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BOUNDS ON PROPAGATORS, COUPLING CONSTANTS AND VERTEX FUNCTIONS*

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ABSTRACT

We construct several bounds on renormalization constants and on the asymptotic behavior of propagation functions and vertices. The inputs are experimental measurements and/or analyticity properties of vertex functions. We also discuss the connection between zeros in propagators, poles in vertex functions, and the values of coupling constants. This is the problem studied by Geshkenbein and Ioffe and by Meiman, and we discuss the possible physical significance of such zeros in terms of an extended Lee model.

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I. INTRODUCTION

Spectral representations for propagators and form factors have been constructed in field theory starting either from the general axioms or from a Feynman graph series. However, their limiting behaviors for large momenta (subtraction constants) and the magnitudes of the renormalization constants are subjects of considerable conjecture. In this paper we construct several bounds on renormalization constants and on the asymptotic behavior of propagation functions and vertices. The inputs are experimental measurements and/or analyticity properties of vertex functions.

The paper is organized as follows: In Section II we first consider the photon propagator and prove that if there is no subtraction term, then the Pauli form factor of the proton, $F_2(q^2)$ must vanish more rapidly than $(\log q^2)^{-\frac{1}{2}}$ for time-like $q^2 \to \infty$. The requirement of no subtractions is necessary if electrodynamics is to predict the observed vacuum polarization contribution to the Lamb shift and other precision measurements without requiring the introduction of new parameters. In Section III we extend techniques, developed by Meiman and Geshkenbein and Ioffe in a different but related study, to construct a lower bound rigorous to all orders of the strong interactions on the pionic contribution to the photon's vacuum polarization. With these same techniques a rigorous bound on the nucleon wave function renormalization due to strong interactions, Z_2, and on the nucleon propagator for space-like momenta is constructed in Section IV. Bounds which can be constructed only after making assumptions on the continuation of amplitudes below physical thresholds are also given for the pion propagator in Section V. Finally in Section VI, we discuss the connection between zeros in propagators, poles in vertex functions and values of coupling constants. This is the problem solved by Geshkenbein and Ioffe, and we discuss the possible physical significance of such zeros. An extension of the Lee model to include in addition an unstable particle field provides a model in terms of which to illustrate these ideas.

II. ASYMPTOTIC BEHAVIOR OF NUCLEON ELECTROMAGNETIC FORM FACTORS

The recently reported experiment[1] on proton anti-proton annihilation to an electron positron pair focuses attention on the behavior of the nucleon electromagnetic form factors $F_1(q^2)$ and $F_2(q^2)$ for time-like momentum transfers $q^2 \geq 4M^2$. Previously electron scattering experiments have measured F_1 and F_2 for increasingly large space-like momentum transfers $q^2 \leq 0$. Analysis of these form factors with dispersion theory has related the observed structures to resonances in two and three pion systems (viz.,ρ,ω,φ) located in the unphysical region $0 \leq q^2 \leq 4M^2$ below the nucleon-anti nucleon threshold. Now with the success of the experimental study at CERN[1] and with the realistic prospect that electron-positron storage rings in the near future will permit study of F_1 and F_2 for larger and larger $q^2 \geq 4M^2$ we look for the possibility of drawing general conclusions on the behavior of these form factors from the structure of field theory. One such result, reported earlier[2] was that a finite value for the charge renormalization constant $Z_3^{-1} = (e_0/e)^2$ requires both the Dirac form factor $F_1(q^2)$ and the Pauli form factor $F_2(q^2)$ to vanish at $q^2 = \infty$. Z_3^{-1} is not a physical observable and this condition that Z_3^{-1} is finite has no direct experimental test. In this paper we derive a new conclusion on F_2

from the observable vacuum polarization. We show below that $F_2(q^2)$ must vanish at $q^2 = \infty$ if the excellent experimental agreement of the calculated vacuum polarization contributions to the Lamb shift and to the g-2 values for electrons and muons is to be a triumph of quantum electrodynamics, and not just an accident. Conditions on the asymptotic behavior of the form factors of the presumed charged vector bosons, $W^{\pm}$, as well as of other particles or resonances as $q^2 \to \infty$ are also summarized.

In reporting these results we recognize their limited value since the energy region which must be probed before "asymptotic conditions" prevail is almost always beyond the range of practicable experiments. Conclusions derived from such arguments are to be viewed primarily as of interest in principle rather than in practice.

In a different approach based only on physical arguments and without a formal theoretical basis, Sachs[3] has suggested stronger conditions on the asymptotic behaviors of F_1 and F_2 for large q^2.

The photon spectral form was first constructed by Källén[4] in 1952. We write it

$$\tilde{D}_F(q)_{\mu\nu} = -g_{\mu\nu}\left[\frac{1}{q^2} + \int_0^\infty \frac{\pi(\sigma^2)\, d\sigma^2}{q^2 - \sigma^2}\right] \equiv + g_{\mu\nu} D_F(q^2) \tag{I}$$

where $\tilde{D}'_F(q)_{\mu\nu}$ is the complete renormalized propagator, except for irrelevant gauge terms proportional to momentun q_μ. The spectral amplitude $\pi(\sigma^2)$ is a real, positive, gauge independent scalar, and related to the renormalized electromagnetic current operator $\underset{\sim}{j}$ by[2]

$$\pi(\sigma^2) = \frac{1}{\sigma^4} \sum_n (2\pi)^3 \, \delta^4 \, (P_n - q) < 0 \left| \underset{\sim}{\epsilon} \cdot \underset{\sim}{j}(0) \right| n >^2$$

$$= - \frac{1}{3\sigma^4} \sum_n (2\pi)^3 \, \delta^4 \, (P_n - q) < 0 \left| j_\mu(0) \right| n > < n \left| j^\mu(0) \right| 0 > \quad . \qquad (2)$$

In Eq. (2), the sum $\sum_n$ includes all physical eigenstates $|n>$ of four momentum $P_n^\mu = q^\mu$, with $q_\mu q^\mu = \sigma^2$. The one photon state with $\sigma^2 = 0$ does not contribute in Eq. (2) and is explicitly separated in Eq. (1).

As remarked by Källén[4] and Lehmann[5] and particularly emphasized in Bogoliubov and Shirkov[6]

$$\lim_{\sigma^2 \to \infty} \pi(\sigma^2) \to 0 \qquad (3)$$

according to the usual assumptions in the renormalization program and indeed Eq. (3) must be satisfied and the integral

$$\int_0^\infty \frac{\pi(\sigma^2)}{\sigma^2} \, d\sigma^2 \qquad (4)$$

must exist if the representation in Eq. (1) is valid; otherwise subtractions are required. If for example

$$\lim_{\sigma^2 \to \infty} \pi(\sigma^2) \to \text{const} > 0 \qquad (5)$$

Eq. (1) would be replaced by

$$\tilde{D}_F'(q)_{\mu\nu} = - g_{\mu\nu} \left[\frac{1}{q^2} + C + (q^2 + a^2) \int_0^\infty \frac{\pi(\sigma^2) \, d\sigma^2}{(\sigma^2 + a^2)(q^2 - \sigma^2)} \right] \qquad (6)$$

where C is a subtraction constant and the subtraction has been made at $q^2 = - a^2$.

The significance of this change from Eq. (1) to Eq. (6) lies in the fact that the vacuum polarization contribution to the Lamb shift and to $g - 2$ is given by Eq. (4) if the integral exists and is thus a calculable prediction of the theory to be tested by experiment.[7] On the other hand if the subtraction of Eq. (6) is required, a new arbitrary parameter, C, is introduced into the theory to be determined by comparing

$$C - a^2 \int_0^\infty \frac{\pi(\sigma^2)\, d\sigma^2}{\sigma^2(\sigma^2 + a^2)}$$

with observation. In this latter case the very beautiful agreement of the measured and calculated Lamb shift in hydrogen, for example, would be lost as a major achievement for quantum electrodynamics. The measured and calculated shifts are 1057.77 ± 0.10 mc and 1057.74 ± 0.22 mc respectively. The calculated value includes $- 27.08$ mc and $- 0.24$ mc from second and fourth order vacuum polarization contributions computed according to Eq. (4) by keeping the electron-positron pair contribution plus radiative correction in Eq. (2) for $\pi(\sigma^2)$.

We now show that Eq. (3) is violated and the subtraction in Eq. (6) must be made if the nucleon electromagnetic form factor $F_2(q^2)$ does not vanish for $q^2 \to \infty$. To see this observe that according to Eq. (2), $\pi(\sigma^2)$ is a sum of positive contributions $\pi^{(n)}(\sigma^2)$ from each physical state $| n >$ and is bounded from below, therefore, by the contribution from any one such state; in particular by the nucleon-antinucleon pair state (proton or neutron). For this pair state the matrix element in Eq. (2) is directly the electromagnetic current of the physical nucleon for $q^2 \geq 4M^2$, i.e., in the notation of Ref. 2

$$< pp'^{(-)} \left| j_\mu \right| 0 > = \frac{1}{(4E_p E_{p'})^{\frac{1}{2}}} \bar{u}_p \left\{ F_1(q^2)\, \gamma_\mu + F_2(q^2) \sigma_{\mu\nu} q_\nu \right\} v_{p'} \quad . \qquad (7)$$

$\pi^{(2)}(\sigma^2)$ is readily computed from Eqs. (2) and (7) and is given in Ref. 2:

$$\pi^{(2)}(\sigma^2) = \frac{1}{12\pi^2\, \sigma^2} \left(1 - \frac{4M^2}{\sigma^2}\right)^{\frac{1}{2}} \left(\left| F_1 - 4MF_2 \right|^2 + \frac{2M^2}{\sigma^2} \left| F_1 - \frac{\sigma^2}{M} F_2 \right|^2 \right) . \qquad (8)$$

Evidently $\pi^{(2)}(\sigma^2)$ is positive and non-vanishing as $\sigma^2 \to \infty$ in violation of Eq. (3) unless $F_2(\sigma^2) \to 0$ as $\sigma^2 \to \infty$ more rapidly than $(\log \sigma^2)^{-\frac{1}{2}}$. This establishes our claim.[8]

This argument against a hard core[9] occurring in $F_2(q^2)$ as $q^2 \to + \infty$ is more of interest in principle than in practice. This is because the vacuum polarization contribution to Eq. (1) has the dimensions of $(\text{mass})^{-2}$. The familiar Uehling term corresponds to $F_1 = e$ and $F_2 = 0$ in Eq. (8) to lowest order in $\alpha = 1/137$, and to $M \to m$, the electron mass. It contributes $\alpha/15\pi m^2$ to the integral, Eq. (4), whereas a baryon pair contribution with $F_1 = e$ and $F_2 = 0$ is reduced by $(m/M_B)^2 < 3 \times 10^{-7}$. The form factor F_2 would have to remain finite and comparable to its static limit $F_2(0) = -(e/4M)\kappa$, with κ the static moment in nucleon Bohr magnetons, up to a very high momentum P far beyond present or projected energies and such that

$$\log \frac{P^2}{M^2} \sim \left(\frac{0.1}{27}\right) \left(\frac{M_B}{m}\right)^2 \sim 10^4$$

before noticeably perturbing the beautiful successes of quantum electrodynamics, which confirms the Uehling term to ~ 0.1 mc/sec out of 27 mc/sec.

The above results for the nucleon form factors (which incidentally also apply to any baryon) can be re-expressed in terms of the "charge" and "magnetic" form factors emphasized[10] in recent analyses. Defining

$$G_E(\sigma^2) \equiv F_1(\sigma^2) - (\sigma^2/M)F_2(\sigma^2) \ ; \quad G_E^{\text{proton}}(0) = e$$

$$G_M(\sigma^2) \equiv F_1(\sigma^2) - 2M\, F_2(\sigma^2) \ ; \quad G_M^{\text{proton}}(0) = e(1 + 1.79) \tag{9}$$

we rewrite Eq. (8) as

$$\pi^{(2)}(\sigma^2) = \frac{1}{12\pi^2\,\sigma^2}\left(1 - \frac{4M^2}{\sigma^2}\right)^{\frac{1}{2}}\left\{\left|G_M(\sigma^2)\right|^2 + \frac{2M^2}{\sigma^2}\left|G_E(\sigma^2)\right|^2\right\} \tag{10}$$

from which it follows that, if Eq. (3) is valid,

$$\left.\begin{aligned} \frac{1}{(\sigma^2)^{\frac{1}{2}}}\left|G_M(\sigma^2)\right| &\to 0 \\ \frac{1}{\sigma^2}\left|G_E(\sigma^2)\right| &\to 0 \end{aligned}\right\} \quad \text{as} \quad \sigma^2 \to \infty$$

G_E and G_M thus require at most one subtraction each in a dispersion analysis. For a finite charge renormalization as discussed in Ref. 2, $\int \pi(\sigma^2)d\sigma^2$ must exist and, by Eq. (10),

$$\left|G_M(\sigma^2)\right| \to 0 \ ; \quad \frac{1}{(\sigma^2)^{\frac{1}{2}}}\left|G_E(\sigma^2)\right| \to 0 \quad \text{as} \quad \sigma^2 \to \infty.$$

This condition assures no subtraction for G_M but still leaves the possibility of one subtraction for $G_E(\sigma^2)$ in constructing dispersion relations. These weaker conditions on G_E and G_M result from the multiplying factor of σ^2 appearing in their definition[11] in Eq. (9).

A similar conclusion is also true for the pion charge form factor. For the vacuum polarization contributions of a pair of the presumed charged vector bosons,[12] $W^{\pm}$, $\pi(\sigma^2) \sim |F_W(\sigma^2)|^2$ as $\sigma^2 \to \infty$ and if Eq. (1) is to be valid the charge form factor $F_W(\sigma^2)$ must vanish as $\sigma^2 \to \infty$ with no hard core or point charge contribution.

In conclusion we compare this result to the earlier related papers of Lehmann, Symanzik and Zimmerman,[13] and of Evans[14] who showed that the irreducible Dyson vertex, defined as in Eq. (7) (with however the important difference that the vacuum polarization contribution on the photon line is removed) must vanish for $q^2 \to \infty$. This is proved in Refs. 13 and 14 to be the necessary condition for the existence of Eq. (1) and hence, as remarked explicitly by Evans,[14] the basis of the vacuum polarization fits. To go from their work to the conclusion drawn in this paper it is necessary to assume that Z_3 is finite.[2,14] Our present argument avoids any such reference to an unobservable renormalization constant.

III. PROPAGATOR BOUNDS WITH APPLICATION TO PIONIC CONTRIBUTION TO VACUUM POLARIZATION

We have seen in the previous section that the finiteness of the vacuum polarization calculation and of renormalization constants is related to the behavior of form factors at large momentum transfer. In this section we give a concise discussion of the method introduced by Meiman,[15] and Geshkenbein and Ioffe[16,17,18] for studying the occurrence of zeros in propagators, and apply it to construct bounds on renormalization constants and propagation functions in general.

Referring back to the photon propagator for concreteness and assuming that Eq. (4) converges and the vacuum polarization is finite we have from Eq. (1)

$$D_F(q^2) = -\frac{1}{q^2} + \int_0^\infty \frac{\pi(\sigma^2)\, d\sigma^2}{\sigma^2 - q^2} \tag{11}$$

which is positive definite for spacelike $q^2 = -|q|^2 < 0$ according to Eq. (2). In particular we have the inequality

$$D_F(-|q|^2) - \frac{1}{|q|^2} \geq \int_0^\infty \frac{\pi^{(n)}(\sigma^2)\, d\sigma^2}{\sigma^2 + |q|^2} \quad , \tag{12}$$

where $\pi^{(n)}(\sigma^2)$ represents the non-negative contribution to the positive definite spectral function of an arbitrary state (n) in the complete state sum in Eq. (2). Our aim in this section is to construct a non-zero lower bound for the right hand side of Eq. (12). As we see in Eqs. (8) and (10) the spectral function can be given as a square root factor for two particle phase space multiplied by form factors if we take a two particle state for n. We restrict ourselves to two body states here since the analyticity properties of these form factors, as established rigorously from formal field theory or to each order of a Feynman graph expansion, are essential ingredients in this development. Suppressing inessential spin complications by considering the contribution of say, a $\pi^+ - \pi^-$ or $K^+ - K^-$ pair in Eq. (12) we find in place of Eq. (10)

$$\pi^{(2b)}(\sigma^2) = \frac{1}{48\pi^2\ \sigma^5}\ (\sigma^2 - 4\mu_b^2)^{3/2}\left|F_b(\sigma^2)\right|^2\ \theta(\sigma^2 - 4\mu_b^2)\quad , \tag{13}$$

where μ_b is the boson mass and $F_b(\sigma^2)$ its electromagnetic form factor. Inserting Eq. (13) into Eq. (12) and introducing dimensionless units $x = \sigma^2/4\mu_b^2$, $y = \left|q\right|^2/4\mu_b^2$, we find

$$4\mu_b^2\ D_F(-\ 4\mu_b^2\ y) - \frac{1}{y}\ \geq \frac{1}{48\pi^2}\int_1^\infty \frac{dx}{x^{5/2}(x+y)}\ (x-1)^{3/2}\left|F_b(x)\right|^2 \equiv \frac{\alpha}{12}\ \Phi(y). \tag{14}$$

The possibility of constructing a minimum

$$\Phi_{min} = \mathrm{Min}(\Phi) > 0$$

was first shown by Geshkenbein and Ioffe[16] and the present discussion is adapted from Meiman.[15] A formal construction is presented in the Appendix. Here we outline the method to illustrate the class of problems to which it is applicable and to give the essential ideas.

In Eq. (14) the integrand is a product of a simple kinematic factor

$$p(x) = x^{-5/2}\ (x-1)^{3/2}\ (x+y)^{-1}$$

and the squared modulus of a form factor analytic in the cut x-plane with a branch cut extending from, say, $x = x_o$ to $x = \infty$. We write then

$$F_b(x) = e + \frac{x}{\pi}\int_{x_o}^\infty \frac{dx'\ \mathrm{Im}\ F_b(x')}{x'(x' - x - i\epsilon)} \tag{15}$$

assuming for simplicity that a once subtracted dispersion relation suffices and that normalization is to $F(0) = e$. The essential point is that $F(x)$ is specified and finite at some point to the left of the branch point at $x = x_o$. The possibility of a finite minimum is suggested if we just look at Eqs. (14) and (15). Φ is clearly larger than zero in the absence of an absorptive part in Eq. (15) as $F_b(x) \rightarrow e$ everywhere. In order to decrease the real part of $F_b(x)$ in Eq. (14), there must be a finite imaginary part present and the most economical balance between real and imaginary parts yields Φ_{min}. Evidently if the branch point x_o in Eq. (15) lies to the left of the threshold of the integral in Eq. (14), i.e., if $x_o < 1$, the most economical balance is achieved if we crowd the contributions to $\mathrm{Im}\, F_b(x')$ into the integral $x_o < x' \leq 1$ in such a way that there is neither a real nor imaginary part of $F_b(x)$ remaining for $x > 1$. This is possible [in the sense of a Riemann-Lebesque integral in Eq. (14)] because the spectral function for the vertex is not positive definite but can oscillate at will. In this case $\Phi_{min} \rightarrow 0$ and no useful bound is obtained for $x_o < 1$, as verified formally in the Appendix. Our considerations apply only to problems with $x_o \geq 1$. A second condition for a finite bound is that $F_b(x)$ be normalized at a point to the left of the branch point $x = 1$. If the normalization point approaches the branch point, an absorptive part of zero width can cancel $F_b(x)$ for $x \geq 1$ without producing a contribution of finite weight to the integrand in Eq. (14). This is also verified explicitly in the Appendix.

A practical deduction from this is that the present techniques are inadequate for constructing general bounds in quantum electrodynamics valid to all orders of the fine structure constant. This is a consequence of the masslessness of a photon which leads to the branch point at $x_o = 0$ in Eq. (15), arising from many photon states, which are coincident with the photon pole. Also in considering the electron propagator the cut for $e \to e + \gamma$ starts at the location of the electron pole.

As an example of a problem for which a bound can be constructed we consider the contribution of a $\pi^+ \pi^-$ pair state to the photon spectral function and find its minimum contribution to Z_3^{-1} and to vacuum polarization, to all orders of strong coupling but to lowest order in e^2. To this order the many photon states coupling to a single photon via the scattering of light by light interaction can be ignored. The propagator and vertex branch points then coincide at $x_o = 1$ (in units of $4\mu_\pi^2$) for a $\pi^+ \pi^-$ pair state and we can find a minimum.

The technique of Meiman is to map the cut x-plane into a unit circle with center at $x = 0$ (the normalization point of $F_\pi(0) = e$) and with the two sides of the cut forming the periphery of the circle as in Fig. 1. The relevant mapping is

$$z = e^{i\theta} = - \frac{(x - 1)^{\frac{1}{2}} - i}{(x - 1)^{\frac{1}{2}} + i} \quad . \tag{16}$$

We then write

$$\Phi = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \; p(\theta) \left| F(e^{i\theta}) \right|^2 \tag{17}$$

where $p(\theta)$ includes the kinematic factors and the Jacobian of the transformation Eq. (16). The behavior of the kinematic quantities and of the form factors can be separated by using the inequality of the arithmetic and geometric means to write

$$\begin{aligned}\Phi &\geq \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, \ln \left[p(\theta)\left|F(e^{i\theta})\right|^2\right]\right] \\ &\geq \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta\, \ln p(\theta)\right] \exp\left[\frac{1}{\pi}\,\mathrm{Re}\int_{-\pi}^{\pi} d\theta\, \ln F(e^{i\theta})\right] \quad .\end{aligned} \tag{18}$$

The first factor in Eq. (18) is integrated directly in terms of known functions. Using the known analytic properties of F and assuming that $F(e^{i\theta})$ vanishes at most at a finite number of points on the circle, the second factor is shown in the Appendix to be $\geq F(0)^2 = e^2$.

The general discussion and formal numerical result in terms of mass parameters is reproduced in the Appendix and here we simply quote the results. The coefficient of $1/y$ in Eq. (14) for $y \to \infty$ defines the charge renormalization $Z_3^{-1} - 1$ due to the pionic contribution to vacuum polarization. For a point pion $F_\pi = e$ and Z_3^{-1} diverges logarithmically. As a lower bound we find

$$Z_3^{-1} \geq 1 + \frac{\alpha}{96} \quad . \tag{19}$$

Similarly a lower bound on pionic contribution to the Lamb shift is obtained by minimizing the integral in Eq. (4) and the result so obtained is

$$\frac{\alpha}{1536\mu^2} \tag{20}$$

This is smaller than the present limit of error by two orders of magnitude. It is reduced by $5\pi/64$ from the value obtained for a point pion and by a factor of $\sim 1/40$ from the enhanced pionic contribution due to the 2π p-wave resonance (or ρ meson).[19]

IV. RIGOROUS BOUND ON NUCLEON PROPAGATOR AND Z_2^{-1}

With the techniques discussed in the preceding section it is possible to bound from below the contributions of strong interactions to the nucleon propagator and wave function renormalization, Z_2^{-1}. This result is rigorous to all orders of the strong interaction.

The spectral representation for the complete renormalized Feynman propagator for the nucleon is, in momentum space,

$$\tilde{S}_F'(p) = \frac{i}{\not{p} - M} + \int_0^\infty d\sigma^2 \frac{\left[\not{p}\rho_1(\sigma^2) + \rho_2(\sigma^2)\right]}{p^2 - \sigma^2} \quad . \tag{21}$$

Since the weight function $\rho_1(\sigma^2)$ is both real and non-negative we may analyze its contribution to the propagator,

$$S_F^*(p^2) = \mathrm{Tr}\left[\frac{-1}{4p_o}\gamma_o \tilde{S}_F'(p)\right] = \frac{1}{M^2 - p^2} + \int_0^\infty d\sigma^2 \frac{\rho_1(\sigma^2)}{\sigma^2 - p^2} \tag{22}$$

and through it

$$Z_2^{-1} \equiv 1 + \int_0^\infty \rho_1(\sigma^2)\, d\sigma^2 \tag{23}$$

as done in Section III. ρ_1 is expressed in terms of the nucleon field operator by

$$\rho_1(\sigma^2) = \left[\frac{1}{4p_o}\sum_n (2\pi)^3 \delta^4(p_n - q) \sum_{\alpha=1}^{4} \left| \langle 0 | \psi_\alpha | n \rangle \right|^2 \right]_{q^2=\sigma^2} \tag{24}$$

in analogy with Eq. (2). We construct our minimum as done earlier by keeping in Eq. (24) the lightest strongly interacting state, the one nucleon, one pion state with a threshold at $\sigma^2 = (M + \mu)^2$.

The matrix element $< 0|\psi|N\,\pi >$ has the form

$$< 0|\psi|p_1 q_1 > = \frac{1}{(2q_{10})^{\frac{1}{2}}} \frac{M}{(p_{10})^{\frac{1}{2}}} \frac{1}{\not{p} - M} \left\{ \gamma_5 f_1 + (\not{p} - M)\, \gamma_5 f_2 \right\} u(p), \tag{25}$$

where f_1 and f_2 are scalar functions of σ^2. At $\sigma^2 = M^2$ and $\not{p} = M$, i.e., for the nucleon line on the mass shell, the f_2 term vanishes and the first one is normalized to the pion nucleon coupling constant $f_1(M^2) = g$; $g^2/4\pi = 14.4$. The contribution of this state to ρ_1 is:

$$\rho_1(\sigma^2)\Big|_{N\pi} = \frac{3Q}{8\pi^2(\sigma^2 - M^2)^2} \left\{ \frac{Q^2}{E} \left| f_1 \right|^2 + E \left| M \left(\frac{1}{E} - \frac{1}{\sigma} \right) f_1 + \left(1 - \frac{M^2}{\sigma^2} \right) f_2 \right|^2 \right\}, \tag{26}$$

where Q is the barycentric three-momentum and E the nucleon energy of the $|N\pi >$ state.

The contribution of the second term in the curly brackets is smaller by two orders of $\frac{\mu}{M}$ than the first at $\sigma = M$ (where we know f_1) so we retain only the first term and write:

$$\rho_1(\sigma^2) \geq \frac{3}{2} \frac{g^2}{4\pi^2} \frac{Q^3}{E} \frac{|G|^2}{(\sigma^2 - M^2)^2} \tag{27}$$

The vertex function G satisfies a dispersion relation, with a cut from $\sigma^2 = (M + \mu)^2$ to ∞, and $G(M^2) = 1$. This is a rigorously established dispersion relation since there is no unphysical region below the physical threshold $(M + \mu)^2$ into which the unitarity equation must be extended.

All the conditions are now met for applying the method of Meiman and we find for spacelike p^2, setting $x = \dfrac{\sigma^2}{(M + \mu^2)}$

$$S_F^*(p^2) \geq \frac{1}{M^2 + |p|^2} + \frac{g^2}{4\pi}$$

$$\text{Min} \left\{ \frac{3}{8\pi} \int_1^\infty dx (x - 1)^{3/2} \left(x - \left[\frac{M-\mu}{M+\mu}\right]^2 \right)^{3/2} x^{-1} \left(x + \frac{M-\mu}{M+\mu} \right)^{-1} \left(x - \frac{M}{M+\mu} \right)^{-2} \right.$$

$$\left. \times \left(x + \frac{|p|^2}{(M + \mu)^2} \right)^{-1} |G(x)|^2 \right\} \qquad (28)$$

$$\geq \frac{1}{M^2 + |p|^2} + \frac{3}{32} \frac{g^2}{4\pi} \frac{\lambda \left[\lambda + \left(1 - \left(\frac{M-\mu}{M+\mu} \right)^2 \right)^{\frac{1}{2}} \right]^3}{(\lambda+1)^2 \left[\lambda + \left(1 + \frac{M-\mu}{M+\mu} \right)^{\frac{1}{2}} \right]^2} \frac{1}{\left[(M+\mu)\lambda + \left([M+\mu]^2 + |p|^2 \right)^{\frac{1}{2}} \right]^2} \qquad (29)$$

with $\lambda = \left(1 - \left(\dfrac{M}{M + \mu} \right)^2 \right)^{\frac{1}{2}}$,

i.e.,

$$S_F^*(p^2) > \frac{1}{M^2 + |p|^2} + \left(\frac{g^2}{4\pi}/100 \right) \left\{ .49(M + \mu) + \left((M + \mu)^2 + |p|^2 \right)^{\frac{1}{2}} \right\}^{-2} . \qquad (30)$$

The lower bound on Z_2^{-1}, the nucleon wave function renormalization, Eq. (23) is read off from Eq. (30) by going to the limit $|p|^2 = \infty$:

$$Z_2^{-1} > 1 + \text{Min} \left[\int \rho_1 |_{\pi N} (k^2) \, dk^2 \right] > \left(1 + \frac{g^2/4\pi}{100} \right) . \qquad (31)$$

V. APPROXIMATE BOUNDS FOR PION PROPAGATOR, Z_3^{-1}, AND $\delta\mu^2$

Beyond this particular application we must make approximations due to the restriction that the branch point of the vertex function must not lie below that of the propagator, i.e., we require $x_o \geq 1$ in Eq. (15). For the π meson propagator, for example, the threshold for the lightest two particle state contributing the weight function ρ in

$$\Delta_F(q^2) = \frac{1}{\mu^2 - q^2} + \int d\sigma^2 \frac{\rho(\sigma^2)}{\sigma^2 - q^2} \tag{32}$$

is $4M^2$, corresponding to a $N\overline{N}$ pair and we can write

$$\rho(\sigma^2) \geq \rho_{N\overline{N}}(\sigma^2) = \frac{1}{4\pi^2}\left(1 - \frac{4M^2}{\sigma^2}\right)^{\frac{1}{2}} \frac{\sigma^2}{(\sigma^2 - \mu^2)^2} \left|F_{\pi N}(\sigma^2)\right|^2 \tag{33}$$

To each finite order in a perturbation calculation $F_{\pi N}(\sigma^2)$ satisfies a dispersion relation in the variable σ^2 with the cut starting at $\sigma^2 = 9\mu^2 < 4M^2$. The branch point at $\sigma^2 = 9\mu^2$ comes from the three pion state which is the lightest strongly interacting one which contributes. No exact proof of a dispersion relation has been constructed for this case because of the necessity of analytically continuing the $N\overline{N}$ scattering amplitude below threshold down into the unphysical region starting at $\sigma^2 = 9\mu^2$. This same problem stops us here, as the cut in $F_{\pi N}$ extends below the threshold of $\rho_{N\overline{N}}$ in Eq. (32) and we can give no exact result.

Although no rigorous conclusions can be drawn it is of interest to establish the approximate ones that can be obtained by keeping only the lightest "two particle cuts" including the contributions of unstable vector resonances. Thus we approximate the 3 pion contribution to $\rho(\sigma^2)$ in Eq. (32) and $F_{\pi N}(\sigma^2)$ in Eq. (33) by a two particle $\rho\pi$ resonant state. The spectral weight function $\rho(\sigma^2)$ in Eq. (33) is replaced by

$$\rho(\sigma^2) > \frac{1}{\pi^2}\frac{\sigma Q^3}{m_\rho^2}\ \frac{1}{(\sigma^2 - \mu^2)^2}\ \left|F_{\rho\pi\pi}(\sigma^2)\right|^2 \tag{34}$$

with Q the barycentric three momentum for the $|\pi\rho>$ state.

The $\rho\pi\pi$ form factor is normalized to the observed $\rho \to 2\pi$ decay width for $\sigma^2 = \mu^2$, which gives[20]

$$F_{\rho\pi\pi}(\sigma^2) = g_{\rho\pi\pi}G(\sigma^2)$$

with

$$\frac{g^2_{\rho\pi\pi}}{4\pi} \sim 1.8 \quad \text{and} \quad G(\mu^2) = 1 \quad . \tag{35}$$

If we neglect all but the lightest two particle $\rho\pi$ intermediate state contribution to the absorptive part of $F_{\rho\pi\pi}$, it is easy to see that the reduced graph, Fig. 2,[21] contributes with branch point at $q^2 = (m_\rho + \mu)^2$ and we can in this case once more apply the method of Meiman.

We can write then for $\Delta_F(q^2)$ with spacelike q^2

$$\Delta_F(|q|^2) > \frac{1}{|q|^2 + \mu^2} + \int \frac{\rho(\sigma^2)\ d\sigma^2}{\sigma^2 + |q|^2} \tag{36}$$

$$> \frac{1}{|q|^2 + \mu^2} + \text{Min}\left\{ \frac{g^2_{\rho\pi\pi}}{m_\rho^2 \pi^2} \int d\sigma^2 \frac{\sigma Q^3}{(\sigma^2 - \mu^2)^2} \frac{|G|^2}{\sigma^2 + |q|^2} \right\} \tag{37}$$

$$> \frac{1}{|q|^2 + \mu^2} + \frac{g^2}{4\pi}\left(\frac{1}{m_\rho^2}\right) \frac{\lambda}{8} \frac{\left[\lambda + \left(1 - \left(\frac{m_\rho - \mu}{m_\rho + \mu}\right)^2\right)^{\frac{1}{2}}\right]^3}{(1 + \lambda)^2} \left[\lambda + \left(1 + \frac{|q|^2}{(m_\rho + \mu)^2}\right)^{\frac{1}{2}}\right]^{-2} \tag{38}$$

with

$$\lambda = \left(1 - \frac{\mu^2}{(m_\rho + \mu)^2}\right)^{\frac{1}{2}} \sim 0.99 \quad ,$$

or

$$\Delta_F(|q|^2) > \frac{1}{|q|^2 + \mu^2} + \left(\frac{g^2}{4\pi}/4.6\right) \left\{(m_\rho + \mu) + \left((m_\rho + \mu)^2 + |q|^2\right)^{\frac{1}{2}}\right\}^{-2} \tag{39}$$

and

$$Z^{-1}_{3_{\text{pion}}} > 1 + \frac{g^2}{4\pi}/4.6 = 1.4 \tag{40}$$

Eq. (39) gives a lower limit, for the correction to the pion propagator as often introduced in peripheral analyses of π-N interactions.[22] For space-like $|q|^2 < m_\rho^2 \sim 30\mu^2$ this increase in the value of the propagator comes to less than 5% and is well within the uncertainties of such analyses.

By a similar calculation we may put a lower bound on the pion self mass

$$\delta\mu^2 = \int \sigma^2 \rho(\sigma^2) d\sigma^2 \, . \tag{41}$$

$\delta\mu^2$ is probably infinite, but again if we assume that the integral exists then it must be larger than

$$\mathrm{Min}\left\{\int \sigma^2\ \rho(\sigma^2)\ d\sigma^2\right\} ,$$

or

$$\delta\mu^2 > (\lambda + 1)^2\ (m_\rho + \mu)^2 \left(\mathrm{Min}\ Z_3^{-1} - 1\right) \approx 66\mu^2 \quad . \tag{42}$$

We note in passing that we may also approximate a lower bound on $\delta\mu^2$ by an entirely different technique similar to that used in Reference 2. The essential assumption now is that the form factor $F_{\pi N}(q^2)$ associated with the $\pi N\bar{N}$ vertex satisfies an unsubtracted spectral representation

$$F(q^2) = \frac{1}{\pi} \int_{9\mu^2}^{\infty} \frac{\mathrm{Im}\ F(q'^2)\ dq'^2}{q'^2 - q^2 - i\epsilon} ;$$

thus

$$g = \frac{1}{\pi} \int_{9\mu^2}^{\infty} \frac{\mathrm{Im}\ F(q^2)\ dq^2}{q^2 - \mu^2} \quad . \tag{43}$$

As in Reference 2, we may use Schwarz' inequality to derive the following inequality for $\mathrm{Im}\ F(q^2)$ above the physical threshold for $N\bar{N}$ production, $q^2 \geq 4M^2$

$$\left[\mathrm{Im}\ F(q^2)\right]^2 \leq \pi(q^2 - \mu^2)^2 \left[(q^2 - 4M^2)/q^2\right]^{\frac{1}{2}} \sigma_T\left((q^2)^{\frac{1}{2}}\right) \rho(q^2) , \tag{44}$$

where $\sigma_T\left((q^2)^{\frac{1}{2}}\right)$ is the total annihilation cross section for the $^1S_0$ state of the $N\bar{N}$ system, and $\rho(q^2)$ is the weight function in the spectral representation of the pion propagator, Eq. (32).

If we now write Eq. (43) in the form

$$g - I \leq \frac{1}{\pi^{\frac{1}{2}}} \int_{4M^2}^{\infty} dq^2 \left[\rho(q^2)\right]^{\frac{1}{2}} \left[(q^2 - 4M^2)/q^2\right]^{\frac{1}{4}} \left[\sigma_T \left((q^2)^{\frac{1}{2}}\right)\right]^{\frac{1}{2}} , \tag{45}$$

where

$$I = \frac{1}{\pi} \int_{9\mu^2}^{4M^2} \frac{\mathrm{Im}\ F(q^2)\ dq^2}{q^2 - \mu^2} \tag{46}$$

is the contribution from below the physical $N\bar{N}$ threshold, and observe that for $q^2 \geq 4M^2$ unitarity bounds σ_T by

$$\sigma_T \left((q^2)^{\frac{1}{2}}\right) \leq \frac{16\pi}{q^2 - 4M^2} \tag{47}$$

then

$$g - I \leq 4 \int_{4M^2}^{\infty} dq^2 \left[\rho(q^2)\right]^{\frac{1}{2}} \left[q^2(q^2 - 4M^2)\right]^{-\frac{1}{4}} . \tag{48}$$

Applying now Schwartz' inequality once again we obtain

$$(g - I)^2 \leq \left(\int_{4M^2}^{\infty} dq^2\ q^2\ \rho(q^2)\right) \left(\int_{4M^2}^{\infty} \frac{dq^2}{q^3} \frac{16}{\left(q^2 - 4M^2\right)^{\frac{1}{2}}}\right) \tag{49}$$

$$\leq \frac{8}{M^2} \int_{4M^2}^{\infty} q^2\ \rho(q^2)\ dq^2$$

$$\leq \frac{8}{M^2}\ \delta\mu^2 \tag{50}$$

or

$$\delta\mu^2 > 80 \frac{(g - I)^2}{4\pi} \mu^2 \tag{51}$$

where $g^2/4\pi = 14.4$.

Any attempt to evaluate I, however, can only be very approximate, as the integral is over a region unphysical for the $N\bar{N}$ process. In the two particle approximation we consider the $\rho\pi$ state as the only one which couples to both the π and $N\bar{N}$ in the mass region $9\mu^2 < \sigma^2 < 4M^2$. Application of unitarity to the amplitude for $\rho\pi \leftrightarrow N\bar{N}$ analytically continued below threshold into this region can then be made as shown by Mandelstam[23] in order to approximate I in this region. We have not carried out this calculation. If it turns out that $I \ll g$, then the bound Eq. (51) is much stronger than Eq. (42).

VI. RELATION BETWEEN ZEROS IN PROPAGATORS, POLES IN IRREDUCIBLE VERTEX FUNCTIONS AND UPPER BOUNDS ON COUPLING CONSTANTS

In this section we discuss the connection between the occurrence of zeros in propagators, poles in the Dyson irreducible (proper) vertex-parts, and upper bounds on the renormalized coupling constants. This is the original problem studied by Geshkenbein and Ioffe[16] and Meiman[15] who bounded coupling constants by the requirement that there be no poles in the proper vertex parts. We present no new limits in this section but rather concern ourselves with the question of whether or not there is physical significance to be attached to the appearance of vertex poles and propagator zeros.

Goebel and Sakita[24] have already pointed out by considerations based on potential models that a pole in the proper vertex part has no direct physical significance and therefore cannot be excluded by general arguments. We present here a further model in support of their argument and in answer to a subsequent communication from Geshkenbein and Ioffe.[18] This is a generalized Lee model with an unstable W particle in addition to the stable V particle both of which couple to the N and θ. It contains a pole in the Dyson irreducible vertex Γ, and a zero in the V particle propagator, but no pole in the scattering amplitude and, hence, no direct observable consequences. Before developing this model let us first review briefly the Geshkenbein-Ioffe argument.

We consider the propagator of a boson with a Källén-Lehmann representation of the form

$$D(x) = \frac{1}{x_p - x} + \int_1^\infty dx' \, \frac{\rho(x')}{x' - x} \, . \tag{52}$$

Here we have introduced dimensionless variables as in previous sections. x_p is the position of the pole.

For $x < x_p$, both terms are positive so that there can be no zero in this region. If $\rho(x)$ does not vanish for $x > 1$ there will also be no zeros in the continuum. We assume this to be the case; i.e., there is always at least one open channel above threshold (no CDD poles). When $x_p < x < 1$ the pole term is negative and the integral is positive leading to a possible zero as illustrated in Fig. 3. The spectral representation Eq. (52) allows at most one such zero. In fact, the necessary and sufficient condition for a zero to exist for $x_p < x < 1$ is

$$\int_1^\infty dx \, \frac{\rho(x)}{x - 1} \geq \frac{1}{1 - x_p} \, . \tag{53}$$

We now relate the existence of a zero to the value of the coupling constant. In the sum of states in $\rho(x)$ we keep only the term corresponding to a two particle state which shall be the state of lowest mass in the sum. Then we have

$$\rho(x) \geq \rho^{(2)}(x) \equiv g^2 p(x) |F(x)|^2 \ , \tag{54}$$

where g^2 measures the strength of the coupling to the two particle state, $p(x)$ is a kinematical factor, and $F(x)$ is the form factor normalized to unity at x_p and assumed to be analytic except for a cut starting at $x = 1$.

Using the inequality in Eq. (53), we see that if

$$(1 - x_p) \int_1^\infty dx \, \frac{\rho^{(2)}(x)}{x - 1} \geq 1 \quad ,$$

it follows from $\rho(x) \geq \rho^{(2)}(x)$ that

$$(1 - x_p) \int_1^\infty dx \, \frac{\rho(x)}{x - 1} \geq 1 \quad ; \tag{55}$$

hence there will be a zero in $D(x)$.

Introducing Eq. (54) into Eq. (55) we see that if $g^2\Omega \geq 1$, where we define

$$\Omega \equiv (1 - x_p) \int_1^\infty p(x) \, \frac{dx}{x - 1} \, |F(x)|^2$$

there will then be a zero in the propagator. Furthermore, if Ω has a minimum $\Omega_{min} > 0$, and if $g^2\Omega_{min} \geq 1$ there will be a zero. We must require then that

$$g^2 \leq \frac{1}{\Omega_{min}} \tag{56}$$

in order to avoid the necessary occurrence of a zero in $D(x)$.

Goshkenbein and Ioffe[16] have shown that if there are no poles in $\Gamma(x)$ where

$$\Gamma(x) = \frac{F(x)}{(x - x_p)\, D(x)} \tag{57}$$

is the proper vertex, then the coupling constant must satisfy the inequality

$$g^2 \leq \frac{1}{I_{min}} \tag{58}$$

where, in our notation,

$$I = \int_1^\infty dx\, p(x) \left| \frac{F(x)}{(x - x_p)\, D(x)} \right|^2 \tag{59}$$

Their result is logically equivalent to the statement that if

$$g^2 \geq \frac{1}{I_{min}} \tag{60}$$

then $\Gamma(x)$ has a pole. However, it follows directly from Eq. (A.2) of the Appendix that

$$\Omega_{min} = I_{min} \quad . \tag{61}$$

Thus we have shown that for $g^2 \geq g_c^2$, where $g_c^2 \equiv \Omega_{min}^{-1} = I_{min}^{-1}$, $D(x)$ develops a zero and $\Gamma(x)$ develops a pole. The crucial assumption required to bound the coupling constant is the absence of a zero in $D(x)$ and/or the absence of a pole in $\Gamma(x)$.

Even if there is a zero of $D(x)$ at $x = x_o$ we can still bound g^2 in terms of the position of the zero as shown by Geshkenbein and Ioffe. Let us assume that $D(x)$ has a zero at $x = x_o$ so that

$$\frac{1}{x_o - x_p} = \int_1^\infty \frac{\rho(x)dx}{x - x_o} . \tag{62}$$

Keeping only the two particle contribution to $\rho(x)$ leads to the inequality

$$g^2(x_o - x_p) \int_1^\infty dx \frac{p(x)}{x - x_o} \left|F(x)\right|^2 \leq 1 . \tag{63}$$

Using the result of Eq. (A.2) of the Appendix we have

$$\left[\int_1^\infty dx \frac{p(x)}{x - x_o} \left|F(x)\right|^2\right]_{min} = \frac{1}{\left[(1 - x_o)^{\frac{1}{2}} + (1 - x_p)^{\frac{1}{2}}\right]^2}\left[\int_1^\infty dx\, p(x) \left|F(x)\right|^2\right]_{min} \tag{64}$$

so that we have

$$g^2 \frac{(x_o - x_p)}{\left[(1 - x_p)^{\frac{1}{2}} + (1 - x_o)^{\frac{1}{2}}\right]^2} \left[\int_1^\infty dx\ p(x) \left|F(x)\right|^2\right]_{min} \leq 1 . \tag{65}$$

If $x \geq 1$ so that there is no zero we obtain our previous result that $g^2\Omega_{min} \leq 1$. As x_o approaches x_p, the bound on g^2 approaches infinity. Since, in general, x_o can be anywhere in the range $x_p \leq x_o \leq 1$ no useful bound is obtained.

This result was constructed with no further assumptions on the form factor than that it is analytic in the cut plane with the branch point at $x = 1$, and with $F(x_p) = 1$, while $x_p < 1$. If we make the additional assumption that $\Gamma(x)$

does not have a pole at the zero of $D(x)$ but has the very same analyticity properties assigned above to $F(x)$, it follows from Eq. (57) that $F(x)$ must have a zero at $x = x_o$.

Then we can obtain a stronger bound by writing

$$F(x) = \frac{x - x_o}{x_p - x_o} \tilde{F}(x) \tag{66}$$

where $\tilde{F}(x)$ has no pole at x_o. We then obtain the inequality

$$g^2 \frac{\left[(1 - x_p)^{\frac{1}{2}} + (1 - x_o)^{\frac{1}{2}}\right]^2}{(x_o - x_p)} \left[\int_1^{\infty} dx\; p(x)\; \left|\tilde{F}(x)\right|^2\right]_{min} \leq 1. \tag{67}$$

For arbitrary x_o between x_p and 1 we again obtain the result that $g^2\Omega_{min} \leq 1$. With this assumption that $F(x)$ has a zero at $x = x_o$ and $\Gamma(x)$ has no pole where $D(x)$ has a zero the bound on g^2 becomes stronger as x_o approaches x_p. This is the case considered by Geshkenbein and Ioffe in Reference 17.

We now argue that there is no compelling physical argument in support of the bound Eq. (58) by considering a generalized Lee model with two fields ψ_V and ψ_W, representing fermions with the same quantum numbers, in addition to the N and θ to which they couple. The Hamiltonian is written as

$$H = M_{V_o}\, \psi_V^\dagger\, \psi_V + M_{W_o}\, \psi_W^\dagger\, \psi_W + \int \frac{d^3k}{(2\pi)^3}\, \omega_k\, a_k^\dagger\, a_k + B(\psi_V^\dagger\, \psi_W + \psi_W^\dagger\, \psi_V)$$

$$+ (\lambda_{V_o}\, \psi_V^\dagger + \lambda_{W_o}\, \psi_W^\dagger)\, \psi_N A + (\lambda_{V_o}\, \psi_V + \lambda_{W_o}\, \psi_W)\, \psi_N^\dagger A^\dagger\ . \tag{68}$$

We have set the mass of the N particle to zero for simplicity and the θ field is written

$$A = \int \frac{d^3k}{(2\pi)^{3/2}}\, \frac{u(k)}{(2\omega_k)^{\frac{1}{2}}}\, a_k\ . \tag{69}$$

We assume the commutation rules

$$\left\{\psi_V^+,\psi_V\right\} = 1, \quad \left\{\psi_W^+,\psi_W\right\} = 1, \quad \left\{\psi_W^+,\psi_V\right\} = 0$$

and all other anticommutators are zero. The a_k satisfy the usual canonical commutation rules $\left[a_k\ ,\ a_{k'}^\dagger\right] = \delta(k - k')$. We further restrict the parameters in the Hamiltonian so that there is only one stable single particle state denoted by $|V>$ plus the continuum of $N\theta$ scattering states. The W field introduces an unstable particle resonance and is of importance here because the mass operator now becomes an infinite series of terms as illustrated in Fig. 4 instead of a single term as in the Lee model, and we therefore have the possibilities of a pole in $\Gamma(x)$ and a zero in $D(x)$.

Furthermore we impose the asymptotic conditions that

$$< 0\left|\psi_V\right|V > = Z_V = \text{constant}$$

and

$$< 0\left|\psi_W\right|V > = 0. \tag{70}$$

This requires that only the ψ_V field will asymptotically generate a stable V state.

We now define the V-propagator by

$$\tilde{D}_V(t - t') \equiv < 0\left|T\left(\psi_V(t)\ \psi_V^+(t')\right)\right|0 > \tag{71}$$

and the Dyson vertex and scattering amplitude in the conventional manner. A direct summation of the graphical series gives for the Fourier transforms of the propagator, proper vertex and transition amplitude

$$D_V^{-1}(\omega) = Z_V^{-1}(\omega - M_V) \left\{ 1 + \lambda_V^2(\omega - M_V) \; \Sigma_2(\omega, M_V, M_V) - \frac{\lambda_{W_0}^2 \; \Sigma_1^2(\omega, M_V)}{(\omega - \omega_0)\left[1 + \lambda_{W_0}^2 \; \Sigma_1(\omega, \omega_0)\right]} \right\} \tag{72}$$

$$\Gamma(\omega) = Z_V^{-\frac{1}{2}} \lambda_V \left[1 - \frac{\lambda_{W_0}^2 (\omega - M_V) \; \Sigma_1(\omega, M_V)}{(\omega - \omega_0)\left[1 + \lambda_{W_0}^2 \; \Sigma_1(\omega, \omega_0)\right]} \right] \tag{73}$$

$$T(\omega) = \Gamma(\omega) \; D_V(\omega) \; \Gamma(\omega) + \frac{\lambda_{W_0}^2}{(\omega - \omega_0)\left[1 + \lambda_{W_0}^2 \; \Sigma_1(\omega, \omega_0)\right]} \tag{74}$$

where M_{W_0} has been eliminated in terms of ω_0 which is the position of the zero in $D_V(\omega)$ and the pole in $\Gamma(\omega)$, M_{V_0} has been eliminated in terms of the stable particle mass M_V, and B was determined by the asymptotic condition $< 0 | \psi_W | V > = 0$. The residue of the pole at $\omega = M_V$ of the scattering amplitude is defined to be λ_V^2 and is related to the bare coupling constant by

$$\lambda_V^2 = \frac{\lambda_{V_0}^2}{1 + \lambda_{V_0}^2 \; \Sigma_1(M_V, M_V)} \tag{75}$$

We have also introduced

$$\Sigma_1(\omega_1, \omega_2) \equiv \frac{1}{4\pi^2} \int \frac{k u^2(\omega) \, d\omega}{(\omega - \omega_1)(\omega - \omega_2)}$$

$$\Sigma_2(\omega_1, \omega_2, \omega_3) \equiv \frac{1}{4\pi^2} \int \frac{k u^2(\omega) \, d\omega}{(\omega - \omega_1)(\omega - \omega_2)(\omega - \omega_3)} \tag{76}$$

which are positive for $\omega < M_t$. We assume that the cutoff $u(\omega)$ falls off sufficiently rapidly for large ω to make $\Sigma_1(\omega_1,\omega_2)$ convergent and the theory finite at each step. The coupling constant λ_V^2 is restricted by its definition and the requirement that $\lambda_{V_o}^2 \geq 0$ to satisfy

$$0 \leq \lambda_V^2 \leq \frac{1}{\Sigma_1(M_V,M_V)} \tag{77}$$

It can be shown that $\lambda_{W_o}^2$ and λ_V^2 can be chosen so that there is only one stable state. We have already assumed this in our discussion since the physical V state and the $N\theta$ continuum were taken to form a complete spectrum of states. Consistency of this assumption is then established by showing that these states do in fact satisfy the completeness condition

$$|V><V| + \sum_k |N\theta_k><N\theta_k| = 1 \tag{78}$$

with appropriate choice of $\lambda_{W_o}^2$ and of $\lambda_{V_o}^2$, or λ_V^2 satisfying restriction Eq. (77). More intuitively we see this by observing that the V propagator Eq. (72) has only one pole at the physical mass M_V of the stable V particle, and that the W propagator

$$<0\left| T\left(\psi_W(t),\ \psi_W^\dagger(t')\right)\right|0>$$

and the "mixing" off diagonal propagator

$$<0\left| T\left(\psi_V(t),\ \psi_W^\dagger(t')\right)\right|0>$$

have no poles. According to Eq. (72) this condition is satisfied if

$$\left\{\frac{\lambda_{V_o}^2\,\Sigma_1(M_t,M_V)}{1+\lambda_{V_o}^2\,\Sigma_1(M_t,M_V)}\right\}\left\{\frac{\lambda_{W_o}^2\,\Sigma_1(M_t,\omega_o)}{1+\lambda_{W_o}^2\,\Sigma_1(M_t,\omega_o)}\right\}\left\{\frac{(M_t-M_V)\;\Sigma_1(M_t,M_V)}{(M_t-\omega_o)\;\Sigma_1(M_t,\omega_o)}\right\}\geq 1\,. \tag{79}$$

Since the first two factors are arbitrarily close to unity for large values of $\lambda_{V_o}^2$ and $\lambda_{W_o}^2$, and the third factor is larger than unity for $M_t > \omega_o > M_V$ according to the defining Eq. (76) for Σ_1 we see that our model with one bound state is consistent. We notice, however, that D_V has a zero and Γ a pole located at ω_o between the pole and the continuum of D_V. These are not present in the scattering amplitude, however, because the pole at $\omega = \omega_o$ in the first term of Eq. (74), i.e.,

$$\Gamma(\omega)\;D_V(\omega)\;\Gamma(\omega)\;\underset{\omega\to\omega_o}{\rightarrow}\;\frac{-\lambda_{W_o}^2}{(\omega-\omega_o)\left[1+\lambda_{W_o}^2\;\Sigma_1(\omega_o,\omega_o)\right]}$$

is cancelled by the pole of the last term leading to a finite $T(\omega_o)$. There is thus no observable effect of the zero in D_V or pole in Γ. Hence there are no physical grounds for ruling out the possibility of zeros in propagators or poles in vertex functions and so the techniques used in this paper lead to no bounds on coupling constants.

In conclusion, we note that if we apply the method of Meiman to bound λ_V^2 in our VW model we obtain the inequality

$$\lambda_V^2 \leq \frac{\left[(M_t-M_V)^{\frac{1}{2}}+(M_t-\omega_o)^{\frac{1}{2}}\right]^2}{\omega_o-M_V}\;R\;\frac{1}{\Sigma_1(M_V,M_V)} \tag{80}$$

where

$$R \equiv \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \, \frac{ku^2(\omega)}{2\pi(\omega - M_V)^2} \, \frac{d\omega}{d\theta}}{\exp \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \, \ln\left[\frac{ku^2(\omega)}{2\pi(\omega - M_V)^2} \, \frac{d\omega}{d\theta}\right]} \tag{81}$$

The inequality of the arithmetic and geometric means implies that

$$R \geq 1$$

If we do not assume that the position of the zero in $D_V(\omega)$ is known but merely that $M_V \leq \omega_o \leq M_t$ then the factor $(\omega_o - M_V)^{-1}$ may be infinite and we obtain no bound. If there were no zero so that $\omega_o \geq M_t$ then we would obtain

$$\lambda_V^2 \leq R \, \frac{1}{\Sigma_1(M_V, M_V)} \tag{82}$$

which by Eq. (81) is consistent with the known bound in λ_V^2 that $\lambda_V^2 \Sigma_1(M_V, M_V) \leq 1$.

An alteration of the VW model to one in which both the V and the W appear as stable particles, as studied by Srivastava[25], no longer yields a zero in D_V or a pole in Γ. This model gives the same equations discussed in a recent paper by Geshkenbein and Ioffe[18] who appealed to this result to support their coupling constant limit and to refute the earlier criticism of Goebel and Sakita.[24] The point is simply that one has additional constraints upon the coupling constants and mass parameters λ_{V_o}, λ_{W_o}, B, M_{V_o} in order to make the <u>two stable</u> V and W particle states mutually orthogonal. The construction of states and propagators in this case has been given by Srivastava as well as by Geshkenbein and Ioffe and we do not repeat it here. The resulting model is thus too restrictive to enable any general conclusions to be drawn.

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APPENDIX

In this appendix we give a simplified resume of the mathematical procedures introduced by Meiman for constructing the limits obtained in this paper.

We prove the following theorem:

Given an integral of the form

$$I = \frac{1}{\pi} \int_{1}^{\infty} dx(x + a_1)^{\alpha_1} (x + a_2)^{\alpha_2} \ldots (x + a_n)^{\alpha_n} |G(x)|^2 \tag{A.1}$$

satisfying:

(1) I exists

(2) the integrand is positive

(3) $G(x)$ is a function which is

(i) analytic in x except for a cut from 1 to ∞

(ii) unity at $x = c$ (c real and < 1)

(iii) bounded at ∞ by some power of $x^{\frac{1}{2}}$

(iv) non-zero on the cut except at a finite number of discrete points,

then

$$I \geq 4\lambda^2 \left[\lambda + (1 + a_1)^{\frac{1}{2}}\right]^{2\alpha_1} \left[\lambda + (1 + a_2)^{\frac{1}{2}}\right]^{2\alpha_2} \ldots \left[\lambda + (1 + a_n)^{\frac{1}{2}}\right]^{2\alpha_n} \tag{A.2}$$

with

$$\lambda = (1 - c)^{\frac{1}{2}} \tag{A.3}$$

Proof: In terms of the variables defined by

$$z = -\frac{t - i}{t + i}$$

$$\tan \theta/2 = t = \left(\frac{x - 1}{1 - c}\right)^{\frac{1}{2}} \tag{A.4}$$

I may be written

$$I = \frac{1 - c}{\pi} \int_0^{\pi} d\theta\, t(1 + t^2) \left[1 + a_1 + (1 - c)t^2\right]^{\alpha_1} \cdots$$

$$\cdots \left[1 + a_n + (1 - c)t^2\right]^{\alpha_n} \left|G(e^{i\theta})\right|^2 \quad . \tag{A.5}$$

Equation (A.4) defines a mapping of the cut x-plane into a unit circle as shown in Fig. 1.

Setting

$$f(\theta) = |t|(1 + t^2) \left[1 + a_1 + (1 - c)t^2\right]^{\alpha_1} \cdots \left[1 + a_n + (1 - c)t^2\right]^{\alpha_n}$$

and

$$g(\theta) = \left|G(e^{i\theta})\right|^2 \quad , \tag{A.6}$$

the inequality of the arithmetic and geometric means[26] gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(\theta)g(\theta)]d\theta \geq \exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \log\, [f(\theta)g(\theta)]d\theta\right] \quad .$$

So

$$I = \frac{\lambda^2}{2\pi} \int_{-\pi}^{\pi} [f(\theta)g(\theta)]d\theta \geq \lambda^2 I_1 I_2 \tag{A.7}$$

with

$$I_1 = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \log f(\theta)\right\}$$

$$I_2 = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \log|G(e^{i\theta})|^2\right\}$$

$$= \exp\left\{\frac{1}{\pi}\,\mathrm{Re}\int_{-\pi}^{\pi} d\theta \log G(e^{i\theta})\right\} \quad . \tag{A.8}$$

Now

$$\log I_2 = 2\mathrm{Re}\left\{\frac{1}{2\pi i}\oint_{\text{unit}\ \odot} \frac{dz}{z}\log G(z)\right\} \quad . \tag{A.9}$$

If $G(z)$ has no zeros within the circle, then its known properties and Cauchy's theorem imply that

$$I_2 = \exp\left[2\mathrm{Re}\log G(0)\right] = 1 \quad . \tag{A.10}$$

On the other hand, if $G(z)$ has zeros at points

$$z_1 = r_1 e^{i\varphi_1} \ , \ z_2 = r_n e^{i\varphi_2} \ \ldots \ z_n = r_n e^{i\varphi_n} \quad (r_j < 1)$$

we can write

$$G(z) = \frac{(z - r_1 e^{i\varphi_1})}{-r_1 e^{i\varphi_1}} \cdots \frac{(z - r_n e^{i\varphi_n})}{-r_n e^{i\varphi_n}}\,\tilde{G}(z) \tag{A.11}$$

where $\tilde{G}(z)$ has no zeros and

$$\tilde{G}(0) = 1 \quad . \tag{A.12}$$

Hence

$$\log I_2 = 2\mathrm{Re}\left\{\frac{1}{2\pi i}\int_{\text{unit }\odot} \frac{dz}{z}\log \tilde{G}(z)\right\}$$

$$+ 2\sum_{j=1}^{n}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \log\left|\frac{z - r_j e^{i\varphi_j}}{-r_j e^{i\varphi_j}}\right|\right\} \tag{A.13}$$

$$= 2\mathrm{Re}\log\frac{\tilde{G}(0)}{r_1 \,..\, r_n} \quad ,$$

or

$$I_2 = \frac{1}{r_1^2 r_2^2 \,..\, r_n^2} \quad . \tag{A.14}$$

So for all allowed $G(z)$ we have[27]

$$I_2 \geq 1 \quad . \tag{A.15}$$

Observing now that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dt}{1+t^2} \log\left|1 + a^2t^2\right| = \log\left|1 + a\right|$$

and

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dt}{1+t^2} \log t = 0 \quad , \tag{A.16}$$

I_1 may be readily evaluated to give

$$\log I_1 = 2\log\left\{2\left[\lambda + (1 + a_1)^{\frac{1}{2}}\right]^{\alpha_1} \ldots \left[\lambda + (1 + a_n)^{\frac{1}{2}}\right]^{\alpha_n}\right\} \quad . \tag{A.17}$$

So, finally,

$$I \geq 4\lambda^2 \left[\lambda + (1 + a_1)^{\frac{1}{2}}\right]^{2\alpha_1} \cdots \left[\lambda + (1 + a_n)^{\frac{1}{2}}\right]^{2\alpha_n} \tag{A.18}$$

which proves the result.

We note that it has not been necessary to make any assumptions on the existence of $\int f(\theta)d\theta$.

Let us now consider the case in which the cut in G extends below $x = 1$ to a point $x = \beta > c$. I_1 remains the same as previously but I_2 must now be evaluated by applying Cauchy's theorem to an integration around a contour Γ as shown in Fig. 5. Then

$$0 = \log G(0)$$

$$= \frac{1}{2\pi i} \int_{\Gamma} \frac{\log G(z)}{z}\, dz$$

$$= \frac{1}{2\pi i} \oint_{\text{unit } \odot} \frac{\log G(z)}{z}\, dz + \frac{1}{2\pi i} \int_{\beta}^{1} \frac{\text{disc} \log G(z)}{z}\, dz \quad , \tag{A.19}$$

where $\text{disc}\, w(z) = w(z + i\epsilon) - w(z - i\epsilon)$. (A.20)

So

$$I_2 = \exp\left\{ - \frac{1}{\pi i} \int_{\beta}^{1} \frac{\text{disc} \log G(z)}{z}\, dz \right\} \tag{A.21}$$

and

$$I_{min} = \exp\left\{ - \frac{1}{\pi i} \int_{\beta}^{1} \frac{\text{disc} \log G(z)}{z}\, dz \right\} 4\lambda^2 \left[\lambda + (1 + a_1)^{\frac{1}{2}}\right]^{2\alpha_1} \cdots \left[\lambda + (1 + a_n)^{\frac{1}{2}}\right]^{2\alpha_n} \tag{A.22}$$

The bound now depends on the unknown function disc $\log G(z)$ and so it cannot be fixed in the same precise manner as previously.

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FIGURE CAPTIONS

1. The transformation $z = -\dfrac{(x-1)^{\frac{1}{2}} - i}{(x-1)^{\frac{1}{2}} + i}$. The lettered points transform as shown.

2. A reduced graph for the process $\pi \to \rho\pi$.

3. Possible occurrence of a zero in the propagator $D(x)$ in Eq. (52).

4. The Dyson expansion for the mass operator Σ of the V field in the extended Lee model described in Sec. VI.

5. The integration contour for the integral I_2 as given in the Appendix when the form factor has a cut starting at a point $x = \beta$ to the left of the propagator cut.

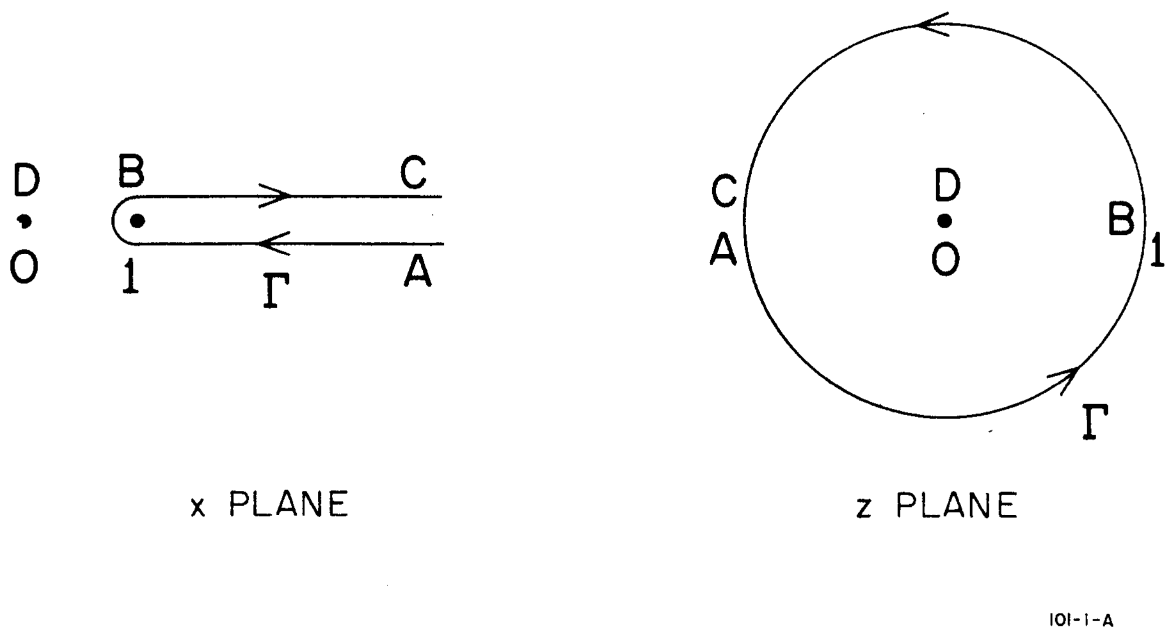


FIGURE 1

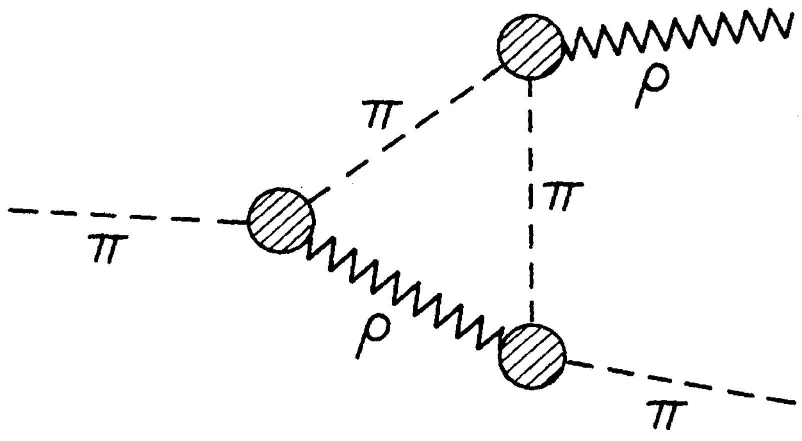


FIGURE 2

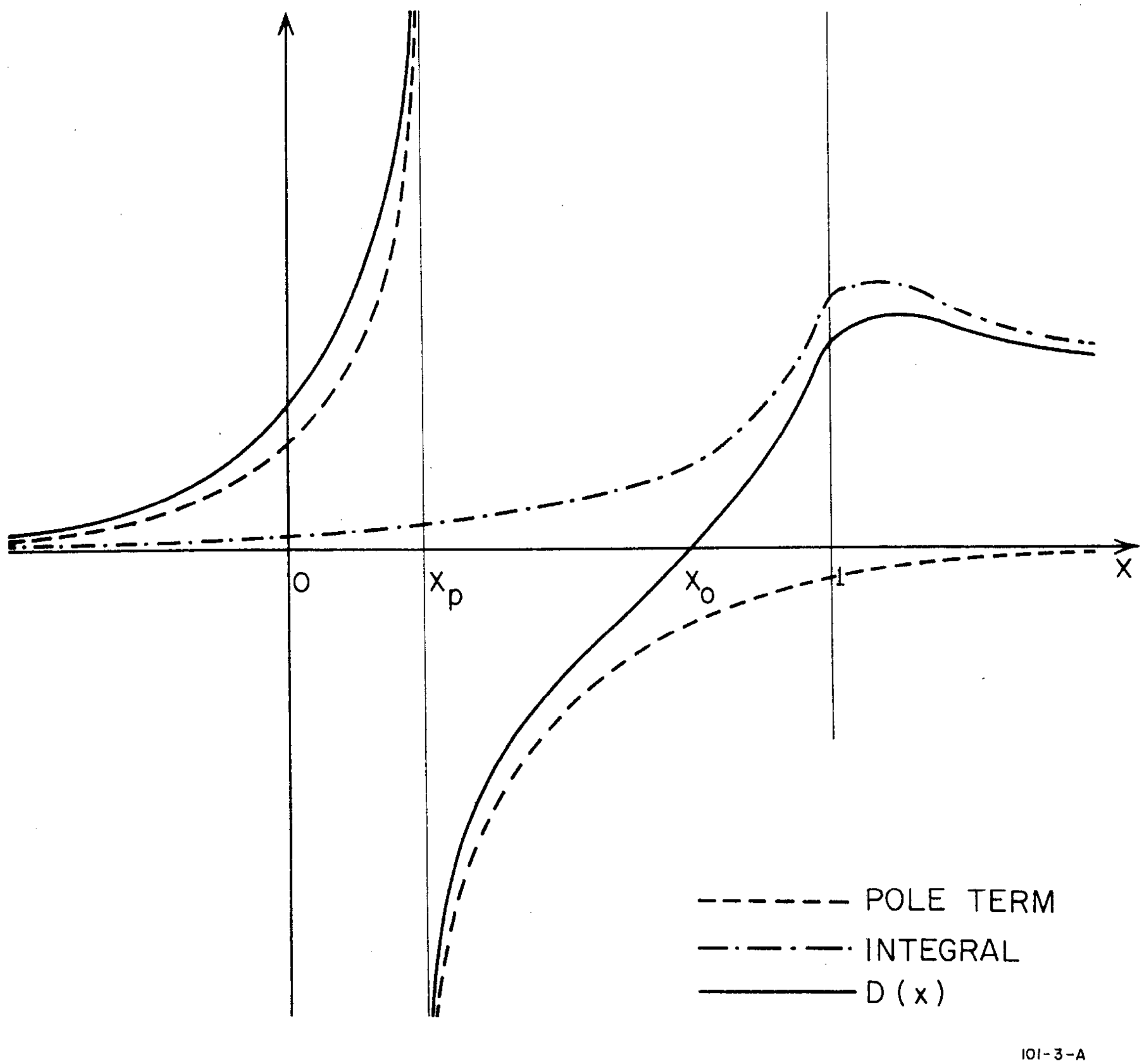


FIGURE 3

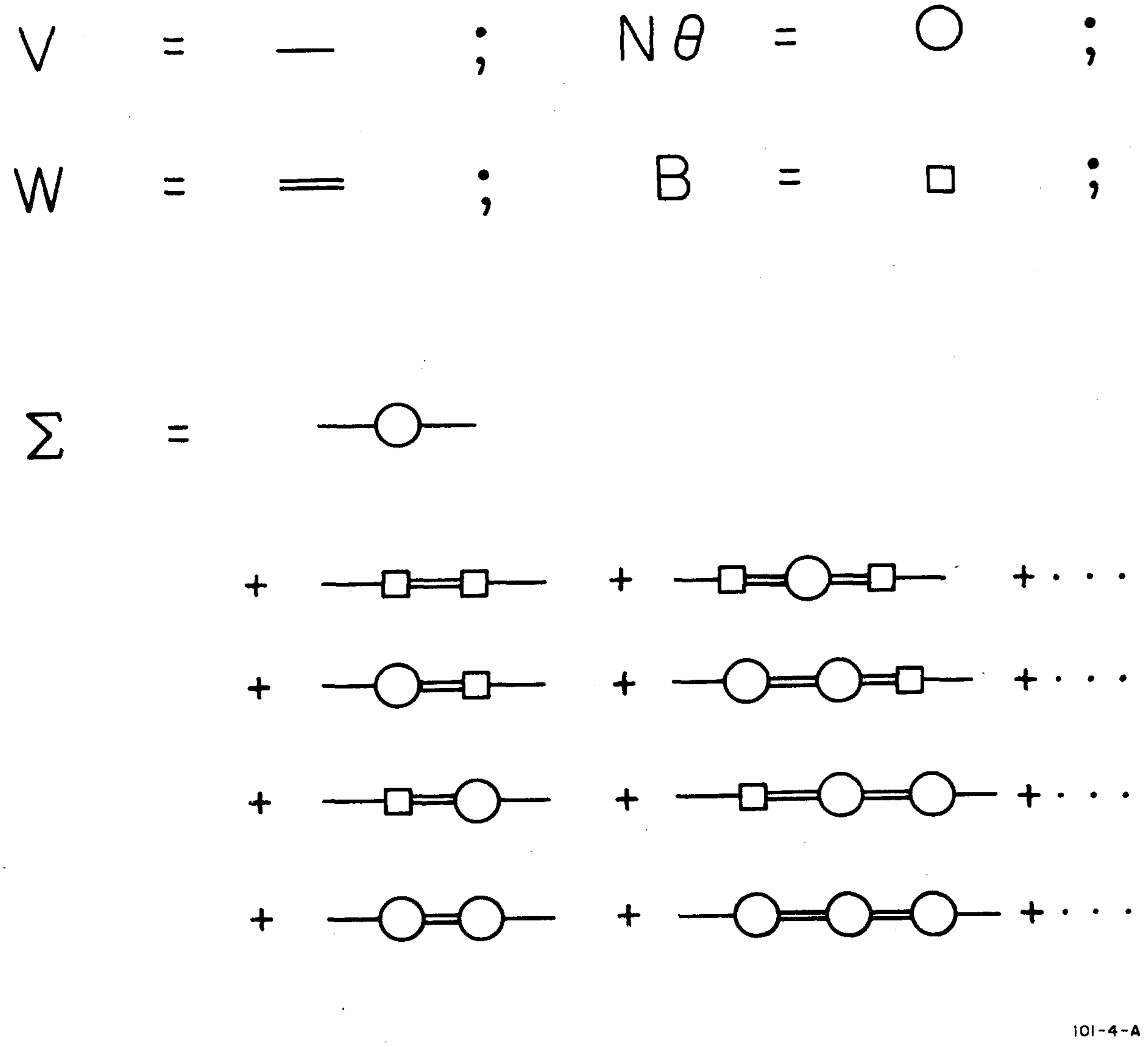
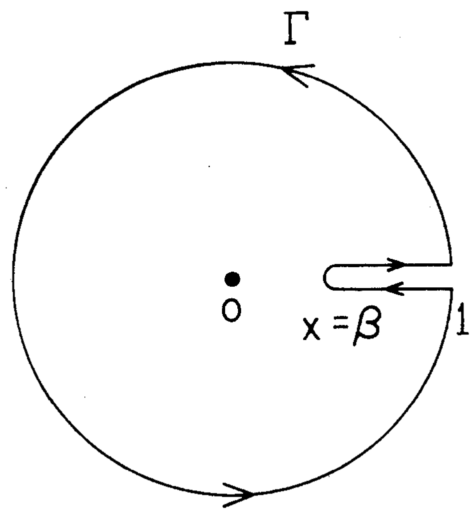


FIGURE 4



z PLANE

101-5-A

FIGURE 5