about -20 fm, in reasonable agreement with experiment. In order to verify the accuracy of this explanation, we have fitted the Bargmann potential[21)] (which gives a single pole in the n-n case) plus the e^2/r coulomb term using the Schroedinger equation, and when the coulomb term is removed find $a_s^{nn} = -17.84$ fm, $r_s^{nn} = 2.78$ fm. The corresponding one pole Wong-Noyes formula (with the above interpretation of the coulomb modification of the residue) predicts $a_s^{nn} = -19.76$ fm, $r_s^{nn} = 2.815$ fm. Since the OPE effect is a small perturbation if the pole position and residue are readjusted to the same data, we conclude that the Wong-Noyes approximation, if correctly applied, is good to about 2 fm for the scattering length and 0.035 fm for the effective range.

Since the p-p scattering length and effective range were fitted empirically, the only effect this correction has on the analysis of the p-p experiments below 3 Mev previously presented[22)] is to increase the predicted residue for the OPE pole by 13%. We have reanalyzed the data using the corrected formula and find as anticipated that the scattering length and effective range change by less than the quoted uncertainty, while the shape parameter predicted for $G^2 = 14$ becomes $P = +0.030$ (instead of 0.023) as compared with the empirical evaluation of $+0.026 \pm 0.010$. Hence the conclusion that the OPE effect has been observed in the $^1S_0$ state remains valid. This is important in what follows, since this analysis is the only direct evidence for the OPE interaction in the $^1S_0$ state. Of course, the indirect evidence is in any case pretty overwhelming. In particular, the same analysis gives (incomplete) evidence for OPE in the central part of the ^3P interaction, when due account is taken of well-established features of the shorter range parts of this interaction, and the quantitative agreement of the high partial waves with OPE is firmly established†.

† For a summary of this evidence and references to the literature, see ref. 23.

If we accept the evidence for charge symmetry given above, the OPE interaction in the n-p $^1S_0$ state may be written as

(2)

$$V_{OPE}^{np}(r) = -2G_{\pm p}^2\left(\frac{\mu_\pm}{2M_{np}}\right)^2\frac{e^{-\mu_\pm r}}{r} + G_{op}^2\left(\frac{\mu_0}{2M_{np}}\right)^2\frac{e^{-\mu_0 r}}{r} \qquad (2)$$

Here $M_{np} = 2M_nM_p/(M_n+M_p)$, and we assume that[24)] $M_p = 938.256$ MeV/c^2, $\mu_\pm = 139.60$ MeV/c^2, $\mu_0 = 135.01$ MeV/c^2. If we keep $G^2 = 14.4$ (and of course $e^2 = 0$), but leave the inner parts of the potential the same, this model predicts $a_s^{np} = -19.07$ fm, $r_s^{np} = 2.762$ fm. If we had not put in the pion mass splitting, the prediction would be essentially the same as that given above for the n-n system, so comparison with the empirical values given below shows that the mass splitting moves the prediction in the right direction, but leaves a significant charge-dependent effect to be explained. Concentrating for the moment on the measured scattering length of -23.68 fm, we first ask if this can be accounted for by allowing a splitting between the charged and neutral coupling constants. The best value for $G_{\pm p}^2$

undoubtedly comes from pion-nucleon scattering, for which Hamilton and Woolcock[25)] quote $\underline{f}^2 = 0.081 \pm 0.002$, corresponding to $\underline{G}^2_{\pm p} = 14.636$. (We could carry through this discussion of the splitting in terms of $\underline{f}^2$ rather than $\underline{G}^2$, but the use of $\underline{G}^2$ strikes us as slightly more natural.) Unfortunately, the value of $\underline{G}^2_{0p}$ is not nearly as well known; a recent analysis of higher partial waves in p-p scattering[26)] yields values in the range from 11.8 to 13.5 with uncertainties of 2 or greater. We therefore fix the charged coupling constant at the value given by Hamilton and Woolcock, and adjust $\underline{G}^2_{0p}$ to fit $\underline{a}_s^{np} = -23.68$ fm. This is achieved for $\underline{G}^2_{0p} = 13.936$, for which value $\underline{r}_s^{np}$ falls to 2.741 fm.

We must now ask if this small splitting of the coupling constants is consistent with other phenomena. If we use $\underline{G}^2 = 14$ in the n-n model given above, we find that $\underline{a}_s^{nn} = -15.68$ fm, $\underline{r}_s^{nn} = 2.853$ fm; however, we have already noted that if we allow the coupling constants to split, we can also expect small charge-dependent effects in the shorter range parts of the interaction, and we find that changing any one of the parameters $\underline{A}_2$ - $\underline{A}_6$ by an MeV or so (in more than 100 MeV) will restore $\underline{a}_s^{nn}$ to -17 fm. Such adjustments might interfere slightly with the fit of the model to the very precise p-p data at low energy. However, we have already seen that the choice of $\underline{G}^2 = 14.4$ for the fit to the p-p experiments at higher energy is not well supported experimentally, and it is obvious that recycling the adjustment would produce an exactly charge-symmetric model fitted to both p-p scattering and the n-n scattering length. At this level of sophistication it would be necessary to include the finite charge and magnetic moment distributions and vacuum polarization, but we feel that such effects are clearly of the same magnitude as charge-dependent corrections to the strong interactions at shorter range, which must be included for consistency once we allow split coupling constants. Hence we feel that this refinement is somewhat pointless until we have some way of calculating the electromagnetic structure of boson systems heavier than the pion which are important in the nuclear force. We conclude that there is good evidence for exact charge symmetry, and that the small failure of charge independence indicated by the n-p scattering length can be equally well accounted for by a small splitting of the pion-nucleon coupling constants (for which there is even some very shaky evidence in the right direction), and/or connected (but currently incalculable) effects of the order of 1% at shorter range. These conclusions are comparable to those reached by Signell, *et al.*[7]. We note also that a measurement of the n-n effective range to ±0.03 fm would provide a valuable test of charge symmetry.

Having shown that once a small coupling constant splitting is allowed, the small discrepancy in the n-p scattering length is "down in the noise" generated by theoretical uncertainties at short range, we turn to the effective range. For this purpose, we can neglect the refinement of using the $\pi^{\pm} - \pi^{0}$ mass splitting, and for convenience collapse the OPE term to a single contribution. As is shown in table 1, the scattering length can still be fitted by very small adjustments of $A_2 - A_6$, and the predicted effective range always lies between 2.72 and 2.73 fm. Taken together with the result given above which used the observed pion masses, we see that the model-dependent uncertainty is less than ±0.02 fm, and since the experimental error in the p-p effective range is only ±0.014 fm, we conclude that regardless of how we account for the observed scattering length, the n-p effective range is predicted by charge independence to be 2.73±0.03 fm. We now compare this prediction with experiment.

As has been discussed in detail in a previous analysis[5], the triplet scattering length and effective range, as well as the singlet scattering length, are almost completely determined by the binding energy of the deuteron, ϵ_B, the coherent neutron-hydrogen scattering length, a_{nH}, and the n-p total cross section for epithermal neutrons, $\sigma_{TOT}(0)$, for which we accept the values[27]

$$\epsilon_B = 2.22452 \text{ MeV}, \quad a_{nH} = \tfrac{1}{2}(a_s+3a_t) = -3.741 \pm 0.011 \text{fm}, \quad \sigma_{TOT}(0) = 2036 \pm 5 \text{fm}^2 \qquad (3)$$

Since higher partial waves give a calculable (and negligibly small) contribution to the total cross section below 5 MeV, r_s can then be directly determined from total cross section measurements in this energy range. It was found in the previous analysis[5] that this determination depended to some extent on the singlet and triplet shape parameters used. However, we have seen above that the p-p singlet shape parameter is correctly predicted from OPE either by dispersion theory or by an appropriate potential model. Since this prediction has been quantitatively verified by experiment[22], we feel fully justified in making use of the same prediction for both singlet and triplet n-p S waves. For the singlet state, we use the approximate formula derived by Cini, Fubini and Stanghellini[28,5], which is quantitatively reliable in the energy range of interest[29]. Since we will require it in the next section we quote the corresponding approximation in the mixed triplet effective range expansion, which is

$$\begin{aligned} k \operatorname{ctn} \delta_{0,1} &= -\frac{1}{a_t} + k^2\left(\frac{1}{\gamma} - \frac{1}{\gamma^2 a_t}\right) - \frac{p k^2 (k^2+\gamma^2)}{(1+q k^2)} \\ &= -\gamma + (k^2+\gamma^2)\left(\frac{1}{\gamma} - \frac{1}{\gamma^2 a_t}\right) - \frac{p k^2(k^2+\gamma^2)}{(1+q k^2)} \end{aligned} \qquad (4)$$

with

$$q = \frac{2 - f^2\left(\frac{M_{np}}{\mu}\right)\left[\frac{3}{2}\sqrt{2} - 2\sqrt{2}\frac{\gamma^2}{\mu^2} - \left(\frac{4}{\mu a_t}\right)\left(1 - \frac{\gamma^2}{\mu^2}\right) - \left(\frac{\mu}{\gamma}\right)\left(1 - \frac{1}{\gamma a_t}\right)\left(1 - \frac{3\gamma^2}{\mu^2}\right)\right]}{\mu^2\left[1 - f^2\left(\frac{M_{np}}{\mu}\right)\left\{\frac{\sqrt{2}}{4} - \frac{1}{\mu a_t} + \left(\frac{\gamma^2}{\mu^3}\right)\left[\frac{\sqrt{2}}{2} - \left(\frac{\mu}{\gamma}\right)\left(1 - \frac{1}{\gamma a_t}\right)\right]\right\}\right]}$$

$$= 3.861 \text{ fm}^2 \qquad (5)$$

$$p = \frac{\left(1 - \frac{1}{2} q \mu^2\right)}{\mu^3\left(1 - \frac{2\gamma^2}{\mu^2}\right)}\left[2\sqrt{2} - \frac{4}{\mu a_t} - 2\left(\frac{\mu}{\gamma}\right)\left(1 - \frac{1}{\gamma a_t}\right)\right] = 0.1147 \text{ fm}^3$$

It may be noted that even though the analysis is slightly dependent on the singlet and triplet shape parameters if these are taken to be independent variables, if both are assumed given by OPE, the two corrections nearly cancel in this energy range, and we get practically the same result with $G^2 = 0$ (that is, by ignoring the shape dependence completely).

The data selection has already been discussed in ref. 5, and we use the same values given there (Table I). The results of the new analysis are reported in table 2a. As can be seen from the χ^2 values, the data are consistent in the sense of a χ^2 test, but only because the older data by themselves are internally more consistent than would be expected if the errors were purely random. We have therefore also performed the analysis subject to the constraint that $r_s = 2.73 \pm 0.03$ fm. The results given in table 2b show that all but two of the measurements are consistent with this hypothesis, but that there is less than a 5% chance that the deviation of the cross sections at 0.4926 and 3.205 MeV as measured by Engelke, Benenson, Melkonian and Lebowitz[30] is due to a statistical fluctuation.

Since we need the results below to test model sensitivity of corrections, we have computed the changes in the coefficients $\underline{A}_2$ - $\underline{A}_6$ taken two at a time needed to fit the scattering length and $\underline{r}_s^{np}$ = 2.68 or 2.4465 fm. The results tabulated in table 1 are not particularly transparent, so we have computed the corresponding changes in the potential energy as a function of distance. Since any simple picture of the two-nucleon interaction ascribes the attractive and repulsive parts to the exchanges of different systems of bosons, we feel it reasonable to refer these changes to the sum of the attractive plus repulsive contributions rather than to the net potential energy. The comparison curve is given in fig. 1, and the changes required to fit n-p low energy scattering in fig. 2. We see that even to fit an effective range of 2.68 fm it is necessary to reduce the attraction by at least 5% in the region between 2 and 4 fm, and to increase the attraction near the core by a corresponding amount. For the shorter effective range indicated by the experiments of Engelke, et al.[30], changes of more than 30% in both regions are required.

Charge-dependent effects of the magnitude found by the above analysis would be very hard to understand on theoretical grounds[6], and to reconcile with other experimental phenomena. In particular, D. H. Wilkinson[31] has recently presented an analysis of isobaric triplets in light nuclei which shows that there are no observable deviations from charge independence in the $^{31}S_0$ state of magnitude greater than 1%. While charge dependent effects in the spin-flip, isospin-flip OPE interactions will be to some extent suppressed in nuclei by the fact that these interactions make no contribution to first-order in nuclear matter, I can think of no mechanism which would cause such a suppression in the shorter range parts of the interaction. We conclude that the experimental results of Engelke et al.[30] are in clear conflict with p-p scattering below 3 MeV interpreted according to the requirements of charge independence, and because of the fundamental importance of this conclusion for both nuclear and elementary particle physics, urgently recommend that new n-p total cross section measurements below 5 MeV of at least comparable precision be attempted.

3. Evaluation of the matrix element by wave function methods

The non-relativistic formula for the n-p magnetic dipole capture cross section may be written (with $\hbar=c=1$) as

$$\sigma = \frac{\pi \alpha (\mu_p - \mu_n)^2 a_s^2 \gamma^6}{v M_p^2 M_{np}^3} N_g^2 \mathcal{M}^2 \tag{6}$$

where v is the laboratory velocity of the neutron (in our case 2200 m/sec), $\gamma^2 = M_{np}\epsilon_B$ and (as above) $M_{np} = 2M_nM_p/(M_n + M_p)$. Note that using the correct combination of nucleon masses in the denominator rather than, say, M_{np}^5, increases the predicted cross section by 0.42 mb, which is almost significant compared with the experimental accuracy quoted by Cox, et al.[4] of ±0.5 mb. Constants not given in the last section are also taken from Rosenfeld, et al.[24] as

$$c = 2.997925 \times 10^8 \text{ m/sec}, \quad \alpha = 0.00729720, \quad \hbar c = 197.322 \text{ MeV} \cdot \text{fm}$$
$$\mu_p = 2.79276, \qquad \mu_n = -1.9128 \tag{7}$$

All authors agree that near threshold electric dipole transitions and transitions to the $^3D_1$ state may be neglected. If the normalized ground state S and D wave functions are called u_g and w_g, and the wave functions which approach $\exp(-\gamma r)$ and $\eta(1 + 3/\gamma r + 3/\gamma^2 r^2) \exp(-\gamma r)$ are called

U_g and W_g, the constant N_g^2 is defined by

$$\int_0^\infty (u_g^2 + w_g^2)\,dr = 1 = N_g^2 \int_0^\infty (U_g^2 + W_g^2)\,dr \tag{8}$$

Hence the percentage D state P_D is given by

$$P_D = \int_0^\infty w_g^2\,dr = N_g^2 \int_0^\infty W_g^2\,dr \tag{9}$$

If the singlet wave function at zero energy which approaches $1 - r/a_s$ asymptotically is called U_s, the matrix element $\mathcal{M}$ appearing in eq. (16) is

$$\mathcal{M} = \int_0^\infty U_g(r)\,U_s(r)\,dr \tag{10}$$

A first approximation to this matrix element can be obtained, following Bethe and Longmire[1)], from the algebraic identity

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}_0 - \tfrac{1}{4}(\rho' + r_s) + C \\
\mathcal{M}_0 &= \int_0^\infty e^{-\gamma r}\left(1 - \frac{r}{a_s}\right)dr = \frac{1}{\gamma} - \frac{1}{\gamma^2 a_s} \\
\rho' &= 2\int_0^\infty \left(e^{-2\gamma r} - U_g^2\right)dr \\
r_s &= 2\int_0^\infty \left[\left(1 - \frac{r}{a_s}\right)^2 - U_s^2\right]dr
\end{aligned} \tag{11}$$

by dropping the shape dependent correction

$$C = \frac{1}{2}\int_0^\infty \left[\left(1 - \frac{r}{a_s} - e^{-\gamma r}\right)^2 - (U_s - U_g)^2\right]dr \tag{12}$$

In this same approximation it is consistent to take

$$\rho' \approx \rho_{s1} = \rho(0, -\epsilon) = \frac{2}{\gamma} - \frac{2}{\gamma^2 a_t} \tag{13}$$

and to assume that the normalization of the ground state wave function is

$$N_g^2 \approx N_{s1}^2 = \frac{2\gamma}{(1-\gamma\rho_{s1})} = \frac{2\gamma^2 a_t}{(2-\gamma a_t)} \tag{14}$$

Hence the shape-independent approximation can be expressed directly in terms of experimental quantities as

$$\begin{aligned} \mathcal{M}_{s1} &= \frac{1}{2\gamma} + \frac{1}{2\gamma a_t} - \frac{1}{\gamma^2 a_s} - \frac{1}{4} r_s \\ \sigma_{s1} &= \frac{\pi\alpha a_s^2 a_t (\mu_p - \mu_n)^2 \gamma^7 \left(1 + \frac{1}{\gamma a_t} - \frac{2}{\gamma a_s} - \frac{1}{2}\gamma r_s\right)^2}{4 v M_p^2 M_{np}^3 (2-\gamma a_t)^2} \end{aligned} \tag{15}$$

Making use of the results of the analysis presented in the last section, including the correlations in error, we obtain the results given in table 3. We note that using the smallest value of the effective range reduces the interaction effect by about 20%, but does not make it disappear, as was erroneously implied in ref. 5. We now examine the shape-dependent corrections to this prediction with an eye to making them as model-independent as possible.

Even though the transitions to the $^3D_1$ state can be neglected, the D state still gives a small correction to $\mathcal{M}$, as can be seen from the effective range theory with tensor forces[2)], because of the relation

$$\rho(-\epsilon, -\epsilon) = 2\int_0^\infty (e^{-2\gamma r} - U_g^2 - W_g^2)\,dr = \rho' - 2 P_D N_g^{-2} \tag{16}$$

obtained by using eqs. (8), (9), and (11). If we designate the difference between $\rho(-\epsilon, -\epsilon)$ and the experimental [see eq. (13)] quantity $\rho(0, -\epsilon)$

by $\underline{\Delta\rho}$, the exact value for the matrix element is clearly given by

$$\mathcal{M} = \mathcal{M}_{SI} - \frac{1}{4}\underline{\Delta\rho} - \frac{1}{2} P_D N_g^{-2} + C \tag{17}$$

We must also replace the shape independent approximation to the ground state normalization by

$$N_g^2 = N_{SI}^2 \frac{(1-\underline{\gamma}\underline{\rho}_{SI})}{[1-\underline{\gamma}(\underline{\rho}_{SI}+\underline{\Delta\rho})]} \tag{18}$$

Fortunately the correction term $\underline{\Delta\rho}$ can be directly evaluated from OPE independent of the short-range behaviour of the wave functions by making use of the theoretical result quoted in the last section [eqs. (4) and (5)]. We find

$$\underline{\Delta\rho} \equiv \underline{\rho}(-\underline{\epsilon},-\underline{\epsilon}) - \underline{\rho}(0,-\underline{\epsilon}) = \frac{2p\underline{\gamma}^2}{1-\underline{q}\gamma^2} = 0.0160\ \text{fm}$$

$$\underline{\rho}(0,0) - \underline{\rho}(0,-\underline{\epsilon}) = -2p\underline{\gamma}^2 = -0.0127\ \text{fm} \tag{19}$$

We might hesitate to use this model (which ignores tensor forces), were it not for the fact that model calculations including the long range OPE tensor force lead to the same result. For example, the nine deuteron models given by Glendenning and Kramer[32)] predict values for $\underline{\Delta\rho}$ lying between 0.006 fm and 0.017 fm. We therefore adopt the value

$$\underline{\Delta\rho} = 0.015 \pm 0.010\ \text{fm} \tag{20}$$

The same models yield values of $\underline{P}_D$ lying between 0.0562 and 0.0742, consistent with other work, so we assume that

$$P_D = 0.07 \pm 0.015 \tag{21}$$

It has been pointed out by Newton[33)] in connection with the interpretation of electron-deuteron scattering that even if we knew the $^3S_1$ phase shift at all energies, the bound state wave function cannot be uniquely determined.† In the specific case where the phase shift is given exactly by

†For a more detailed discussion and references to the literature, see ref. 34.

the shape independent approximation, one effect is to multiply the normalization constant N_{SI}^2 by a factor $2/z$, where z can have any value between 0 and ∞! As has been pointed out by Austern[35)], this is somewhat of a mathematical quibble, since the effective range expansion has a much more intimate connection with the wave functions than is implied by the usual derivation in terms of static, local potentials. In fact, if one looks at the family of phase-equivalent potentials given by Newton, one finds that for $z \neq 2$ they all have exponential tails which fall off exponentially with half the deuteron radius, thus violating the firmly established result that the longest range part of the nuclear force is due to OPE. However, these tails go to zero smoothly as z approaches 2, and since a departure from 2 by as little as 0.005 would introduce appreciable uncertainty in σ, we thought it worthwhile to examine this question quantitatively. For example, the values of the $^3D_1$ phase predicted by these potentials at three energies are given in table 4, together with the dependence on z for z near 2. Since this phase shift is now known empirically to about 1° or 2° at five energies in this range[26)], we see from these results that a potential fitted to n-p scattering data with $z = 2$ would already predict D phases in conflict with experiments at individual energies if z departed from 2 by as little as 0.05. Coupling this fact with the characteristically different energy dependence of the D phase shifts for such potentials, we see that the $^3\underline{D}$ phase shifts alone are probably enough to bring the allowed uncertainty in $\underline{z}$ down into the range assigned above as tolerable. To strengthen this conclusion further, we have computed the ratio of these potentials for $\underline{z} = 2 \pm 0.005$ to OPE in the region between 5 and 10 fm, and find departures from OPE by as much as 50%. This is in direct conflict with the well-established agreement between the predictions of OPE and the highest ℓ phase shifts in nucleon-nucleon scattering[23]. We conclude that the mathematical possibility raised by Newton can be completely ruled out on the basis of these experimental results.

At first sight, the remaining correction term $\underline{C}$ will require explicit assumptions about the wave functions, although Austern[2] has shown that it comes mainly from the intermediate range part of the wave functions and can be expected to be reasonably small. A "shape independent" value of $\underline{C}$ can be obtained by using wave functions corresponding to the (unique) potentials which exactly reproduce the shape-independent approximation for the phase shifts (see discussion in the next section and table 6), but these models do not have the required OPE tail. Austern and Rost[3] use an analytic fit to the Gartenhaus[36] deuteron wave function given by Moravcsik[37] and their own analytic fit to the Gammel-Thaler[38] singlet wave function. For this combination, the correct† value of $\underline{C}$ is + 0.022 fm.

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†The value of 0.048 fm quoted in ref. 3 is in error. I am indebted to N. Austern for an independent calculation confirming my value of 0.022 fm.

Since the Gammel-Thaler potential does not have the required OPE tail, we prefer to use the various modifications of the Signell-Yoder-Heller[7] model discussed in the last section. To explore to some extent the dependence on the triplet interaction, we have used, in addition to the Gartenhaus model, the analytic fit to the Yale[39] potential wave function given by Kottler and Kowalski[40], and an analytic interpolation between tabulated values[41] of the wave function computed by Glendenning and Kramer[32] for their model No. 8. The results collected in table 1 show that $\underline{C}$ is reasonably insensitive both to how we choose to adjust the SYH model to the n-p scattering length and effective range, and to which model we adopt for the deuteron, but does depend to a significant extent on the value of $\underline{r}_s^{np}$ which is assumed. We therefore adopt

$$\begin{aligned} C &= 0.0058 \pm 0.020 \text{ fm} \quad (\text{for } r_s^{np} = 2.4465 \text{ fm}) \\ C &= 0.028 \pm 0.020 \text{ fm} \quad (\text{for } r_s^{np} = 2.68 \text{ fm}) \end{aligned} \qquad (22)$$

and interpolate linearly for other values of the singlet effective range.

The matrix element and cross section predicted by these parameters are given in table 3 for the three different selections of n-p total cross section data discussed above. In order to facilitate comparison with other calculations, we give results both for the shape-independent approximation and with the corrections included; for the same reason, we give results for $\underline{r}_s = 2.68$ fm, assuming the same errors as in the data selection which comes closest to this value. If we accept the charge-independent prediction of $\underline{r}_s = 2.73$ fm (which implies rejecting the experiments of Engelke, et al.[30]), the uncertainty in the prediction is seen to come almost entirely from the corrections. Comparison of these results with the measured value obtained by Cox, Wynchank and Collie[4] gives a discrepancy with the prediction of from 7.7% to 9.5%, depending on the value of the singlet effective range assumed. Since the statistical error is only 1.5% we conclude that the interaction effect is very firmly established, independent of the systematic uncertainty arising from the conflict between two measurements of the n-p total cross section and the hypothesis of charge independence.

In order to allow ready adjustment of these predictions for new values of the parameters, and to exhibit the sensitivity explicitly, we write the prediction as

$$\sigma = \sigma_0 + t\left(\frac{a_t - a_t^0}{\delta a_t}\right) + s\left(\frac{a_s - a_s^0}{\delta a_s}\right) + r\left(\frac{r_s - r_s^0}{\delta r}\right) + \rho\left(\frac{\Delta\rho - \Delta\rho^0}{\delta\rho}\right) + p\left(\frac{P_D - P_D^0}{\delta P}\right) + c\left(\frac{C - C^0}{\delta C}\right) \tag{23}$$

The quantities appearing here are given in table 5.

4. Evaluation of the matrix element by dispersion theory

Since the non-relativistic prediction clearly fails to account for the observed result, we are forced to conclude that physical effects which modify the static electromagnetic properties of the neutron and the proton exist in the overlap between the deuteron and the singlet scattering state even at threshold. The strong coupling of pions to nucleons certainly will produce such effects, and we would not be surprised by, say, a 3% discrepancy, but an effect of nearly 10% is rather startling. Three different calculations of the cross section using dispersion theory have been published. Sakita and Goebel[42)] start from the covariant S-matrix for the problem but show in an appendix that their final result can also be derived from a non-relativistic dispersion theory. Bosco, Ciocchetti, and Molinari[43] provide a different derivation of the same non-relativistic formula which has the advantage of showing that, due to the equality of the OPE interaction in singlet and triplet S states, there is an exact cancellation of long-range effects which should increase the accuracy to be expected for this approximation. M. H. Skolnick[44] gives a covariant S-matrix calculation which in his "Born Approximation" reduces to the same formula; he extends this calculation by a "pole approximation" to the first strip of the double spectral functions, giving a meson current contribution we will discuss below. Except for this last refinement we note that all three calculations make the same prediction for $\mathcal{M}$ at threshold. As this prediction is 321 mb (for $\underline{r}_s$ = 2.68 fm) rather than the 304 mb given by the comparable Bethe-Longmire approximation, and in particular since Skolnick claims that his final result is in agreement with experiment, we must obviously decide whether or not the formula derived from dispersion theory should be used in place of the approach discussed in the last section.

Since all three authors make use of the shape-independent approximation to evaluate the formula, there is a straightforward way to evaluate the adequacy of the approximation as a calculation of the non-relativistic matrix element. As noted above, the potentials corresponding to this assumption can be explicitly constructed. In the language of dispersion theory, the assumption made is that the triplet scattering amplitude has a bound-state pole at $\underline{k} = +i\gamma$ and an interaction pole at $\underline{k} = +i\varphi$, while the singlet amplitude has a virtual state pole at $\underline{k} = -i\alpha$ and an interaction pole at $k = +i\beta$. This leads immediately to the exact result for the effective range expansion

$$k \operatorname{ctn} \delta_{0,1} = -\frac{\gamma\phi}{\phi+\gamma} + \frac{k^2}{\phi+\gamma}$$
$$k \operatorname{ctn} \delta_0 = \frac{\alpha\beta}{\beta-\alpha} + \frac{k^2}{\beta-\alpha} \tag{24}$$

and allows us to compute the pole positions from the empirical effective range expansion parameters. The construction not only gives explicit analytic forms for the potentials, but also for the wave functions[34], namely

$$U_q = \frac{(\phi-\gamma)\sinh\phi r}{\phi\cosh\phi r - \gamma\sinh\phi r}\, e^{-\gamma r}$$
$$U_s = \frac{(\beta+\alpha)\sinh\beta r + \frac{\alpha\beta r}{(\beta-\alpha)}(\beta\sinh\beta r + \alpha\cosh\beta r)}{\beta\cosh\beta r + \alpha\sinh\beta r} \tag{25}$$

so the precise value of $\mathcal{M}$ in this "shape-independent" approximation is easily computed. Since the zero range result at threshold is

$$\mathcal{M}_0 = \frac{1}{\gamma} - \frac{1}{\gamma^2 a_s} = \frac{\alpha\beta + \gamma(\beta-\alpha)}{\gamma^2(\beta-\alpha)} \tag{26}$$

we can clarify the content of the formula obtained by dispersion theory by writing it as

$$\mathcal{M}_{DT} = \mathcal{M}_0 - \frac{1}{(\beta+\gamma)} = \frac{\beta^2(\alpha+\gamma)}{\gamma^2(\beta-\alpha)(\beta+\gamma)} \tag{27}$$

and comparing it with the Bethe-Longmire approximation, which in the same notation is given by

$$\mathcal{M}_{BL} = \mathcal{M}_0 - \frac{1}{2}\left[\frac{1}{(\phi+\gamma)} + \frac{1}{(\beta-\alpha)}\right]$$
$$= \frac{(\beta\phi - \frac{1}{2}\gamma^2)(\alpha+\gamma) + (\alpha\gamma + \frac{1}{2}\gamma^2)(\beta-\phi)}{\gamma^2(\beta-\alpha)(\phi+\gamma)} \tag{28}$$

We note that the range constant φ for the triplet interaction enters the result obtained from dispersion theory only through the normalization constant N_g^2 and that the range correction to $\mathcal{M}$ comes only from the singlet final state interaction. In contrast, the Bethe-Longmire approximation has a range correction which depends both on the singlet and the triplet interaction; further, $\mathcal{M}_{BL}$ reduces to $\mathcal{M}_{DT}$ if we let $\beta = \varphi$ and neglect γ^2 compared to $\beta\varphi$. Since the Bethe-Longmire approximation apparently contains more detailed information, we are not surprised that the comparison of the two approximations with the exact result given in table 6 definitely favors the Bethe-Longmire approximation.

This would settle the matter were it not for the argument of Bosco, *et al.*[43] that the cancellation between the singularities coming from OPE should improve the accuracy of the dispersion theory formula. While the above model does not have the OPE tail, all models quoted in table 2 do, and are even closer to the Bethe-Longmire approximation than the model just given (and hence still farther from the dispersion theory formula). We believe this is decisive from a practical point of view, particularly since we have seen in the last section that the correction terms $\Delta\rho$, P_D and C are needed for the level of precision required here, and no one has yet succeeded in showing how to include these in the dispersion theory calculation.

It might still be objected that these hard core models fail to satisfy the conditions of the derivation from dispersion theory because they have an essential singularity at infinity coming from the core. We have therefore fitted a three-Yukawa model to the same parameters and recomputed C. (The repulsive short range potential is required in order to allow the intermediate range attraction to have a range shorter than $(2m_\pi)^{-1}$.) The results given in table 7 show that the Bethe-Longmire formula is superior in this case also. Further, they show that omitting the OPE tail has essentially no effect on the conclusion, contrary to the argument given by Bosco, et al.[43], presumably because it is quantitatively so weak compared to the intermediate range attraction. We note that the dispersion theory formula is good in the sense that it is only $2\frac{1}{2}\%$ different from the exact result. However, this produces a 5% error in the predicted cross section, which is not tolerable when the entire effect we are looking for is only 8-10%, and is known experimentally to better than 2%. We conclude that the dispersion theory calculation is, so far, not quantitatively adequate for the problem at hand, and that the agreement with experiment claimed by Skolnick[44] is spurious.

Having rejected the claim that about 5% of the observed cross section can be accounted for by using $\mathcal{M}_{DT}$ instead of $\mathcal{M}_{BL}$, we are still left with an interaction effect of 8-10%, and must ask whether this effect can be understood theoretically. G. Stranahan[45] has attempted to do this by introducing the pion-nucleon 33 resonance as an intermediate state (evaluated using the cutoff model) into a Heitler-London type calculation. The three diagrams which he includes that go beyond the non-relativistic approximation are given in fig. 3. The result of the calculation is to increase the cross section by 2.9%, so it accounts for about one-third of the observed effect.

We have already seen that the covariant calculation of M. H. Skolnick[44] is unreliable considered as a calculation of the entire cross section, since the treatment of the two-nucleon final state interaction reduces to the quantitatively inadequate dispersion-theoretic formula just discussed. However, our discussion also shows that this formula is close enough to the truth so that if the additional terms coming from mesonic effects are assumed to be corrections to the non-relativistic matrix element, this inaccuracy will have negligible effect on the magnitude of the corrections. Skolnick shows that the first strip of the double spectral functions contributes only one such term, coming from the diagram given in fig. 4. He shows that this term may be approximated by a single pole of residue $\underline{R} = -0.68$ MeV at a position $-\underline{\nu}_p = -50.5$ MeV, which gives an additive correction to the matrix element

$$\Delta \mathcal{M}_\pi = \frac{M_{np} R}{\gamma^2 a_s} \left\{ \frac{1}{M_{np} \nu_p + \gamma^2} + \left(\frac{\beta}{\alpha}\right)\left(\frac{\beta-\alpha}{\beta^2+\gamma^2}\right)\left[\frac{\gamma}{M_{np}\nu_p} - \frac{\gamma^2}{M_{np}\nu_p+\gamma^2-\beta^2}\left(\frac{1}{\beta} - \frac{(\beta^2-\gamma^2)}{M_{np}\nu_p(M_{np}\nu_p+\gamma^2)^{1/2}}\right)\right]\right\} \quad (29)$$

This formula differs from eq. 4.41 of ref. 44 in that a missing bracket and factors of M_{np} (which are also missing in eqs. 4.33 and 4.39) have been supplied; these changes are needed to reproduce the numerical results given in ref. 44, and can be verified by referring to eq. 5.40 of the original preprint. This term increases the cross section by 2.2%, so if our arguments are accepted it again fails to account for the observed effect.

Comparison of the diagrams given in figs. 3 and 4 show that these two calculations refer to rather distinct physical processes. The first comes from the direct excitation of a resonant state of the pion-nucleon system by the γ ray, coupled with the modifications arising because this state interacts differently with the remaining nucleon than does the unexcited nucleon. The second calculation comes from the interaction of γ ray with the pion being exchanged between the two nucleons, and takes no account of the 33 resonance (which would enter the calculation only if more of the double spectral functions were included). One is tempted, therefore, to consider the two effects as additive, and say that together they account for over half the observed interaction effect. However, there is no guarantee that if both processes were considered together that they would not interfere, and the best we can say is that at most only 5.1% of the observed cross section can be accounted for in this way, leaving a sizable effect still to be explained. It has recently been shown by Adler and Drell[46] that the difference between the static magnetic moment of the deuteron and the non-relativistic prediction of that quantity can be computed by using the same π-γ-ρ coupling which is in agreement with the large photopion production cross section at forward angles observed in the multi-GeV region. We therefore suggest that at least part of the interaction effect observed in n-p capture is due to the same cause.

5. Conclusions

We confirm the conclusion reached by Signell, Yoder, and Heller[7] that the measured n-n scattering length of -17 fm is strong evidence for exact charge symmetry. If we use the same model for n-p scattering using (in OPE) the observed $\pi^{\pm} - \pi^{0}$ mass splitting, $G^2_{\pm p} = 14.636$ from π-N scattering, and adjust G^2_{0p} to fit an n-p scattering length of -23.68 fm, we find $G^2_{0p} = 13.936$. If we allow such a splitting between the coupling constants (for which there is shaky evidence from p-p scattering), we can anticipate charge dependent effects of the order of 1% in the strong interactions at shorter range, which frustrate a more detailed treatment of the problem. However, within these theoretical uncertainties and the experimental uncertainty in the p-p effective range, it is still possible to conclude that charge independence requires the n-p singlet effective range to be 2.73 ± 0.03 fm. A new analysis of n-p total cross sections below 5 MeV, taking advantage of the OPE shape dependence which has been empirically confirmed in p-p scattering, shows that all but two measurements are consistent with this prediction. The n-p total cross sections measured by Engelke, et al.[30] at 0.4926 and 3.205 MeV have less than a 5% chance of being consistent with the charge-independent prediction given by the above analysis, and by themselves would imply $r_s^{np} = 2.44 \pm 0.11$ fm. In order to fit this value, the p-p singlet potential model would have to be more than 30% less attractive in the region between 2 and 4 fm, and correspondingly more attractive near the core. The recent analysis of light nuclei by Wilkinson[31], which finds no charge-dependent effects in this state of magnitude greater than 1%, makes charge dependent effects in the two-nucleon system of this magnitude very hard to understand. We therefore strongly recommend that new high precision measurements of the n-p total cross section below 5 MeV be attempted.

We now make use of these parameters to evaluate the non-relativistic prediction of the n-p capture cross section, and compare this prediction with the experimental value recently determined by Cox, Wynchank, and Collie[4]. The discrepancy between the prediction and experiment is interpreted as the departure of the electromagnetic properties of the neutron and proton in the overlap between the deuteron and singlet scattering states from those measured for the free particles, usually called the interaction effect. The corrections to the shape-independent Bethe-Longmire[1] approximation from the shape-dependence of the triplet effective range and the $^3D_1$ state probability in the deuteron are shown from published model calculations to be determined to reasonable precision by the requirement that the long range part of the interaction is given by OPE. The third correction coming from the intermediate range part of the wave functions is also found to be reasonably model insensitive, but to depend to some extent on the value chosen for r_s in a direction that decreases the sensitivity of the prediction to this parameter. The theoretical ambiguity in the construction of the bound state wave function from the $^3S_1$ phase shift is eliminated by using experimental information from other angular momentum states. A recent claim that most of the interaction effect can be accounted for by replacing the Bethe-Longmire formula by a dispersion-theory approximation is examined. We find by direct calculation, using models which satisfy the postulates of the dispersion-theory derivation, that it is not quantitatively adequate for the problem at hand, and therefore that this claim must be rejected as fallacious. Since the treatment of the final state interaction in covariant calculations of the capture cross section reduces to this approximation at threshold, we also conclude that these calculations cannot be accepted as sufficiently accurate to predict the observed cross section. We conclude that the interaction effect is 9.5 ± 1.2% of the observed cross section if we adopt the charge independent value of $r_s = 2.73 \pm 0.03$ fm, or 7.7 ± 1.5% if we abandon charge independence and accept the value of $r_s = 2.44 \pm 0.11$ fm implied by the measurements of Engelke, et al.[30). We therefore confirm the previous evaluation of Austern and Rost[3), with improved precision.

Attempts to explain this effect are briefly examined. According to G. Stranahan[45], including the excitation of the pion-nucleon 33 resonance in the calculation increases the predicted cross section by 2.9%. We interpret the calculation of M. H. Skolnick[44] as giving the correction, due to the non-resonant interaction of the γ-ray with a single pion exchanged between the two nucleons, which increases the cross section by only 2.2%. Even if we made the dubious assumption that these different physical effects are strictly additive, a substantial proportion of the observed interaction effect is still unaccounted for. We suggest that this residual effect might be due to the strong π-γ-ρ coupling which has been shown by Adler and Drell[46] to offer an explanation of the static magnetic moment of the deuteron, and which is also consistent with the large cross section for the photoproduction of pions at forward angles observed in the multi-GeV region.

We are indebted to Cox, Wynchank, and Collie[4] for discussion of their measurement prior to publication, and for allowing us to present this theoretical discussion in conjunction with their work. We are grateful to N. Austern, D. H. Wilkinson, C. Goebel, B. Bosco, and M. H. Skolnick for comments on and criticisms of earlier versions of this paper, and to P. Signell for informing us of the results of Signell, Yoder and Heller[7] prior to publication. We wish to thank T. Osborn for the calculation of phase shifts for the Newton potential, and C. Moore for the least-squares code used in the *n*-*p* analysis.

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Table 1. Changes in the parameters of the Signell-Yoder-Heller p-p potential model required to fit an n-p scattering length of -23.68 fm and two values of the n-p effective range, and corresponding values of the correction C (eq. (11)) to the magnetic dipole capture matrix element. The potential is of the form

$$V(r) = +\infty,\ r \le 0.48\,\mathrm{fm};\quad V(r) = \sum_{n=1}^{6} A_n\left[\exp(-x)/x\right]^n,\quad r > 0.48\,\mathrm{fm}$$

with

$$x = m_\pi r,\quad A_1 = -14.4\,(m_\pi/2M_{np})^2 m_\pi,\quad m_\pi = 137.35\ \mathrm{MeV}/c^2$$

and initial values of the parameters (in MeV)

$$A_2^\circ = -105.76,\quad A_3^\circ = 128.69,\quad A_4^\circ = -577.01,\quad A_5^\circ = 326.05,\quad A_6^\circ = -7.9366$$

The triplet models used for computing C are: G(M), the Gartenhaus[36] model as fitted by Moravcsik[37]; Y(KK), the Yale[39] model as fitted by Kottler and Kowalski[40]; GK8(N), the author's analytic interpolation between tabulated[41] values of Model No. 8 given by Glendenning and Kramer[32].

								value of C for		
a_s	r_s	x	y	A_x	A_x/A_x°	A_y	A_y/A_y°	G(M)	Y(KK)	GK8(N)
-17.32 fm	2.826 fm			same as for proton-proton model				0.03838	0.04552	0.04108
-23.68	2.7394	2		-108.32	1.0242			0.02703	0.03787	0.03372
	2.7323	3		125.12	0.9722			0.02640	0.03723	0.03309
	2.7302	4		-580.82	1.0066			0.02625	0.03706	0.03293
	2.7294	5		322.56	0.9893			0.02619	0.03699	0.03287
	2.7290	6		- 10.7932	1.3599			0.02618	0.03696	0.03237
	2.68	2	3	- 87.104	0.8236	99.369	0.7722	0.02167	0.03239	0.02838
		2	4	- 91.895	0.8689	-601.09	1.0417	0.02185	0.03247	0.02848
		2	5	- 93.292	0.8821	306.13	0.9389	0.02196	0.03249	0.02852
		2	6	- 93.907	0.8879	- 23.433	2.9525	0.02204	0.03251	0.02854
		3	4	213.09	1.6558	-669.85	1.1609	0.02233	0.03266	0.02874
		3	5	187.22	1.4548	267.28	0.8197	0.02250	0.03266	0.02876
		3	6	179.16	1.3922	- 48.957	6.1685	0.02265	0.03260	0.02876
		4	5	-368.95	0.6394	137.04	0.4203	0.02289	0.03264	0.02878
		4	6	-440.86	0.7640	-107.17	13.503	0.01330	0.02932	0.02551
	2.4465	2	3	- 10.156	0.0960	- 3.0653	- 0.0238	-0.00094	0.00924	0.00574
		2	4	- 32.867	0.3108	-680.60	1.1795	-0.00043	0.00925	0.00586
		2	5	- 39.654	0.3749	244.09	0.7486	-0.00011	0.00913	0.00580
		2	6	- 42.760	0.4043	- 68.872	8.6777	0.00013	0.00900	0.00570
		3	4	537.63	4.1777	-990.73	1.7170	0.00069	0.00860	0.00558
		3	5	411.18	3.1951	84.432	0.2590	0.00089	0.00784	0.00488
		3	6	370.67	2.8803	-163.41	20.589	0.00100	0.00713	0.00419
		4	5	313.72	-0.5437	-416.18	- 1.2764	0.00071	0.00535	0.00252
		4	6	- 6.2345	0.0108	-357.37	45.028	0.00047	0.00386	0.00101
		5	6	2813.84	7.2192	-2353.90	296.59	-0.00978	0.00288	0.00045

Table 2. Analysis of ϵ_B, a_{nH}, $\sigma_{TOT}(0)$ and the 8 n-p total cross section measurements below 5 MeV selected in ref. 5. The OPE shape dependence is assumed for both singlet and triplet amplitudes, but due to a cancellation has practically no influence on the results. Data selection "ALL - EBML" omits the cross sections given in ref. 30, while "EBML" makes use only of these. Part a) gives the results when r_s is a free parameter, while part b) includes the charge-independent prediction $r_s = 2.73 \pm 0.03$ fm in the analysis

	Data	r_s(fm)	a_s(fm)	a_t(fm)	$\frac{\langle \delta a_s \delta r_s \rangle}{\lvert\delta a_s\rvert \lvert\delta r_s\rvert}$	$\frac{\langle \delta r_s \delta a_t \rangle}{\lvert\delta r_s\rvert \lvert\delta a_t\rvert}$	$\frac{\langle \delta a_s \delta a_t \rangle}{\lvert\delta a_s\rvert \lvert\delta a_t\rvert}$	D.F.	χ^2	Probability %
a)	ALL	2.5166	-23.679	5.3987	-0.6691	0.8430	-0.7462	6	4.45	62.1
		±0.1036	±0.028	±0.0108						
	ALL-EBML	2.6393	-23.678	5.3987	-0.5370	0.7004	-0.7446	4	1.27	86.5
		±0.1259	±0.028	±0.0108						
	EBML	2.4427	-23.678	5.3989	-0.6297	0.7767	-0.7477	1	0.15	68.5
		±0.1122	±0.028	±0.0108						
b)	ALL	2.7134	-23.714	5.4161	-0.2427	0.3997	-0.5019	7	8.40	33.8
		±0.0288	±0.021	±0.0064						
	ALL-EBML	2.7251	-23.688	5.4039	-0.1460	0.2218	-0.6225	5	1.77	87.8
		±0.0292	±0.023	±0.0079						
	EBML	2.7108	-23.720	5.4191	-0.2043	0.3032	-0.5545	2	6.31	4.25
		±0.0290	±0.022	±0.0072						

Table 3. Non-relativistic predictions of the n-p magnetic dipole capture matrix element and the interaction effect at a neutron laboratory velocity of 2200 m/sec. The experimental value for the cross section used is 334.2 ± 0.5 mb as measured by Cox, Wynchank, and Collie[4]. Values are given both in the shape-independent (Bethe-Longmire) approximation, and with corrections. In the latter case it is assumed that $\underline{\Delta\rho}$ = 0.015±0.010 fm and P_D = 0.07±0.015. Values for a_s, a_t, errors, and correlations in error are given in table 2.

n-p $\underline{\sigma}_{TOT}$ Data	M_{SI} (fm)	$\underline{\sigma}_{SI}$ (mb)	r_s (fm)	C (fm)	M (fm)	$\underline{\sigma}$ (mb)	$\underline{\sigma}_{int}$ (mb)	(%)
EBML	4.062±0.031	313.34±3.22	2.4427±0.1122	0.0054±0.020	4.018±0.038	308.47±4.96	25.73	(7.7±1.5)
ALL	4.043±0.030	310.50±2.77	2.5166±0.1036	0.0125	4.007±0.037	306.73±4.66	27.47	(8.2±1.4)
ALL-EBML	4.013±0.034	305.80±3.87	2.6393±0.1259	0.0241	3.988±0.041	303.81±5.38	30.39	(9.1±1.6)
(same errors)	4.003±0.034	304.25±3.86	2.68 ±0.1259	0.028	3.982±0.041	302.86±5.37	31.34	(9.4±1.6)
ALL-EBML	3.989±0.009	303.24±0.14	2.7251±0.0292	0.0323	3.973±0.024	302.50±3.99	31.70	(9.5±1.2)

(r_s = 2.73±0.03 fm)

Table 4. Dependence of the $^3D_1$ phase on the parameter z.

Lab Energy	$\delta_{2,1}$ z=1	$\delta_{2,1}$ z=2	$\delta_{2,1}$ z=3	$d\delta_{2,1}/dz\big\|_{z=2}$
100 MeV	18.615°	7.372°	-1.649°	-10.44°
200 MeV	19.001°	13.287°	7.324°	-5.96°
300 MeV	18.852°	15.713°	12.041°	-3.46°

Table 5. Sensitivity of the non-relativistic prediction to the parameters using the definition given by eq. 23.

σ_o	308.47 mb	306.73	303.81	302.85	302.51
a_t^o	5.399 fm	5.399	5.399	5.399	5.40386
δa_t	0.01088 fm	0.01083	0.01085	0.01085	0.0079419
t	1.1522 mb	1.1340	1.1149	1.1080	0.8077
a_s^o	-23.678 fm	-23.678	-23.678	-23.678	-23.6879
δa_s	0.02767 fm	0.02756	0.02757	0.02757	0.023502
s	-0.5901 mb	-0.5818	-0.5721	-0.5689	-0.4828
r_s^o	2.4427 fm	2.5166	2.6393	2.68	2.7251
δr_s	0.1122 fm	0.1036	0.1259	0.1259	0.029131
r	-4.3127 mb	-3.9650	-4.7784	-4.7663	-1.1080
$\Delta\rho^o$	0.015 fm				
$\delta\rho$	0.010 fm				
ρ	1.5889 mb	1.5810	1.5678	1.5635	1.5654
P_D^o	0.07				
δP	0.015				
p	-1.4796 mb	-1.4754	-1.4684	-1.4660	-1.4640
C^o	0.0054 fm	0.0125	0.0241	0.028	0.03228
δC	0.020 fm				
c	3.0781 mb	3.0694	3.0549	3.0501	3.0533

Table 6. Comparison of the exact value of the matrix element with the Bethe-Longmire and the Dispersion Theory approximations using "shape-independent" wave functions.

Triplet parameters:

$\underline{\epsilon}_B = 2.22452$ MeV $\quad a_t = 5.39992$ fm $\quad \underline{\gamma} = 0.231608\ \mathrm{fm}^{-1}$ $\quad \underline{\varphi} = 0.923981\ \mathrm{fm}^{-1}$

Singlet parameters:

$a_s = -23.6809$ fm $\quad r_s = 2.68$ fm $\quad \underline{\alpha} = 0.040076\ \mathrm{fm}^{-1}$ $\quad \underline{\beta} = 0.806899\ \mathrm{fm}^{-1}$

$\quad\quad = 2.4465 \quad\quad = 0.040247 \quad\quad = 0.857741$

r_s	$C = \mathcal{M} - \mathcal{M}_{BL}$	$\mathcal{M}$	$\mathcal{M} - \mathcal{M}_{DT}$
2.68 fm	0.04865 fm	4.05084 fm	-0.07166 fm
2.4465	0.01731 fm	4.08752 fm	-0.09937 fm

Table 7. Comparison of the exact value of the matrix element with the Bethe-Longmire and the Dispersion Theory approximations for a 3-Yukawa model.

Parameters of the model are defined by:

$$V(r) = -G^2\left(\frac{\mu}{2M}\right)^2 \frac{e^{-\mu r}}{r} - f_\sigma^2 \frac{e^{-m_\sigma r}}{r} + f_\omega^2 \frac{e^{-m_\omega r}}{r}$$

Triplet model, $a_t = 5.397F$; $\epsilon_B = 2.22452$ MeV.

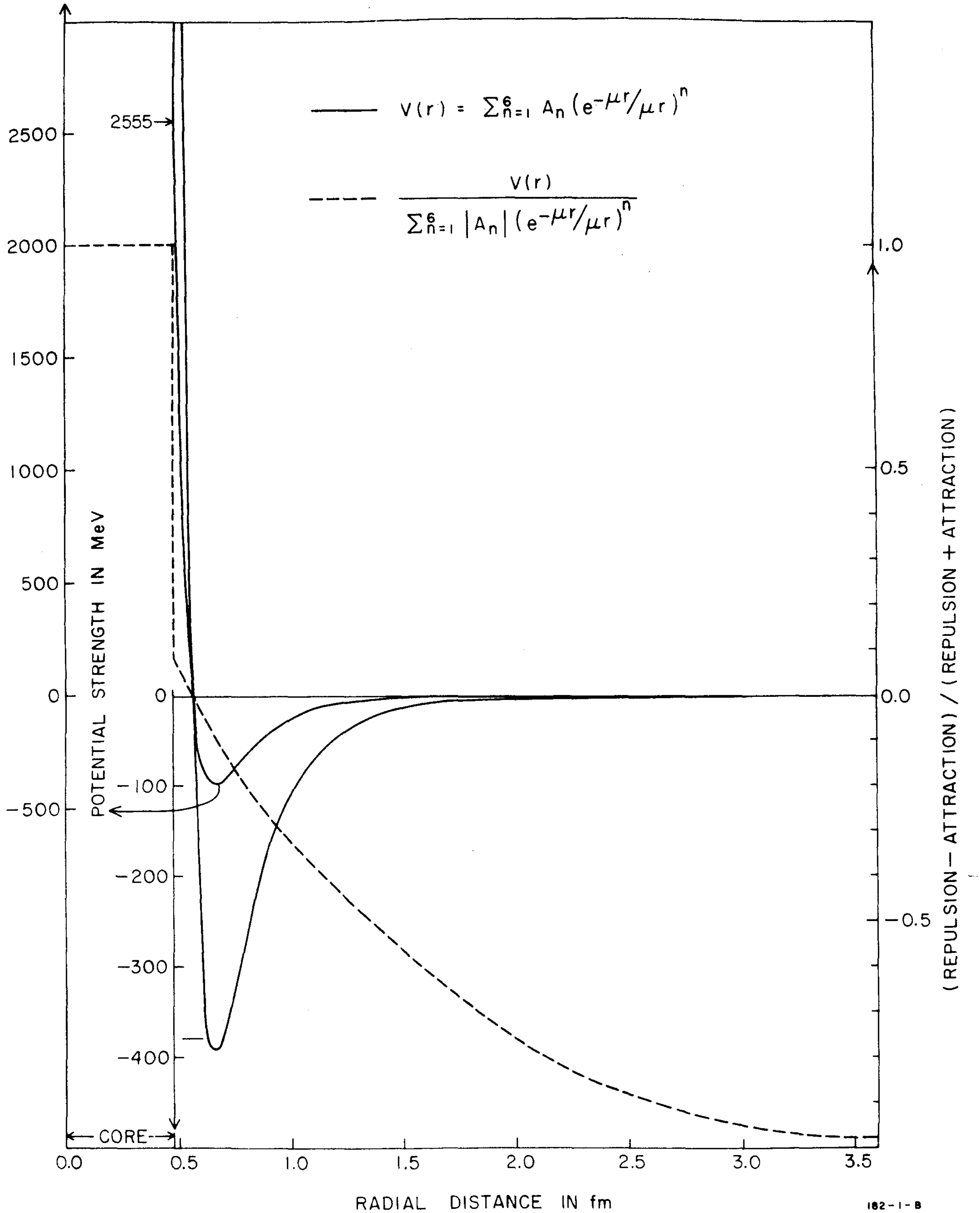
G^2	μ MeV/c^2	f_σ^2	m_σ MeV/c^2	f_ω^2	m_ω MeV/c^2	$\rho(0,-\epsilon)$	$\Delta\rho$	$\rho(0,0) - \rho(0,-\epsilon)$
14	138.07	1.33480	283.126	3.5	782.8	1.72698 fm	0.02243 fm	-0.01780 fm

Singlet models, $a_s = -23.68F$

G^2	μ MeV/c^2	f_σ^2	m_σ MeV/c^2	f_ω^2	m_ω MeV/c^2	$\mathcal{M}-\mathcal{M}_{BL}$	$\mathcal{M}-\mathcal{M}_{DT}$	C
				$r_s = 2.4465$ fm				
0		0.311683	184.970	0		0.023529	-0.101863	0.29134
14	138.07	0.264884	229.007	0		0.025931	-0.099462	0.031536
14	138.07	1.66208	390.525	3.5	782.8	0.023609	-0.101779	0.0292141
				$r_s = 2.68$ fm				
0		0.284516	169.738	0		0.048923	-0.070459	0.054528
14	138.07	0.220515	193.922	0		0.050080	-0.069334	0.055686
14	138.07	1.23917	326.868	3.5	782.8	0.042819	-0.076562	0.048424

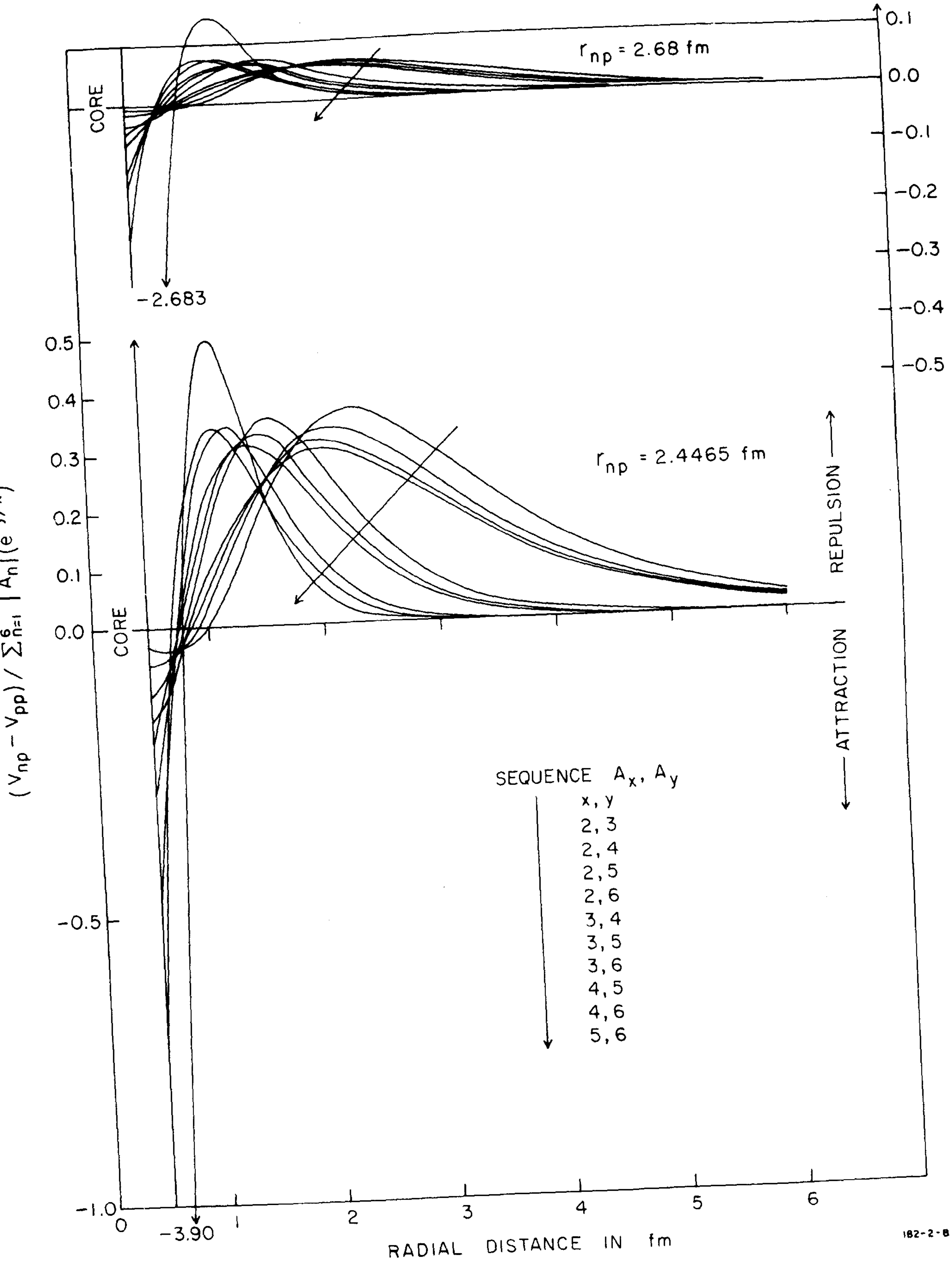
Figure captions

1. The Signell-Yoder-Heller potential, and the ratio of the net potential energy to the sum of the (absolute value of) the attractive and repulsive contributions as a function of radial distance.
2. Changes in the Signell-Yoder-Heller potential needed to fit $\underline{a}_s^{np} = -23.68$ fm and two different values of $\underline{r}_s^{np}$, referred to the sum of the (absolute value of) the attractive and repulsive parts of the original potential. The sequence indicated by the arrow corresponds to the (pairwise) choice of the two adjusted parameters which will modify the potential at successively shorter distances.
3. Diagrams included by G. Stranahan[45)] which contribute to the magnetic dipole capture matrix element due to the excitation of the pion-nucleon 33 resonance by the $\underline{\gamma}$-ray.
4. The only diagram which contributes to the first strip of the double spectral function in the magnetic dipole capture matrix element, according to M.H. Skolnick[44)].



$V(r) = \sum_{n=1}^{6} A_n (e^{-\mu r}/\mu r)^n$
$\frac{V(r)}{\sum_{n=1}^{6} |A_n| (e^{-\mu r}/\mu r)^n}$
2555
2500
2000
1500
1000
500
0
-500
0
-100
-200
-300
-400
POTENTIAL STRENGTH IN MeV
1.0
0.5
0.0
-0.5
(REPULSION − ATTRACTION) / (REPULSION + ATTRACTION)
CORE
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
RADIAL DISTANCE IN fm
182-1-B

FIG. 1



0.1
0.0
-0.1
-0.2
-0.3
-0.4
-0.5
r_{np} = 2.68 fm
CORE
-2.683
0.5
0.4
0.3
0.2
0.1
0.0
-0.5
-1.0
$(V_{np} - V_{pp}) / \sum_{n=1}^{6} |A_n| (e^{-y/x})^n$
r_{np} = 2.4465 fm
REPULSION
ATTRACTION
CORE
SEQUENCE A_x, A_y
x, y
2, 3
2, 4
2, 5
2, 6
3, 4
3, 5
3, 6
4, 5
4, 6
5, 6
0
-3.90
1
2
3
4
5
6
RADIAL DISTANCE IN fm
182-2-8

FIG. 2

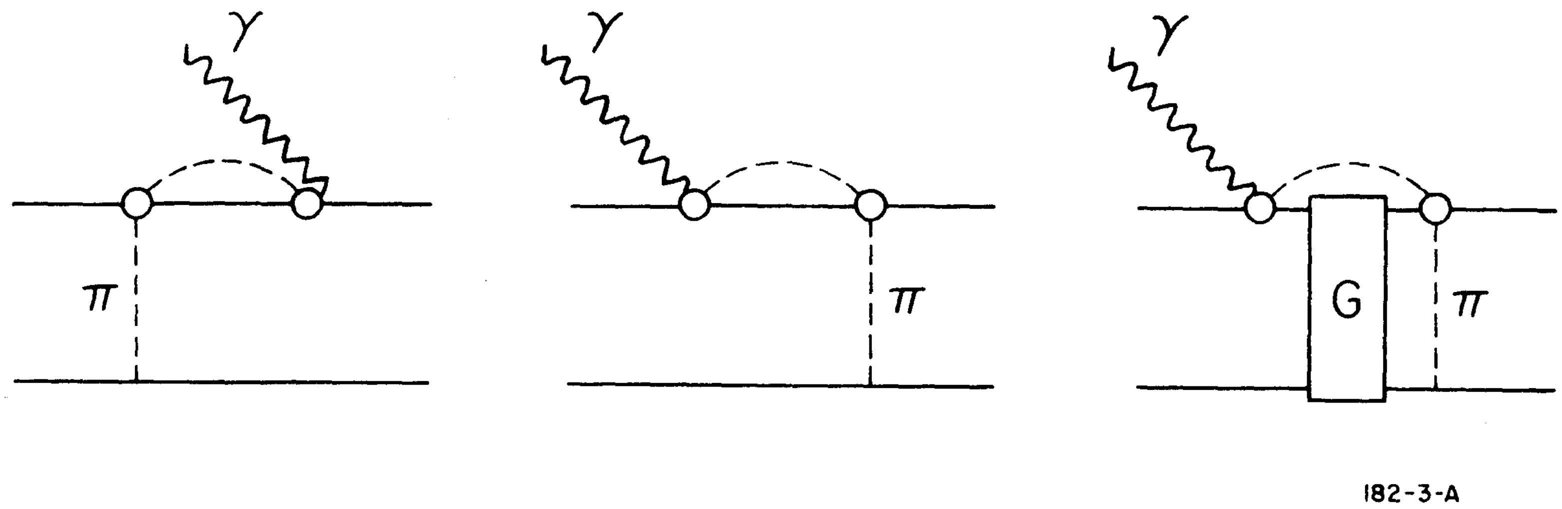


FIG. 3

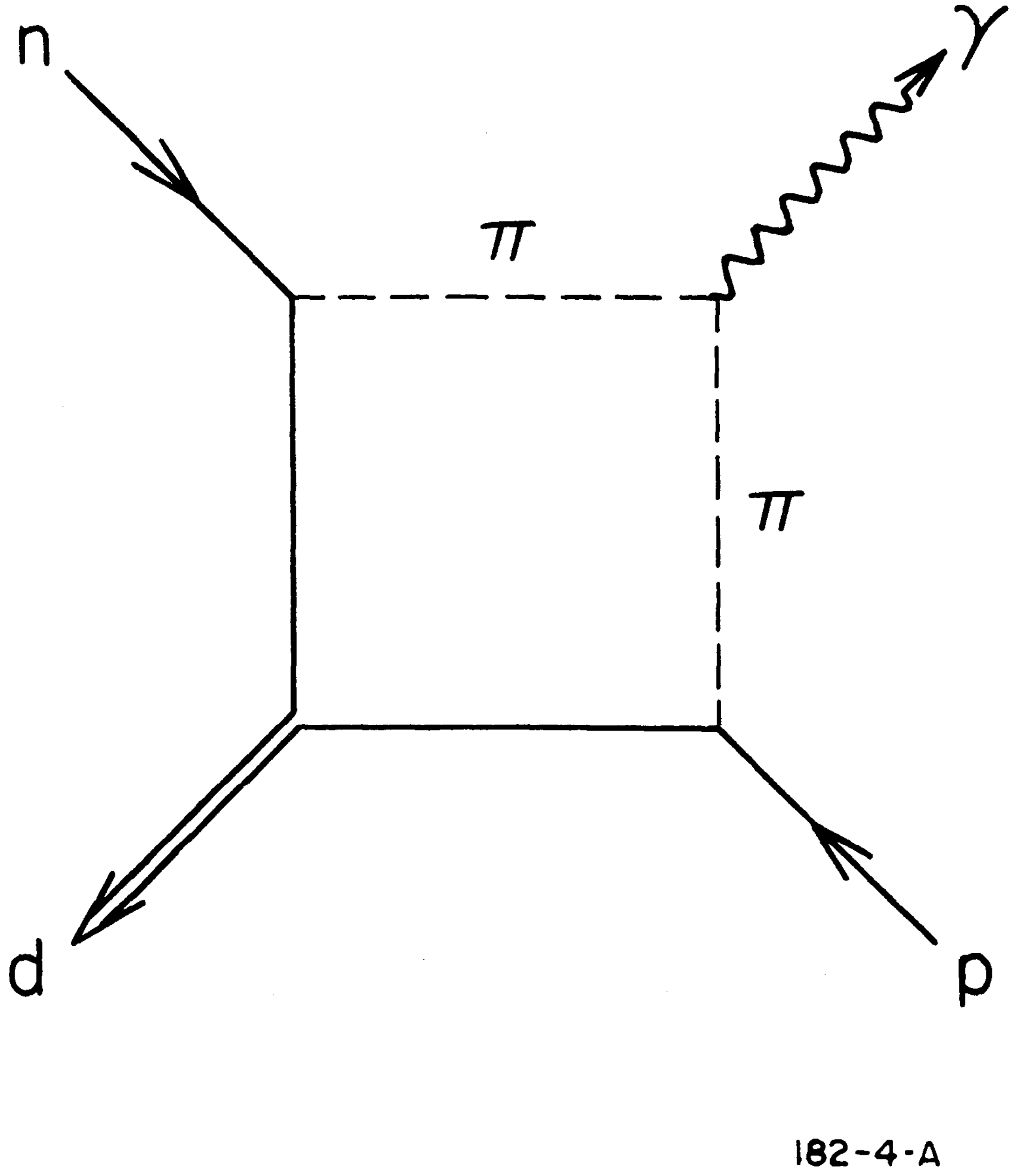


FIG. 4