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ON THE PROBLEM OF HIDDEN VARIABLES IN QUANTUM MECHANICS*

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ABSTRACT

The demonstrations of von Neumann and others, that quantum mechanics does not permit a hidden variable interpretation, are reconsidered. It is shown that their essential axioms are unreasonable. It is urged that in further examination of this problem an interesting axiom would be that mutually distant systems are independent of one another.

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I. INTRODUCTION

To know the quantum mechanical state of a system implies, in general, only statistical restrictions on the results of measurements. It seems interesting to ask if this statistical element be thought of as arising, as in classical statistical mechanics, because the states in question are averages over better defined states for which individually the results would be quite determined. These hypothetical "dispersion free" states would be specified not only by the quantum mechanical state vector but also by additional "hidden variables" — "hidden" because if states with prescribed values of these variables could actually be prepared, quantum mechanics would be observably inadequate.

Whether this question is indeed interesting has been the subject of debate.[1,2] The present paper does not contribute to that debate. It is addressed to those who do find the question interesting, and more particularly to those among them who believe that[3] "the question concerning the existence of such hidden variables received an early and rather decisive answer in the form of von Neumann's proof on the mathematical impossibility of such variables in quantum theory." An attempt will be made to clarify what von Neumann and his successors actually demonstrated. This will cover, as well as von Neumann's treatment, the recent version of the argument by Jauch and Piron,[3] and the stronger result consequent on the work of Gleason.[4] It will be urged that these analyses leave the real question untouched. In fact it will be seen that these demonstrations require from the hypothetical dispersion free states, not only that appropriate ensembles thereof should have all measurable properties of quantum mechanical states, but certain other properties as well. These additional demands appear reasonable when results of measurement are loosely identified with properties of isolated systems. They are seen to be quite unreasonable when one remembers with Bohr[5] "the impossibility of any sharp distinction between the behavior of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear."

The realization that von Neumann's proof is of limited relevance has been gaining ground since the 1952 work of Bohm.[6] However it is far from universal. Moreover the writer has not found in the literature any adequate analysis of what went wrong.[7] Like all authors of non-commissioned reviews he thinks that he can restate the position with such clarity and simplicity that all previous discussions will be eclipsed.

II. ASSUMPTIONS, AND A SIMPLE EXAMPLE

The authors of the demonstrations to be reviewed were concerned to assume as little as possible about quantum mechanics. This is valuable for some purposes, but not for ours. We are interested only in the possibility of hidden variables in ordinary quantum mechanics, and will use freely all the usual notions. Thereby the demonstrations will be substantially shortened.

A quantum mechanical "system" is supposed to have "observables" represented by Hermitian operators in a complex linear vector space. Every "measurement" of an observable yields one of the eigenvalues of the corresponding operator. Observables with commuting operators can be measured simultaneously.[9] A quantum mechanical "state" is represented by a vector in the linear state space. For a state vector ψ the statistical expectation value of an observable with operator O is the normalized inner product $(\psi, O\psi)/(\psi,\psi)$.

The question at issue is whether the quantum mechanical states can be regarded as ensembles of states further specified by additional variables, such that given values of these variables together with the state vector determine precisely the results of individual measurements. These hypothetical well-specified states are said to be "dispersion free."

In the following discussion it will be useful to keep in mind as a simple example a system with a 2-dimensional state space. Consider for definiteness a spin-$\frac{1}{2}$ particle without translational motion. A quantum mechanical state is represented by a 2-component state vector, or spinor, ψ. The observables are represented by 2×2 Hermitian matrices

$$\alpha + \underset{\sim}{\beta} \cdot \underset{\sim}{\sigma} \tag{1}$$

where α is a real number, $\underset{\sim}{\beta}$ a real vector, and $\underset{\sim}{\sigma}$ has for components the Pauli matrices; α is understood to multiply the unit matrix. Measurement of such an observable yields one of the eigenvalues

$$\alpha \pm \left|\underset{\sim}{\beta}\right| \tag{2}$$

with relative probabilities that can be inferred from the expectation value

$$\left\langle \alpha + \underset{\sim}{\beta} \cdot \underset{\sim}{\sigma} \right\rangle = (\psi, [\alpha + \underset{\sim}{\beta} \cdot \underset{\sim}{\sigma}]\, \psi)$$

For this system a hidden variable scheme can be supplied as follows: The dispersion free states are specified by a real number λ, in the interval $-\frac{1}{2} \leq \lambda \leq \frac{1}{2}$, as well as the spinor ψ. To describe how λ determines which eigenvalue the measurement gives, we note that by a rotation of coordinates ψ can be brought to the form

$$\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Let β_x, β_y, β_z, be the components of $\underset{\sim}{\beta}$ in the new coordinate system. Then measurement of $\alpha + \underset{\sim}{\beta} \cdot \underset{\sim}{\sigma}$ on the state specified by ψ and λ results with certainty in the eigenvalue

$$\alpha + \left|\underset{\sim}{\beta}\right| \text{ sign } \left(\lambda \left|\underset{\sim}{\beta}\right| + \tfrac{1}{2}\left|\beta_z\right|\right) \text{ sign } X \tag{3}$$

where

$$\begin{aligned} X &= \beta_z \quad \text{if} \quad \beta_z \neq 0 \\ &= \beta_x \quad \text{if} \quad \beta_z = 0, \quad \beta_x \neq 0 \\ &= \beta_y \quad \text{if} \quad \beta_z = 0, \quad \text{and} \quad \beta_x = 0 \end{aligned}$$

and

$$\begin{aligned} \text{sign } X &= +1 \quad \text{if} \quad X \geq 0 \\ &= -1 \quad \text{if} \quad X < 0 \end{aligned}$$

The quantum mechanical state specified by ψ is obtained by uniform averaging over λ. This gives the expectation value

$$\left\langle \alpha + \underset{\sim}{\beta} \cdot \underset{\sim}{\sigma} \right\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} d\lambda \left\{ \alpha + \left|\underset{\sim}{\beta}\right| \text{ sign } \left(\lambda \left|\underset{\sim}{\beta}\right| + \tfrac{1}{2}\left|\beta_z\right|\right) \text{ sign } X \right\} = \alpha + \beta_z$$

as required.

It should be stressed that no physical significance is attributed here to the parameter λ. All that is offered is a trivial example of a possibility which von Neumann's reasoning was for long thought to exclude.

III. VON NEUMANN

Consider now the proof of von Neumann[10] that dispersion free states, and so hidden variables, are impossible. His essential assumption[11] is: <u>Any real linear combination of any two Hermitian operators represents an observable, and the same linear combination of expectation values is the expectation value of the combination</u>. This is true for quantum mechanical states; it is required by von Neumann of the hypothetical dispersion free states also. In the 2-dimensional example of Section II the expectation value must then be a linear function of α and $\boldsymbol{\beta}$. But for a dispersion free state (which has no statistical character) the expectation value of an observable must equal one of its eigenvalues. The eigenvalues (2) are certainly not linear in $\boldsymbol{\beta}$. Therefore dispersion free states are impossible. If the state space has more dimensions we can always consider a 2-dimensional subspace; therefore the demonstration is quite general.

The essential assumption can be criticized as follows. At first sight the required additivity of expectation values seems very reasonable, and it is rather the non-additivity of allowed values (eigenvalues) which requires explanation. Of course the explanation is well known: A measurement of a sum of non-commuting observables cannot be made by combining trivially the results of separate observations on the two terms — it requires a quite distinct experiment. For example the measurement of σ_x for a magnetic particle might be made with a suitably oriented Stern Gerlach magnet. The measurement of σ_y would require a different orientation, and of $(\sigma_x + \sigma_y)$ a third and different orientation. But this explanation of the non-additivity of allowed values also establishes the non-triviality of the additivity of expectation values. The latter is a quite peculiar property of quantum mechanical states, not to be expected a priori.

There is no reason to demand it individually of the hypothetical dispersion free states, whose function it is to reproduce the measurable peculiarities of quantum mechanics when averaged over.

In the trivial example of Section II the dispersion free states (specified λ) have additive expectation values only for commuting operators. Nevertheless they give logically consistent and precise predictions for the results of all possible measurements, which when averaged over λ are fully equivalent to the quantum mechanical predictions. In fact for this trivial example the hidden variable question as posed informally by von Neumann[12] in his book is answered in the affirmative.

Thus the formal proof does not justify the informal conclusion[13]: "It is therefore not, as is often assumed, a question of reinterpretation of quantum mechanics — the present system of quantum mechanics would have to be objectively false in order that another description of the elementary process than the statistical one be possible." It was not the objective measurable predictions of quantum mechanics which ruled out hidden variables. It was the arbitrary assumption of a particular (and impossible) relation between the results of incompatible measurements that might be made on a given occasion but only one of which can in fact be made.

IV. JAUCH AND PIRON

A new version of the argument has been given by Jauch and Piron.[3] Like von Neumann they are interested in generalized forms of quantum mechanics and do not assume the usual connection of quantum mechanical expectation values with state vectors and operators. We will assume the latter and shorten the argument, for we are concerned here only with possible interpretations of ordinary quantum mechanics.

Consider only observables represented by projection operators. The eigenvalues of projection operators are 0 and 1. Their expectation values are equal to the probabilities that 1 rather than 0 is the result of measurement. For any two projection operators, a and b, a third $(a \cap b)$ is defined as the projection on to the intersection of the corresponding subspaces. The essential axioms of Jauch and Piron are the following:

(A) Expectation values of <u>commuting</u> projection operators are additive.

(B) If, for some state and two projections a and b,

$$\langle a \rangle = \langle b \rangle = 1$$

then for that state

$$\langle a \cap b \rangle = 1$$

Jauch and Piron are led to this last axiom (4° in their numbering) by an analogy with the calculus of propositions in ordinary logic. The projections are to some extent analogous to logical propositions, with the allowed value 1 corresponding to "truth" and 0 to "falsehood," and the construction $(a \cap b)$ to (a "and" b). In logic we have, of course, if a is true and b is true then (a and b) is true. The axiom has this same structure.

Now we can quickly rule out dispersion free states by considering a 2-dimensional subspace. In that the projection operators are the zero, the unit operator, and those of the form

$$\frac{1}{2} + \frac{1}{2} \hat{\underset{\sim}{\alpha}} \cdot \underset{\sim}{\sigma}$$

where $\hat{\alpha}$ is a unit vector. In a dispersion free state the expectation value of an operator must be one of its eigenvalues, 0 or 1 for projections. Since from A

$$\left\langle \frac{1}{2} + \frac{1}{2}\hat{\alpha} \cdot \sigma \right\rangle + \left\langle \frac{1}{2} - \frac{1}{2}\hat{\alpha} \cdot \sigma \right\rangle = 1$$

we have that for a dispersion free state either

$$\left\langle \frac{1}{2} + \frac{1}{2}\hat{\alpha} \cdot \sigma \right\rangle = 1 \quad \text{or} \quad \left\langle \frac{1}{2} - \frac{1}{2}\hat{\alpha} \cdot \sigma \right\rangle = 1.$$

Let $\hat{\alpha}$ and $\hat{\beta}$ be any non-collinear unit vectors and

$$a = \frac{1}{2} \pm \frac{1}{2}\hat{\alpha} \cdot \sigma \qquad b = \frac{1}{2} \pm \frac{1}{2}\hat{\beta} \cdot \sigma$$

with the signs chosen so that $< a > = < b > = 1$. Then B requires

$$< a \cap b > = 1$$

But with $\hat{\alpha}$ and $\hat{\beta}$ non-collinear one readily sees that

$$a \cap b = 0$$

so that

$$< a \cap b > = 0$$

So there can be no dispersion free states.

The objection to this is the same as before. We are not dealing in B with logical propositions, but with measurements involving, for example, differently oriented magnets. The axiom holds for quantum mechanical states.[14] But it is a quite peculiar property of them, in no way a necessity of thought. Only the quantum mechanical averages over the dispersion free states need reproduce this property, as in the example of Section II.

V. GLEASON

The remarkable mathematical work of Gleason[4] was not explicitly addressed to the hidden variable problem. It was directed to reducing the axiomatic basis of quantum mechanics. However, as it apparently enables von Neumann's result to be obtained without objectionable assumptions about non-commuting operators, we must clearly consider it. The relevant corollary of Gleason's work is that, if the dimensionality of the state space is greater than 2, the additivity requirement for expectation values of commuting operators cannot be met by dispersion free states. This will now be proved, and then its significance discussed. It should be stressed that Gleason obtained more than this, by a lengthier argument, but this is all that is essential here.

It suffices to consider projection operators. Let $P(\Phi)$ be the projector on to the Hilbert space vector Φ, i.e., acting on any vector ψ

$$P(\Phi)\psi = (\Phi,\Phi)^{-1} (\Phi,\psi)\Phi$$

If a set Φ_i are complete and orthogonal

$$\sum_i P(\Phi_i) = 1$$

Since the $P(\Phi_i)$ commute, by hypothesis then

$$\sum_i \langle P(\Phi_i) \rangle = 1 \tag{4}$$

Since the expectation value of a projector is non-negative (each measurement yields one of the allowed values 0 or 1), and since any two orthogonal vectors can be regarded as members of a complete set, we have

(A) If with some vector Φ, $\langle P(\Phi)\rangle = 1$ for a given state, then for that state $\langle P(\psi)\rangle = 0$ for any ψ orthogonal on Φ.

If ψ_1 and ψ_2 are another orthogonal basis for the subspace spanned by some vectors Φ_1 and Φ_2 then from (4)

$$\langle P(\psi_1)\rangle + \langle P(\psi_2)\rangle = 1 - \sum_{\substack{i\neq 1\\ i\neq 2}} \langle P(\Phi_i)\rangle$$

or

$$\langle P(\psi_1)\rangle + \langle P(\psi_2)\rangle = \langle P(\Phi_1)\rangle + \langle P(\Phi_2)\rangle$$

Since ψ_1 may be any combination of Φ_1 and Φ_2 we have

(B) If for a given state

$$\langle P(\Phi_1)\rangle = \langle P(\Phi_2)\rangle = 0$$

for some pair of orthogonal vectors, then

$$\langle P(\alpha\Phi_1 + \beta\Phi_2)\rangle = 0$$

for all α and β.

(A) and (B) will now be used repeatedly to establish the following. Let Φ and ψ be some vectors such that for a given state

$$\langle P(\psi)\rangle = 1 \qquad (5)$$

$$\langle P(\Phi)\rangle = 0 \qquad (6)$$

Then Φ and ψ cannot be arbitrarily close; in fact

$$\left|\Phi - \psi\right| > \frac{1}{2}\left|\psi\right| \tag{7}$$

To see this let us normalize ψ and write Φ in the form

$$\Phi = \psi + \epsilon\psi'$$

where ψ' is orthogonal to ψ and normalized, and ϵ is a real number. Let ψ'' be a normalized vector orthogonal to both ψ and ψ' (it is here that we need three dimensions at least) and so to Φ. By A and (5)

$$\left\langle P(\psi')\right\rangle = 0, \left\langle P(\psi'')\right\rangle = 0$$

Then by B and (6)

$$\left\langle P(\Phi + \gamma^{-1}\epsilon\psi'')\right\rangle = 0$$

where γ is any real number, and also by B

$$\left\langle P(-\epsilon\psi' + \gamma\epsilon\psi'')\right\rangle = 0$$

The vector arguments in the last two formulae are orthogonal; so we may add them, again using B:

$$\left\langle P\left(\psi + \epsilon(\gamma + \gamma^{-1})\psi''\right)\right\rangle = 0$$

Now if ϵ is less than $\frac{1}{2}$, there are real γ such that

$$\epsilon(\gamma + \gamma^{-1}) = \pm 1$$

Therefore

$$\left\langle P(\psi + \psi'')\right\rangle = \left\langle P(\psi - \psi'')\right\rangle = 0$$

The vectors $\psi \pm \psi''$ are orthogonal; adding them and again using B

$$\left\langle P(\psi) \right\rangle = 0$$

This contradicts the assumption 5. Therefore

$$\epsilon > \frac{1}{2}$$

as announced in (7)

Consider now the possibility of dispersion free states. For such states each projector has expectation value either 0 or 1. It is clear from (4) that both values must occur, and since there are no other values possible there must be arbitrarily close pairs ψ, Φ with different expectation values 0 and 1 respectively. But we saw above such pairs could not be arbitrarily close. Therefore there are no dispersion free states.

That so much follows from such apparently innocent assumptions leads us to question their innocence. Are the requirements imposed, which are satisfied by quantum mechanical states, reasonable requirements on the dispersion free states? Indeed they are not. Consider the statement B. The operator $P(\alpha\Phi_1 + \beta\Phi_2)$ commutes with $P(\Phi_1)$ and $P(\Phi_2)$ only if either α or β is zero. Thus in general measurement of $P(\alpha\Phi_1 + \beta\Phi_2)$ requires a quite distinct experimental arrangement. We can therefore reject B on the grounds already used: It is a peculiar feature of quantum mechanical states; it relates in a non-trivial way the results of experiments which cannot be performed simultaneously; the dispersion free states need not have this property, it will suffice if the quantum mechanical averages over them do. How did it come about that B was a consequence of assumptions in which only commuting operators were explicitly mentioned? The danger in fact was not in the explicit but in the implicit assumptions. It was tacitly assumed that measurement of an observable must yeild the same value independently of what other measurements may be made simultaneously. Thus as well as $P(\Phi_3)$ say, one might measure <u>either</u> $P(\Phi_2)$ <u>or</u> $P(\psi_2)$ where Φ_2 and ψ_2 are orthogonal to Φ_3 but not to one another. These different possibilities require different experimental arrangements; there is no a priori reason to believe that the results for $P(\Phi_3)$ should be the same. The result of an observation may reasonably depend not only on the state of the system (including hidden variables) but also on the complete disposition of the apparatus; see again the quotation from Bohr at the end of Section I.

To illustrate these remarks we construct a very artificial but simple hidden variable decomposition. If we regard all observables as functions of commuting projectors, it will suffice to consider measurements of the latter. Let $P_1, P_2 \ldots$ be the set of projectors measured by a given apparatus, and for a given quantum mechanical state let their expectation values be $\lambda_1, \lambda_2 - \lambda_1, \lambda_3 - \lambda_2, \ldots$ As hidden variable we take a real number $0 < \lambda \leq 1$; we specify that measurement on a state with specified λ yields the value 1 for P_n if $\lambda_{n-1} < \lambda \leq \lambda_n$, and zero otherwise. The quantum mechanical state is obtained by uniform averaging over λ. There is no contradiction with Gleason's corollary, because the result for a given P_n depends also on the choice of the others. Of course it would be silly to let the result be affected by a mere permutation of the other P's, so we specify that the same order is taken (however defined) when the P's are in fact the same set.

Reflection will deepen the initial impression of artificiality here. However the example suffices to show that the implicit assumption of the impossibility proof was essential to its conclusion. A more serious hidden variable decomposition will be taken up in Section VI.[15]

VI. LOCALITY AND SEPARABILITY

Up till now we have been resisting arbitrary demands upon the hypothetical dispersion free states. However, as well as reproducing quantum mechanics on averaging, there are features which can reasonably be desired in a hidden variable scheme. The hidden variables should surely have some spacial significance and should evolve in time according to prescribed laws. These are prejudices, but it is just this possibility of interpolating some (preferably causal) space-time picture, between preparation of and measurements on states, that makes the quest for hidden variables interesting to the unsophisticated.[2] The ideas of space, time, and causality are not prominent in the kind of discussion we have been considering. To the writer's knowledge the most successful attempt so far in that direction is the 1952 scheme of Bohm for elementary wave mechanics. By way of conclusion this will be sketched briefly and a curious feature of it stressed.

Consider for example a system of two spin-$\frac{1}{2}$ particles. The quantum mechanical state is represented by a wave function

$$\psi_{ij}\,(\underset{\sim}{r}_1, \underset{\sim}{r}_2)$$

where i and j are spin indices which will be suppressed. This is governed by the Schrodinger equation

$$\frac{\partial\psi}{\partial t} = -\,i\left(-\frac{\partial^2}{\partial \underset{\sim}{r}_1^2} - \frac{\partial^2}{\partial \underset{\sim}{r}_2^2} + V(\underset{\sim}{r}_1 - \underset{\sim}{r}_2) + a\underset{\sim}{\sigma}_1 \cdot \underset{\sim}{H}(\underset{\sim}{r}_1) + b\underset{\sim}{\sigma}_2 \cdot \underset{\sim}{H}(\underset{\sim}{r}_2)\right)\psi \qquad (8)$$

where V is the interparticle potential. For simplicity we have taken neutral particles with magnetic moments, and an external magnetic field $\underset{\sim}{H}$ has been allowed to represent spin analyzing magnets. The hidden variables are then two vectors $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$, which give directly the results of position measurements. Other measurements are reduced ultimately to position measurements.[16] For example measurement of a spin component means observing whether the particle emerges with an upward or downward deflection from a Stern Gerlach magnet. The variables $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ are supposed to be distributed in configuration space with the probability density

$$\rho(\underset{\sim}{X}_1, \underset{\sim}{X}_2) = \sum_{ij} \left| \psi_{ij} (\underset{\sim}{X}_1, \underset{\sim}{X}_2) \right|^2$$

appropriate to the quantum mechanical state. Consistently with this $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ are supposed to vary with time according to

$$\left.\begin{aligned} \frac{d\underset{\sim}{X}_1}{dt} &= \rho(\underset{\sim}{X}_1,\underset{\sim}{X}_2)^{-1} \, \mathrm{Im} \sum_{ij} \psi^*_{ij} (\underset{\sim}{X}_1,\underset{\sim}{X}_2) \frac{\partial}{\partial \underset{\sim}{X}_1} \psi_{ij} (\underset{\sim}{X}_1,\underset{\sim}{X}_2) \\ \frac{d\underset{\sim}{X}_2}{dt} &= \rho(\underset{\sim}{X}_1,\underset{\sim}{X}_2)^{-1} \, \mathrm{Im} \sum_{ij} \psi^*_{ij} (\underset{\sim}{X}_1,\underset{\sim}{X}_2) \frac{\partial}{\partial \underset{\sim}{X}_2} \psi_{ij} (\underset{\sim}{X}_1,\underset{\sim}{X}_2) \end{aligned}\right\} \qquad (9)$$

The curious feature is that the trajectory equations (9) for the hidden variables have in general a grossly nonlocal character. If the wave function is factorable before the analyzing fields become effective (the particles being far apart)

$$\psi_{ij} (\underset{\sim}{X}_1,\underset{\sim}{X}_2) = \Phi_i (\underset{\sim}{X}_1) \, \chi_j (\underset{\sim}{X}_2)$$

this factorability will be preserved. Equations (8) then reduce to

$$\frac{d\underset{\sim}{X}_1}{dt} = \left(\sum_i \Phi_i^* (\underset{\sim}{X}_1)\, \Phi_i(X_1) \right)^{-1} \operatorname{Im} \sum_i \Phi_i^* (\underset{\sim}{X}_1) \frac{\partial}{\partial \underset{\sim}{X}_1} \Phi_i(X_1)$$

$$\frac{d\underset{\sim}{X}_2}{dt} = \left(\sum_j \chi_j^* (\underset{\sim}{X}_2)\, \chi_j(X_2) \right)^{-1} \operatorname{Im} \sum_j \chi_j^* (\underset{\sim}{X}_2) \frac{\partial}{\partial \underset{\sim}{X}_2} \chi(X_2)$$

The Schrodinger equation (8) also separates, and the trajectories of $\underset{\sim}{X}_1$ and $\underset{\sim}{X}_2$ are determined separately by equations involving $\underset{\sim}{H}(\underset{\sim}{X}_1)$ and $\underset{\sim}{H}(\underset{\sim}{X}_2)$ respectively. However in general the wave function is not factorable. The trajectory of 1 then depends in a complicated way on the trajectory and wave function of 2, and so on the analyzing fields acting on 2 — however remote these may be from particle 1. So in this theory an explicit causal mechanism exists whereby the disposition of one piece of apparatus affects the results obtained with a distant piece. In fact the Einstein-Podolsky-Rosen paradox is resolved in the way which Einstein would have liked least (Ref. 2, p. 85).

More generally, the hidden variable account of a given system becomes entirely different when we remember that it has undoubtedly interacted with numerous other systems in the past and that the total wave function will certainly not be factorable. The same effect complicates the hidden variable account of the theory of measurement, when it is desired to include part of the "apparatus" in the system.

Bohm of course was well aware[6,17,18,19] of these features of his scheme, and has given them much attention. However it must be stressed that, to the present writer's knowledge, there is no proof that any hidden variable account of quantum mechanics must have this extraordinary character.[20] It would therefore be interesting, perhaps,[1] to pursue some further "impossibility proofs," replacing the arbitrary axioms objected to above by some condition of locality, or of separability of distant systems.

ACKNOWLEDGMENTS

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REFERENCES AND FOOTNOTES

1. The following works contain discussions of and references on the hidden variable problem: Louis de Broglie, Physicien et Penseur, (Albin Michel, Paris, 1953); W. Heisenberg, in Niels Bohr and the Development of Physics, (W. Pauli, Ed., McGraw Hill, New York, and Pergamon, London, 1955); Observation and Interpretation, (S. Körner, Ed., Academic Press, New York, and Butterworth, London, 1957); N. R. Hansen, The Concept of the Positron, (Cambridge University Press, Cambridge, 1963). See also the various works by D. Bohm cited later. For the view that the possibility of hidden variables has little interest, see especially the contributions of Rosenfeld to the first and third of these references, of Pauli to the first, the article of Heisenberg, and many passages in Hansen.
2. Albert Einstein, Philosopher Scientist, (P. A. Schilp, Ed., Library of Living Philosophers, Evanston, Illinois, 1949). Einstein's "Autobiographical Notes" and "Reply to Critics" suggest that the hidden variable problem has some interest.
3. J. M. Jauch and C. Piron, Helvetica Physica Acta, 36, 827 (1963).
4. A. M. Gleason, Journal of Mathematics and Mechanics, 6, 885 (1957). I am much indebted to Professor Jauch for drawing my attention to this work.
5. N. Bohr, in his contribution to the volume cited in Reference 2.
6. D. Bohm, Physical Review, 85, 166, 180 (1952).
7. In particular the analysis of Bohm[6] seems to lack clarity, or else accuracy. He fully emphasizes the role of the experimental arrangement. However, it seems to be implied (p. 187 of Ref. 6) that the circumvention of the theorem *requires* the association of *hidden* variables with the apparatus

as well as with the system observed. The scheme of Section II is a counter example to this. Moreover it will be seen in Section III that if the essential additivity assumption of von Neumann were granted, hidden variables wherever located would not avail. Bohm's further remarks in Reference 17 (p. 95) and Reference 18 (p. 358) are also unconvincing. Other critiques of the theorem are cited, and some of them rebutted, by Albertson.[8]

8. J. Albertson, American Journal of Physics, 29, 478 (1961).

9. The notion of measurement can bear much analysis. No more will be made here than is strictly necessary. Recent papers on this subject, with further references, are: E. P. Wigner, American Journal of Physics, 31, 6 (1963); A. Shimony, American Journal of Physics, 31, 755 (1963).

10. J. von Neumann, Mathematische Grundlagen der Quantenmechanik, (Verlag Julius-Springer, Berlin, 1932), [English trans.: Princeton University Press, 1955]. All page numbers quoted are those of the English edition. The problem is posed in the preface, and on page 209. The formal proof occupies essentially pages 305-324, and is followed by several pages of commentary. A self-contained exposition of the proof has been presented by J. Albertson.[8]

11. This is contained in von Neumann's B' (p. 311), I (p. 313), and II (p. 314).

12. Reference 10, p. 209.

13. Reference 10, p. 325.

14. In the two dimensional case $\langle a \rangle = \langle b \rangle = 1$ (for some quantum mechanical state) is possible only if the two projectors are identical $(\hat{\alpha} = \hat{\beta})$. Then $a \cap b = a = b$ and $\langle a \cap b \rangle = \langle a \rangle = \langle b \rangle = 1$.

15. The simplest example for illustrating the discussion of Section V would then be a particle of spin 1, postulating a sufficient variety of spin-external-field interactions to permit arbitrary complete sets of spin states to be spacially separated.

16. There are clearly enough measurements to be interesting that can be made in this way. We will not consider whether there are others.

17. D. Bohm, Causality and Chance in Modern Physics,(D. Van Nostrand, Princeton and New York, 1957).

18. D. Bohm, in Quantum Theory, (D. R. Bates, Ed., Academic Press, New York and London, 1962).

19. D. Bohm and Y. Aharonov, Phys. Rev., 108, 1070 (1957).

20. Since the completion of this paper such a proof has been found for the example of two spin-$\frac{1}{2}$ particles (J. S. Bell, to be published).