BNS Damping of Beam Breakup Instability*

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Abstract

We study BNS damping of the beam breakup instability in a simple model assuming a constant beam energy, flat bunch distribution, and a smooth transverse focusing. The model allows an analytic solution for a constant and linear wake functions. Scaling dimensionless parameters are derived and the beam dynamics is illustrated for the range of parameters relevant to the Stanford Linear Collider.

1 Introduction

Cumulative beam breakup (BBU) instability for a single bunch beam in a linear accelerator arises due to the interaction of the tail of an accelerating bunch with the transverse wakefield generated by the bunch head. It leads to, at best, a growth of the emittance of the beam and at worst, to total beam disruption. Theoretical studies of BBU has been performed in a number of papers [1-4].

One of the most effective techniques for alleviating BBU was proposed by Balakin, Novokhatsky and Smirnov (BNS) [5]. The BNS damping in linacs is accomplished by introducing a variation of the focusing strength along the beam by way of RF quadrupole or correlated energy variation along the beam. It has since been analyzed by a numbers of workers [6-9] and has proven invaluable for practical operation in a large collider [10].

For a given arbitrary wakefield function, in principle, BBU can be suppressed by setting the focusing strength along the bunch such that it completely compensates for the wakefield force. This criterion is often called an *autophasing condition* (see, e. g., [11]) and, in general, requires the focusing strength to be a nonlinear function of position inside the beam. However, for short bunches, typical for high energy linear accelerators, one usually deals with a linear "chirp" in betatron tune for which autophasing can only be achieved if the wake function would be a constant. On the other hand, the transverse wake function for short bunches can well be approximated by a linear dependence, which means that the autophasing condition cannot be satisfied in this case.

Computer codes are now available for beam dynamics simulations in linear accelerators [12], which treat in great detail many important features of the real machines including wakes, feedbacks, errors, etc. However, in our opinion, even relying on such powerful tool as a good simulation code, it is always desirable to have a relatively simple qualitative picture of the beam behavior which could be a guiding line in understanding more realistic situations. The main goal of this paper is to illustrate important elements of the beam dynamics associated with the BNS damping using a relatively simple model which, at the same time, contains essential physics elements of the phenomenon.

We chose a special case of a linear chirp in the focusing strength and two types of wakes: a constant wakefunction, and a wake that is a linear function of the longitudinal coordinate. Using a method previously developed by D. Whittum [9] we show that in these cases the solution can be found in the analytic form, and allows easy computation of the instantaneous beam profiles together with averaged beam displacement and effective emittance increase due to the filamentation. Also we find dimensionless beam parameters that give scaling laws for the development of the BBU. The model assumes constant beam energy, a flat particle distribution within the bunch, and linear chirp that does not vary along the linac. Some of these assumptions can actually be dropped [9] by the price of a slightly complicated math.

Our results partially overlap with previous studies reported in [13, 14]. We hope, however, that independent calculation of the beam dynamics in application to the current SLC program makes this work useful.

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2 General Equations

We start from equation of betatron oscillations for a constant energy beam under the influence of the transverse wakefield in a smooth focusing approximation,

$$\ddot{x}(s, z) + (k_0 + \Delta k(z))^2 x(s, z) = \frac{Ne^2}{E} \int_{-\infty}^z dz' x(s, z') w(z - z') \rho(z'), \quad (1)$$

where x is the beam offset in either horizontal or vertical direction, s is the path length along the beam trajectory, z is the longitudinal coordinate measured from the head of the bunch toward the tail, N is the number of particles in the bunch, E is the particle's energy, k_0 is the focusing strength, $\Delta k(z)$ accounts for the variation of the focusing along the bunch and is responsible for the BNS effect, $w(z)$ is the wake function, $\rho(z)$ is the density distribution in the bunch, and the dot stands for the derivative with respect to s . The effects of the wake and BNS focusing are usually small compared to the lattice focusing,

$$\Delta k, \frac{Ne^2 w}{Ek_0} \ll k_0. \quad (2)$$

This allows us to seek a solution of Eq. (1) in the form

$$x(s, z) = a(s, z) e^{ik_0 s}, \quad (3)$$

where the complex amplitude $a(s, z)$ is a slowly varying function of coordinate s . Substituting Eq. (3) into Eq. (1) and keeping only first order terms gives

$$\dot{a} - i\Delta k(z)a = -i \frac{Ne^2}{2k_0 E} \int_{-\infty}^z dz' a(s, z') w(z - z') \rho(z'). \quad (4)$$

In what follows we consider a bunch of length l_b having a constant density, $\rho = 1/l_b$ for $0 \leq z \leq l_b$, and introduce a new variable $u(s, z)$ instead of $a(s, z)$,

$$a(s, z) = u(s, z) e^{i\Delta k(z)s}. \quad (5)$$

Note that u and a differ only in a phase factor and have the same absolute values. From Eq. (4) we find that u satisfies the following equation,

$$\dot{u} = -i \frac{Ne^2}{2k_0 El_b} \int_{-\infty}^z dz' u(s, z') w(z - z') e^{i\alpha[\Delta k(z') - \Delta k(z)]}. \quad (6)$$

Now, assume a linear focusing chirp, $\Delta k(z) = \alpha z$, and make the Laplace transform of Eq. (6),

$$\begin{aligned} \frac{\partial \hat{u}(s, p)}{\partial s} &= -i \frac{Ne^2}{2k_0 El_b} \int_0^\infty dz e^{-pz} \int_{-\infty}^z dz' u(s, z') w(z - z') e^{i\alpha s(z' - z)} \\ &= -i \frac{Ne^2}{2k_0 El_b} \hat{u}(s, p) \hat{w}(p + i\alpha s), \end{aligned} \quad (7)$$

where the cap denotes the Laplace transform. For a wide range of wakes, $\hat{w}(p)$ can be found analytically and Eq. (7) can be integrated [9]. Then the inverse Laplace transform gives an analytic solution for the evolution of the amplitude of the beam oscillations. In the next two sections, we will solve Eq. (7) for constant and linear wakes, respectively, and find beam oscillations due to an initial offset and slope in the x -direction.

3 Constant Wake

For a constant wake, $w(z) = W_0$, we have

$$\hat{w}(p) = W_0 \int_0^\infty e^{-pz} dz = W_0/p, \quad (8)$$

and introducing the parameter r_0 ,

$$r_0 = \frac{Ne^2 W_0}{2k_0 El_b}, \quad (9)$$

which has dimension of m^{-2} , Eq. (7) reduces to

$$\frac{\partial \hat{u}}{\partial s} = \frac{-ir_0}{p + i\alpha s} \hat{u}. \quad (10)$$

Solution to Eq. (10) is

$$\hat{u}(s, p) = \hat{u}_0(p) \left(\frac{p}{p + i\alpha s} \right)^{r_0/\alpha}, \quad (11)$$

where $\hat{u}_0(p) = \hat{u}(0, p)$ is the initial value of $\hat{u}$. Making the inverse Laplace transform, we find that

$$u(s, z) = \int_0^\infty dz' u_0(z') G(s, z - z'), \quad (12)$$

where the Green's function G is given by the following formula

$$G(s, \zeta) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} dp e^{p\zeta} \left(\frac{p}{p + i\alpha s} \right)^{r_0/\alpha}, \quad (13)$$

with the contour of the integration located to the right of all poles of the integrand in the complex p -plane. Closing the integration path by an infinitely large semicircle in the right half plane, one can show that $G(s, \zeta) = 0$ for $\zeta < 0$, which means that the effect of the initial perturbation propagates only downstream, in agreement with the causality principle.

Consider first the case of autophasing, when $\alpha = r_0$. In this case, for $\zeta > 0$,

$$G(s, \zeta) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} dp e^{p\zeta} \left(1 - \frac{i\alpha s}{p + i\alpha s} \right) = \delta(\zeta) - i\alpha s e^{-i\alpha s \zeta}. \quad (14)$$

Assume an initial perturbation that is a combination of an offset x_0 and a tilt θ ,

$$u(s=0, z) = x_0 + \theta z. \quad (15)$$

Doing integration in Eq. (12) gives

$$u(s, z) = x_0 e^{-i\alpha s z} - \frac{i\theta}{\alpha s} (1 - e^{-i\alpha s z}). \quad (16)$$

The first term here shows that an initial offset causes betatron oscillations with a constant amplitude equal to the initial offset. This is a direct manifestation of the autophasing condition. The second term describes evolution of the initial slope. It is seen that this part of the oscillations decays inversely proportional to s and vanishes in the limit $s \rightarrow \infty$.

We now proceed to the case when the autophasing condition is not satisfied and limit our consideration by the initial conditions given by Eq. (15). For this initial condition, we have

$$\hat{u}_0(p) = \frac{x_0}{p} + \frac{\theta}{p^2}, \quad (17)$$

which gives for $u(s, z)$ the following formula

$$u(s, z) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} dp e^{pz} \left(\frac{x_0}{p} + \frac{\theta}{p^2} \right) \left(\frac{p}{p+i\alpha s} \right)^{r_0/\alpha}. \quad (18)$$

Evaluation of this integral in the asymptotic limit of large s , $s\alpha r_0 \gg 1$, is performed in Appendix A and is given by Eq. (32).

We see that the bunch is unstable for small α such that $r_0/\alpha > 1$, that is when the chirp is weaker than required by the autophasing condition. In this case, asymptotically, the largest term in Eq. (32) is the second one,

$$u(s, z) \approx \frac{x_0}{\pi} \Gamma\left(1 - \frac{r_0}{\alpha}\right) e^{-i\alpha s z} (-i)^{r_0/\alpha-1} \sin\left(\frac{\pi r_0}{\alpha}\right) (\alpha s z)^{r_0/\alpha-1}, \quad (19)$$

with the absolute value of the amplitude $|u| \propto z^{r_0/\alpha-1}$.

The beam is stable for $r_0/\alpha < 1$, with initial offset decaying as $s^{-\lambda}$, where $\lambda = \min(r_0/\alpha, 1 - r_0/\alpha)$. It is interesting to note that making the chirp very strong does not improve the stability, because in the limit $r_0/\alpha \rightarrow 0$ the asymptotic dependence of u is dominated by the first term in Eq. (32) and $|u| \propto z^{-r_0/\alpha}$. The optimal value of the ratio r_0/α ratio for the fastest damping is equal to 1/2 (two times stronger than the autophasing) with the initial offset falling off as $s^{-1/2}$.

For the initial tilt, given by the last two terms in Eq. (32) we see that it is unstable if $r_0/\alpha > 2$ and stable in the opposite case. The most rapid fall off of the initial tilt is achieved at autophasing, $r_0/\alpha = 1$, and results in the damping $\propto s^{-1}$.

4 Linear Wake

A linear function,

$$w(z) = W_1 z, \quad (20)$$

usually very well approximates the transverse wake for short bunches such as at the SLC and NLC. It has a Laplace transform $\hat{w}(p) = W/p^2$. Introducing a parameter r_1 ,

$$r_1 = \frac{N e^2 W_1}{2 k_0 E l_b}, \quad (21)$$

which has dimension of m^{-3} , Eq. (7) now takes the form,

$$\frac{\partial \hat{u}}{\partial s} = \frac{-ir_1}{(p+i\alpha s)^2} \hat{u}, \quad (22)$$

with the solution

$$\hat{u}(s, p) = \hat{u}_0(p) \exp\left[-\frac{ir_1 s}{p(p+i\alpha s)}\right]. \quad (23)$$

Making the inverse Laplace transform of Eq. (23), for the initial condition (12) we obtain

$$u(s, z) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} dp e^{pz} \left(\frac{x_0}{p} + \frac{\theta}{p^2} \right) \exp\left[-\frac{ir_1 s}{p(p+i\alpha s)}\right]. \quad (24)$$

Eq. (24) can be easily integrated numerically for different values of r_1 , α , s and z . It is convenient to introduce dimensionless parameters,

$$f = \frac{r_1 l_b}{\alpha}, \quad \bar{z} = \frac{z}{l_b}, \quad \bar{s} = s\alpha l_b, \quad (25)$$

and reduce Eq. (24) to

$$u(s, z) = \frac{x_0}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{dt}{t} \exp\left[t\bar{s}\bar{z} - \frac{if}{\bar{s}} \frac{1}{t(t+i)}\right] + \frac{\theta l_b}{2\pi i \bar{s}} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{dt}{t^2} \exp\left[t\bar{s}\bar{z} - \frac{if}{\bar{s}} \frac{1}{t(t+i)}\right], \quad (26)$$

where $t = p/s\alpha$.

For the first third of the SLC, taking the average energy $E = 7\text{GeV}$, $W_1 = 1.4\text{ V/pC/m/mm}^2$, $k_0 = 1/25\text{ m}^{-1}$, $l_b = 2\sqrt{3}\sigma_z = 3.5\text{ mm}$, $N = 3 \times 10^{10}$, $s = 1000\text{ m}$, the correlated energy spread $\sigma_\delta = 3\%$ and assuming a 90 degrees FODO lattice gives $f = 4$ and $\bar{s} = 8.4$. With these parameters in mind, we computed the beam oscillations for the values of f equal to 1, 2 and 5.

Figures 1–3 show the absolute value of the amplitude of the oscillations as a function of position inside the bunch at different distances s for the initial condition $u(s=0, z) = x_0$.

Figure 1: Absolute value of u along the beam for an initially displaced bunch for $f = 1$ and different values of $\bar{s}$. Dashed line - $\bar{s} = 1$, dotted line - $\bar{s} = 5$, dash-dotted line - $\bar{s} = 10$, solid line - $\bar{s} = 30$.

It is seen, that for $f = 5$, the amplitude of the oscillation in the tail grows to the values large compared with the initial offset before it begins to damp. For smaller values of f (1 and 2) the BNS damping effectively suppresses the instability. The same set of pictures for the initial condition $u(s = 0, z) = \theta z$ is shown in Figures 4-6.

Note that Figs. 1-6 show the amplitude of the betatron oscillations of the bunch. To demonstrate the relation between this amplitude and the beam shape at different betatron phases we plot several snapshots of the oscillating beam for different betatron phase. Fig. 7 shows snapshots corresponding to phases of 0, 90, 180 and 270 degrees for $f = 2$ and $\bar{s} = 10$. Fig. 8 shows 20 snapshots for equally spaced phases between 0 and 360 degrees. We see that the absolute value of the amplitude plotted by dash-dotted curve in Fig. 2 represents an envelope for the betatron oscillations at each location inside the bunch and give the maximum deviation of the particles as a function of z .

Asymptotical expressions for the function $u(s, z)$ is obtained in Appendix B and is given by Eq. (36).

One can also calculate the amplitude of the average offset of the beam, $\langle a \rangle$, for various initial conditions as a function of parameter $\bar{s}$. These are the quantities that will be measured by the beam position monitors in the linac. Using Eq. (5) we find

$$\begin{aligned} \langle a(s) \rangle &= \frac{1}{l_b} \int_0^{l_b} dz u(s, z) = \int_0^{l_b} d\bar{z} u(s, z) e^{i\bar{z}s} \\ &= \frac{x_0}{2\pi i \bar{s}} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{dt}{t(t+i)} \exp \left[\left(e^{(t+i)\bar{s}} - 1 \right) - \frac{if}{\bar{s}} \frac{1}{t(t+i)} \right] \end{aligned} \quad (27)$$

Figure 2: Absolute value of u along the beam for an initially displaced bunch for $f = 2$ and different values of $\bar{s}$. Dashed line - $\bar{s} = 1$, dotted line - $\bar{s} = 5$, dash-dotted line - $\bar{s} = 10$, solid line - $\bar{s} = 30$.

Figure 3: Absolute value of u along the beam for an initially displaced bunch for $f = 5$ and different values of $\bar{s}$. Dashed line - $\bar{s} = 1$, dotted line - $\bar{s} = 3$, dash-dotted line - $\bar{s} = 15$, dash-dot-dotted line - $\bar{s} = 60$, solid line - $\bar{s} = 200$.

Figure 4: Absolute value of u along the beam for an initially tilted bunch for $f = 1$ and different values of $\bar{s}$. Dashed line $-\bar{s} = 1$, dotted line $-\bar{s} = 5$, dash-dotted line $-\bar{s} = 10$, solid line $-\bar{s} = 30$.

Figure 5: Absolute value of u along the beam for an initially tilted bunch for $f = 2$ and different values of $\bar{s}$. Dashed line $-\bar{s} = 1$, dotted line $-\bar{s} = 5$, dash-dotted line $-\bar{s} = 10$, solid line $-\bar{s} = 30$.

Figure 6: Absolute value of u along the beam for an initially tilted bunch for $f = 5$ and different values of $\bar{s}$. Dashed line $-\bar{s} = 1$, dotted line $-\bar{s} = 3$, dash-dotted line $-\bar{s} = 7$, solid line $-\bar{s} = 60$.

Figure 7: Four snapshots of the beam for $f = 5$ and $\bar{s} = 10$.

Figure 8: 20 snapshots of the beam for $f = 5$ and $\bar{s} = 10$.

$$+ \frac{\theta l_b}{2\pi i \bar{s}^2} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{dt}{t^2(t+i)} \exp \left[\left(e^{(t+i)\bar{s}} - 1 \right) - \frac{if}{\bar{s}} \frac{1}{t(t+i)} \right]$$

The plots of the absolute value of this function is shown in Figs. 9 and 10 for the initial offset and tilt, respectively.

Finally, we calculate the contribution of the BBU to the effective emittance increase of the beam. Due to filamentation of the betatron oscillations, they add up in quadrature to the transverse size of the beam causing a dilution of the beam emittance. The effect can be estimated by calculating the averaged square of the beam deviation from the average offset:

$$\langle (x(s, z) - \langle x \rangle)^2 \rangle_{t.av.} = \frac{1}{2} \left(\int_0^1 d\bar{z} |u|^2 - \left| \int_0^1 d\bar{z} u \right|^2 \right) \quad (28)$$

where the superscript "t. av." indicated averaging over a period of the betatron oscillations. This quantity divided by the beta function gives the emittance dilution of the beam due to the BBU. Figs. 11 and 12 show the square root of the left hand side of Eq. (28) for the initial offset and tilt, respectively.

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Figure 9: Absolute value of $\langle a(s) \rangle$ for the initial offset $a = x_0$. Dashed line - $f = 1$, dotted line - $f = 2$, dash-dotted line - $f = 5$.

Figure 10: Absolute value of $\langle a(s) \rangle$ for the initial offset $a = \theta(z - 0.5l_b)$. Dashed line - $f = 1$, dotted line - $f = 2$, dash-dotted line - $f = 5$.

Figure 11: Plot of $\left(\langle (x(s, z) - \langle x \rangle)^2 \rangle_{l.av.}\right)^{1/2}$ for the initial offset $a = x_0$. Dashed line - $f = 1$, dotted line - $f = 2$, dash-dotted line - $f = 5$.

Figure 12: Plot of $\left(\langle (x(s, z) - \langle x \rangle)^2 \rangle_{l.av.}\right)^{1/2}$ for the initial offset $a = \theta(z - 0.5l_b)$. Dashed line - $f = 1$, dotted line - $f = 2$, dash-dotted line - $f = 5$.

A Evaluation of the Integral (18)

To estimate the integral (18) we assume that $\alpha s > 0$ and introduce a new variable $t = p\alpha s$. We have

$$u(s, z) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} dt e^{\alpha s t} \left(\frac{x_0}{t} + \frac{\theta}{\alpha s t^2} \right) \left(\frac{t}{t+i} \right)^{r_0/\alpha}. \quad (29)$$

This integral can be expressed in terms of the Whittaker function whose asymptotic expansions are known [15]. We will use the following asymptotic representation of the function

$$f(\zeta) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} dt e^{\zeta t} t^{\lambda-1} (t-d)^{\mu-1} \quad (30)$$

in the limit $\zeta \rightarrow \infty$ [16],

$$f(\zeta) \approx \frac{1}{\pi} \left(\Gamma(\lambda) (-d)^{\mu-1} \zeta^{-\lambda} \sin(\pi\lambda) + \Gamma(\mu) e^{d\zeta} d^{\lambda-1} \zeta^{-\mu} \sin(\pi\mu) \right), \quad (31)$$

where Γ is the gamma function. Applying Eq. (31) to the function (29) gives

$$\begin{aligned} u(s, z) \approx & \frac{x_0}{\pi} \Gamma\left(\frac{r_0}{\alpha}\right) i^{-r_0/\alpha} \sin\left(\frac{\pi r_0}{\alpha}\right) (\alpha s z)^{-r_0/\alpha} \\ & + \frac{x_0}{\pi} \Gamma\left(1 - \frac{r_0}{\alpha}\right) e^{-i\alpha s z} (-i)^{r_0/\alpha-1} \sin\left(\frac{\pi r_0}{\alpha}\right) (\alpha s z)^{r_0/\alpha-1} \\ & - \frac{\theta}{\pi \alpha s} \Gamma\left(\frac{r_0}{\alpha} - 1\right) i^{-r_0/\alpha} \sin\left(\frac{\pi r_0}{\alpha}\right) (\alpha s z)^{1-r_0/\alpha} \\ & - \frac{\theta}{\pi \alpha s} \Gamma\left(1 - \frac{r_0}{\alpha}\right) e^{-i\alpha s z} (-i)^{r_0/\alpha} \sin\left(\frac{\pi r_0}{\alpha}\right) (\alpha s z)^{r_0/\alpha-1}. \end{aligned} \quad (32)$$

B Asymptotic Evaluation of the Integral (24)

Asymptotically, for large s , Eq. (24) can be transformed into

$$\begin{aligned} u(s, z) = & \frac{1}{2\pi i} \int_{C_1} dp \left(\frac{x_0}{p} + \frac{\theta}{p^2} \right) \exp\left[pz - \frac{r_1}{\alpha p}\right] \\ & + \frac{1}{2\pi i} \int_{C_2} dp \left(\frac{x_0}{-i s \alpha} + \frac{\theta}{(-i s \alpha)^2} \right) \exp\left[pz + \frac{r_1}{\alpha(p + i s \alpha)}\right], \end{aligned} \quad (33)$$

where the contour C_1 encloses the point $p = 0$ in the complex plane, and the contour C_2 encloses the point $p = -i s \alpha$. In the second integral we change the variable $t = p + i s \alpha$

$$\frac{1}{2\pi i} e^{-i\alpha s z} \int_{C_1} dp \left(x_0 + \frac{i\theta}{s\alpha} \right) \exp\left[tz + \frac{r_1}{\alpha t}\right]. \quad (34)$$

Using identities

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{dt}{t} e^{\zeta t - k/t} = J_0(2\sqrt{k\zeta}), \quad \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{dt}{t} e^{\zeta t + k/t} = I_0(2\sqrt{k\zeta}), \quad (35)$$

we find

$$u(s, z) \approx x_0 J_0\left(2\sqrt{\frac{zr_1}{\alpha}}\right) + \theta \sqrt{\frac{z\alpha}{r_1}} J_1\left(2\sqrt{\frac{zr_1}{\alpha}}\right) + \frac{i}{s\alpha} \left(x_0 + \frac{i\theta}{s\alpha}\right) e^{-i\alpha s z} \sqrt{\frac{r_1}{z\alpha}} I_1\left(2\sqrt{\frac{zr_1}{\alpha}}\right). \quad (36)$$

References

- [1] W. K. H. Panofsky and M. Bander, *Rev. Sci. Instrum.*, **39**, 206 (1968).
- [2] V. K. Neil, L. S. Hall and R.K. Cooper, *Part. Accel.*, **9**, 213 (1979).
- [3] A. W. Chao, B. Richter, and C. Y. Yao, *Nucl. Instrum. Methods*, **178**, 1, (1980).
- [4] Y. Y. Lau, *Phys. Rev. Lett.*, **63**, 1141, (1989).
- [5] V. Balakin, A. Novohatsky, and V. Smirnov, in *Proceedings of the 12th International Conference on High Energy Accelerators*, Batavia, Illinois, 1983, edited by F. T. Cole and R. Donaldson (Fermilab, Batavia, IL, 1984), p. 119-120.
- [6] K. A. Thompson and R. D. Ruth, *Phys. Rev. D*, **41**, 964 (1990).
- [7] W. L. Spence, "BNS Damping-Autophasing and Discrete Focusing", in *Proceedings of the International Workshop on Emittance Preservation in Linear Colliders*, edited by J. Urakawa and K. Oide, KEK Conference Proceedings 93-13, (KEK, Tsukuba, 1993), pp. 510-513.
- [8] T. O. Raubenheimer and K. Kubo, *Nucl. Instr. and Meth. A*, **370**, 303 (1996).
- [9] D. Whittum, unpublished.
- [10] J. Seeman et al., *Proceedings of the 1993 IEEE Particle Accelerator Conference*, (IEEE, New York, 1994), p.3288.
- [11] A. W. Chao. *Physics of Collective Beam Instabilities in High Energy Accelerators*, (John Wiley & Sons, New York 1993), p. 142.
- [12] R. Assmann et al. LIAR - A Computer Program for the Modeling and Simulation of High Performance Linacs, SLAC/AP-103, 1997.
- [13] A. W. Chao, unpublished.
- [14] D. Chernin and A. Mondelli, *Particle Accelerators*, **24**, 177, (1989)
- [15] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, (McGraw-Hill, New-York, 1953) part 1, p. 612.
- [16] V. I. Smirnov, *Calculus*, v. 3, part 2, (Moscow, 1950) – in Russian.