Study of the energy gain and the beam loading of the detuned structure with a simple model

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Abstract

A circuit model for the longitudinal case from which to study the field pattern, energy gain and beam loading of a detuned structure is derived from Maxwell’s equations. The results obtained with the model are compared to numerical results of the code PROGON. The model gives reasonable scaling of the group velocity and voltage with geometric parameters. The energy gain and beam loading are compared and are shown to depend on the same factor. This is true even for periodic variations of the boundary. Finally, a way to find the shape of the rf pulse envelope for the beam loading compensation is suggested.

Introduction

The energy gain of a particle in a RF wave and the energy loss in a structure are intimately related because both processes are caused mostly by the interaction of a particle with the synchronous wave. If the main contribution to the energy gain and loss comes from interaction with a single EM mode, they are, obviously, related. This is true in two important cases: for a standing wave in an rf cavity, and for a single wave in a traveling wave structure. Any modification of the structure in this case changes energy gain and the beam loading simultaneously. The questions remain, however, how good is the approximation of a single mode, and is it possible to minimize the beam loading without deteriorating the energy gain. One scheme has been proposed recently and is considered below.

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To study the problem, we use the code PROGON\textsuperscript{[1]} and compare its results with a circuit model for the longitudinal case derived from Maxwell's equations. The numerical calculations are performed for two structures: the detuned 204 cell structure and a constant impedance 184 cell structure. The variations of the iris radius $a(n)$ and cavity radius $b(n)$ for the detuned structure are shown in Fig. 1. Both structures have the same synchronous frequency 11.4309 GHz. Matching couplers are added at the beginnings and ends of the structures to minimize reflection.

**Energy Gain**

The code PROGON calculates the longitudinal EM fields in each cell for a given stepwise variation of the beam pipe radius. This allows us to calculate the energy gain $V(N)$, shown in Fig. 2; stored energy $W(N)$, power flow $P(N)$ and group velocity, shown by solid lines in Fig. 3 as a function of cell number $N$ for a lossless detuned structure. Dependence of $v_g(N)$ is in a very good agreement with previous calculations\textsuperscript{[1]}. The small ripples in the curves of Fig. 3 are caused by the reflected wave. The ripple amplitude decreases with better matching of the output coupler (cf. Fig. 2). These results use $J^C = 24$ radial modes in the cavities and $J^W = 30$ radial modes in the irises. The results remain practically the same with further increases of the number of modes.

It could be expected that, for the fundamental pass-band, the main contribution is given by a propagating mode which is the result of coupling between the first radial modes in the cavities and which is, in this case, the only propagating mode in the structure. This is confirmed numerically: the function shown in Fig. 3 by dotted lines is calculated using only the propagating mode. The mode, however, was found using the full $24 \times 30$ matrices. Attempting to find $v_g$ using $J^C = 1$ and $J^W = 30$, the approximation used in the circuit model, gives results very different from those in Fig. 3. This means that evanescent fields in cavities are important for field matching and may change the coupling coefficients. Hence, a circuit model, to
be correct, should be supplied with coupling coefficients found independently, either numerically or phenomenologically. A single propagating mode is sufficient for the field description, provided the coupling coefficients are calculated correctly.

It is worthwhile, therefore, to compare numerical code with a simple analytic model. Such a model, which is similar to a circuit model for the transverse case\textsuperscript{[6]}, can be derived from Maxwell’s equations. This derivation is given in the Appendix (see Eq. A4).

The voltage $V^N$ in the $N$-th cavity satisfies the following equation:

$$
(k^2 - k_1^2 - \Delta^N + \Delta^N) V^N + \xi^{N+1} V^{N+1} + \xi^N V^{N-1} = -i R_H e^{ikz_N} \quad (1),
$$

where $k = \omega/c$ and $k_1$ is given similarly by the fundamental frequency $\omega_1 = \nu_1/b$ of a closed cavity with gap $g$ and radius $b$. Equation 5 in the Appendix defines $\Delta^N$ and $\xi^N$. The term $R_H$

$$
R_H = \frac{2\pi c \sqrt{g}}{c} \frac{\nu_1}{b^2} \frac{\sin(kg/2)}{k^2/2} e^{ik(g/2-s)} \quad (2)
$$

defines the excitation of the field by a point-like particle located at $z = s$ at the time $t = 0$.

For a periodic structure, the solution of the homogeneous equation is $V(N) = V(1)e^{iN\psi}$. This gives the field in the structure

$$
\tilde{E}_\omega(r, z) = V(N) \tilde{e}_1, \quad (3)
$$

where $\tilde{e}_1$ is the field of the first radial mode of a closed cavity and the phase advance $\psi$ is defined for the frequency $\omega = 2\pi f = kc$ by the dispersion relation

$$
k^2 = (k_1^2 + 2\Delta - 2\xi) + 2\xi(1 - \cos \psi). \quad (4)
$$

For a large number of cells in a structure with period $d$, Eq. 1 can be replaced by the
differential equation

\[
\frac{\partial}{\partial z}(\xi d^2) \frac{\partial V}{\partial z} + [k^2 - k_1^2 - 2\Delta(z) + 2\xi(z)]V(z) = -iR_k e^{ikz}.
\] (5)

This is the equation of a circuit model for the longitudinal case. It also describes the vibration of a string with variable elasticity. The equation can be derived from the Hamiltonian

\[
H(p(z), x(z), ct) = \int dz \left[ \frac{p^2}{2} + \frac{\xi d^2}{2} \left( \frac{dx}{dz} \right)^2 + \frac{k_0^2 x^2}{2} \right].
\] (6)

The Hamiltonian equations

\[
\frac{dx}{d(\xi d^2)} = p \quad \text{and} \quad \frac{d}{d(\xi d^2)} \left( \frac{dx}{dz} \right) - k_0^2 x
\] (7)

give the equation of motion for the Fourier harmonic \( x(\omega, z) \)

\[
\frac{d}{dz} \left( \frac{\xi d^2}{dz} \frac{dx}{dz} \right) + (k^2 - k_0^2) x = 0
\] (8)

which is equivalent to Eq. 5 where \( x(z) \) is replaced by the voltage \( V(z) \). Eq. 5 can be used to describe an aperiodic structure provided that cell to cell variation of the parameters is adiabatic.

The expression for the stored energy per cell \( W(z) \) follows from Eq. 6:

\[
W(z) = \int_z^{z+d} dz \left[ \frac{p^2}{2} + \frac{\xi d^2}{2} \left( \frac{dx}{dz} \right)^2 + \frac{k_0^2 x^2}{2} \right].
\] (9)

Variation of the stored energy with respect to \( t \) is related to the power flow \( P \) by the continuity equation

\[
\frac{dW}{d(\xi d^2)} + P = 0.
\] (10)

An explicit expression for \( P \) follows from Eqs. 9 and 7:

\[
P = -p\xi(z) d\frac{dx}{dz} = -\xi(z) \frac{dx}{dz} \frac{dx}{d(\xi d^2)}.
\] (11)
Equation 8, with an adiabatic variation of $\xi(z)$,

$$\left(\frac{\xi'}{\xi}\right)^2 \ll q^2, \quad \text{and} \quad \frac{\xi''}{\xi} \ll q^2$$

where $q(z)$ is defined by

$$k^2 = k_0^2 + \xi(z) \frac{d^2 q^2}{dz^2}, \quad (12)$$

is given by the WKB method:

$$x(z) = \sqrt{\frac{<P>}{2k\xi(z)q(z)}} e^{i \int_0^z d\xi q(z)} \quad (13)$$

Note that, for a periodic structure, the propagating constant $q$ is related to the phase advance Eq. 4:

$$qd = 2 \sin\left(\frac{\psi}{2}\right).$$

This relation is valid for a detuned structure with $\psi$ being a local phase advance (i.e. the phase advance of a periodic structure made out of cells with local parameters) provided that the cell parameters vary slowly along the structure.

The solution to Eq. 13 is normalized by the time average of the power flow $<P>$ which is constant along the structure, as it follows from Eq. 10. The time average of the stored energy can be obtained substituting the solution to Eq. 13 into Eq. 9:

$$<W> = \frac{\omega}{\xi q d} <P> . \quad (14)$$

This gives the group velocity $v_g$

$$\beta_g = \frac{v_g}{c} = \frac{d <P>}{c <W>} = \frac{\xi q d^2}{k}. \quad (15)$$

The group velocity defined in this way is the same as that defined from the dispersion relation, Eq. 13,

$$\beta_g = \frac{dk}{dq}.$$

Note that the group velocity is related to the coupling coefficient $\xi$ and, hence, to the width of the pass-band. For a synchronous wave $q = k$ and $\beta_g^* = \xi d^2$. 

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The amplitude of the voltage $V(z)$ (cf. Eqs. 5 and 13) varies along the structure and is inversely proportional to $v_g^{-1/2}$:

$$V(z) = \sqrt{\frac{<P> d^2}{2k^2 \beta_g^2}} e^{i \int_0^z d\zeta(\zeta)}.$$  

Equation (16)

The coupling coefficient can be found by perturbation theory for small openings, as shown in the Appendix Eq. 15. Equation A5 gives the following dependence of the group velocity on the cell parameters, the ratio $p = a/b$ of the radii of an iris $a$ and a cavity $b$, and the iris thickness $l$:

$$\beta_g^s = \frac{2\nu_1^2}{\pi g k a} \frac{d}{\nu_1 p} J_0^2(\nu_1 p) \sum_{m} \frac{\mu_m^N}{\sinh \alpha_m^N (\nu_m^2 - \nu_1^2 p^2)^2}.$$  

Equation (17)

The sum is performed over the radial modes of an iris where

$$\alpha_m = l \mu_m, \quad \mu_m^N = \sqrt{(\nu_m/a_N)^2 - k^2},$$

and $\nu_m$ are roots of the Bessel function $J_0(\nu_m) = 0$.

Note that the main factor in Eq. 17 is $(a/b)^4$, although for small $a$, scaling changes to $(a/b)^3$. Figure 4 shows the dependence of the group velocity on Helm's parameter $F = 0.1(f/\text{GHz})(a/cm)$. The solid line calculated according to Eq. 17 deviates from the exact Helm's numeric result[6] for large iris radii $a$.

Fig. 5 shows the energy gain along the structure multiplied by $\sqrt{v_g}$ with three different scalings of the group velocity: $v_g \propto (a/b)^4$, $v_g \propto (a/b)^3$, and as in Eq. 17. The latter scaling gives the best result: variation is of the order of 2%. Scaling $v_g \propto (a/b)^4$ is slightly worse, giving variation of 5%.

The total energy gain $\Delta E$ in the structure with $N_c$ cells is given by the integral

$$\Delta E = \int_0^{N_c d} E_0^z e^{-ikz} dz + c.c.,$$
or

\[ \Delta E = \sqrt{\frac{< P > d^2}{2k^2}} \int_0^{N_c d} \frac{dz}{\sqrt{\beta_g}} e^{i \int \delta_s [g(\xi) - k]} + c.c. \]  \hspace{1cm} (18)\\

If the rf frequency is the synchronous frequency \( k_s \), \( q(k_s, z) - k_s \), and

\[ \Delta E = \sqrt{\frac{2 < P > d^2}{k}} \int_0^{N_c d} \frac{dz}{\sqrt{\beta_g}}. \]  \hspace{1cm} (19)

For a constant impedance structure, the energy gain in Eq. 19 is constant along the structure (cf. Fig. 2). For \( k = k_s + \Delta k \),

\[ \Delta E = \sqrt{\frac{2 < P > d^2}{k}} \int_0^{N_c d} \frac{dz}{\sqrt{\beta_g}} \cos[\Delta k \int_0^z \frac{d\zeta}{\beta_g(\zeta)}], \]  \hspace{1cm} (20)

and the energy gain decreases for large detuning \( \Delta k > (c\tau_f)^{-1} \) where \( \tau_f \) is the filling time

\[ \tau_f = \int_0^{N_c d} \frac{d\zeta}{v_g^s(\zeta)}. \]  \hspace{1cm} (21)

Calculation with PROGON of the dependence of the energy gain \( \Delta E \) on frequency confirms this result (cf. Fig. 6).

Equations 3 and 16 describe the field in the detuned structure quite well. It should be noted, however, that this wave is not a synchronous wave if the latter is defined as a solution of the wave equation with \( e^{ikz} \) dependent on \( z \). The field given by Eqs. 3 and 16:

\[ E^z_\omega(r, z) = V(z) e^{i k z}, \]  \hspace{1cm} (21)

has, for a periodic array, the structure required by the Floquet theorem. The periodic function \( e_z(r, z) \) can be expanded in the Fourier series, giving

\[ E^z_\omega(r, z) = e^{i \omega z} \sum_n c_n(r) e^{2 \pi i n z / d}. \]

If \( q_\omega = k \), the zero harmonic \( n = 0 \) is the synchronous wave. It defines the energy
gain in an infinite structure while all spatial harmonics except \( n = 0 \) do not interact with the beam. The amplitude \( \epsilon_0 \) of the synchronous harmonic for a monopole mode is independent of \( r \). This follows from the wave equation for the synchronous component \( e^{ikz} \epsilon_0 \).

**Beam Loading**

Energy loss due to the longitudinal wake field is described by the inhomogeneous Eq. 5. The solution can be obtained with the Green’s function \( G(z, z') \) of the equation

\[
\frac{\partial}{\partial z} (\xi d^2) \frac{\partial G}{\partial z} + (k^2 - k_0^2) G(z, z') = 2i d^2 \delta(z - z')
\]  

(22)

which in the WKB approximation is

\[
G(z, z') = \frac{1}{\sqrt{\xi(z)q(k, z)\xi(z')q(k, z')}} e^{i\int_{z}^{z'} d\zeta q(k, \zeta)}.
\]  

(23)

The voltage excited in a structure by a particle is then

\[
V(z) = -\frac{1}{2d^2} \int_{0}^{N_c d} dz' G(z, z') R_H(z') e^{ikz'}.
\]  

(24)

or, using Eqs. 22 and 15,

\[
V(z) = -\frac{R_H}{2d^2} \frac{e^{ikz}}{\sqrt{\beta_g(z)}} \int_{0}^{N_c d} \frac{dz'}{\sqrt{\beta_g(z')}} e^{-ik(z-z')+i\int_{z}^{z'} d\zeta q(k, \zeta)}.
\]  

(25)

The main contribution comes from \( z > z' \):

\[
V(z) = -\frac{R_H}{2d^2} \frac{e^{ikz}}{\sqrt{\beta_g(z)}} \int_{0}^{z} \frac{dz'}{\sqrt{\beta_g(z')}} e^{i\int_{z}^{z'} d\zeta q(k, \zeta)}.
\]  

(26)

The induced voltage at the synchronous frequency \( k = k_s = q(k_s, z) \) grows linearly with \( z \) for the constant impedance structure \( u_g = \text{const.} \), and the energy loss at this frequency is quadratic in the \( N_c d \). The energy gain Eq. 19 in this case is constant along the structure. This is confirmed by PROGON (see Figs. 7 and 8).
The energy loss of a particle is

\[ \kappa_L = \int_0^{N_c d} dz V(z) e^{-ikz} + c.c. \]

For a small detuning, \( k - k_s = \Delta k \), it takes the form

\[ \kappa_L \propto \left| \int_0^{N_c d} \frac{dz}{\sqrt{\beta_g(z)}} e^{i\Delta k \int_0^z \frac{dc}{\beta_g(c)}} \right|^2. \]  

The total loss, defined as the integral of \( \kappa_L \) over the frequencies, is dominated by the contribution of frequencies in the range \( \Delta \omega \sim 1/N_c d \) around the synchronous frequency. Therefore, the total loss is proportional to the total length \( N_c d \), corresponding to a constant loss per cell.

**Beam Loading Compensation**

The energy loss of the \( n \)-th bunch in the long train of bunches with bunch spacing \( s_B \) depends on the contribution of all previous bunches in the train:

\[ R_H \propto \sum_{m=1}^{n} a_m e^{-ik(s_n-s_m)}, \]

where \( a_m = 1/2 \) and \( a_m = 1 \) for \( m \neq n \). The sum has a sharp maximum at the resonance frequency, \( k_r = s_B = 2\pi l \) where \( l \) is an integer. Because \( s_B \) is a multiple of the rf wave length, \( s_B = m\lambda_{RF} \), the resonance frequency is \( k_r = (l/m)k_{RF} \). With the proper choice of bunch spacing, only one resonance frequency \( k_r = k_{RF} \) contributes to Eq. 27. If the RF frequency is equal to the synchronous frequency of a structure, the beam loading

\[ B_L \propto \left| \int_0^{N_c d} \frac{dz}{\sqrt{\beta_g(z)}} \right|^2 \]

is defined by the same factor as the energy gain Eq. 19. Therefore, any variations of the group velocity along the structure would affect both energy gain and beam loading.
It is worth noting that the beam loading in a long train of bunches driven by the resonance frequency, is quadratic in the length $N_c d$ of a structure, while the energy gain is linear. Since the cost of a collider decreases for large $N_c$, there is an optimum length for an accelerating section. Other factors favoring short sections are the transverse bunch size $\sigma_\perp$, which depends on the length $N_c d$ as $\sigma_\perp \propto \beta_\perp^{1/2} \propto \sqrt{N_c d}$, and tolerances.

The beam loading not only reduces the energy gain, but generates the bunch to bunch energy variation, the main adverse effect of beam loading. The shape of the rf pulse can be tailored to reduce this variation \cite{2} in such a way that the increase of the beam loading to the end of the bunch train is compensated by the increase of the rf amplitude in time. The model of the rf wave in the structure allows to optimize the shape of the rf pulse.

The dependence of the energy loss on the bunch number $n$, in a train of bunches with the bunch spacing $s_b$, due to beam loading can be approximated by a polynom

$$B_L(n) = \sum_{m=0}^{m_{max}} W_m n^m.$$

With a good accuracy, in a constant impedance structure without damping $m_{max} = 2^{\text{[3]}}$. Variation of the group velocity along the detuned structure, damping and dispersion may request a higher order polynom to describe the beam loading variation with $n$. The problem is to find the envelope $F(t)$ of an rf pulse

$$E_{rf} = F(t) \sin(\omega_0 t),$$

which would compensate the variation of $B_L(n)$. The time dependence of the envelope is given by the polynomial

$$F(t) = \sum_m F_m t^m.$$

where the coefficients $F_m$ have to be related to the coefficients $W_m$ which can be calculated for any given structure.
To find this relationship, we write the energy gain of a particle in the \(n\)-th bunch as

\[
\Delta E_n = \int \frac{d\omega}{2\pi} F(\omega) \int_0^z dz \Psi(z)e^{i \int_0^z d\zeta \gamma(\omega, \zeta) - i\omega t_n(z)}. \tag{32}
\]

Here

\[
\Psi(z) = \sqrt{\frac{v_g(0)}{v_g(z)}} e^{-\int_0^z d\zeta \gamma(\omega, \zeta)}, \tag{33}
\]

where \(F(\omega)\) is the Fourier component of the envelope \(F(t)\), \(\gamma\) describes the damping of the wave, and \(t_n(z) = z/c + t_d(n)\) is given by the time delay

\[
t_d(n) = t_d(1) + n s_B/c
\]

when the \(n\)-th bunch enters the accelerating structure.

The frequency width of the rf pulse is small. Neglecting the frequency dependences of the group velocity \(v_g\) and the damping \(\gamma\), Eq. 32 can be written as

\[
\Delta E_n = \frac{i}{2} e^{-i\omega t_d(n)} \int_0^z dz F[t_d(n) - \tau(z)] \sqrt{\frac{v_g(0)}{v_g(z)}} e^{-\int_0^z d\zeta \gamma(\zeta)}, \tag{34}
\]

where

\[
\tau(z) = \int_0^z \frac{d\zeta}{v_g(\zeta)}. \tag{35}
\]

The filling time \(\tau_F = \sigma(N_c d)\).

For a constant impedance structure without damping the envelope

\[
F(t) = F_0 + F_1 t, \quad t < t_0 \quad \text{and}
\]

\[
F(t) = F_{max} - F_0 + F_1 t_0, \quad t > t_0
\]

give the quadratic dependence of \(\Delta E_n\) on \(n\) for \(n < n_F = c\tau_F/s_B\), provided that \(t_d(1) > \text{maxim}(t_0, \tau_F)\).
For a polynomial pulse, Eq. 31 becomes

$$\Delta E_n = ie^{-i\omega_0 t_d(n)} \sum_{k=0}^{m_{max}} C_k n^k$$

where

$$C_k = \frac{1}{2} \left( \frac{s_b}{c} \right)^k \sum_{m=k}^{m_{max}} M_{m,k} F_m$$

and

$$M_{m,k} = \frac{m!}{k!(m-k)!} \int_0^{N_c \delta} dz \left[ t_d(1) - \frac{s_b}{c} - \tau(z) \right]^{m-k} \psi(z). \quad (36)$$

The beam loading compensation is $C_k = W_k$, for $k > 0$. Then coefficients $F_m$ can be found from the linear system of equations

$$\sum_{m=k}^{m_{max}} M_{k,m} F_m = 2W_k \left( \frac{c}{s_b} \right)^k. \quad (37)$$

Practically, it may be better to use Chebyshev's polynomials in the expansions of Eqs. 29 and 31.

**Periodic Variation of the Boundary**

The conclusion of the previous section, drawn from Eq. 28, should be true for a variation of the cell radii proposed recently by Gao [6]. To study this idea we find first the dependence of the propagating constant $q(k, \delta)$ on the the cell radius variation $\Delta b/b$ at the synchronous frequency.

A small change of $\Delta b/b$ of a single cell at $z = z_0$ produces a reflected wave upstream of the cell without changing the wave propagating downstream:

$$V(z) = e^{iq(z-z_0)} + Re^{-iqz}, \quad z < z_0$$

$$V(z) = \sqrt{1 - R^2} e^{iq(z-z_0)}, \quad z > z_0$$

where $r$ is reflection coefficient. If $|V(z) e^{-ikz}|$ vs $z$ is plotted, the reflected wave manifests itself as upstream ripples with amplitude determined by $\Delta b/b$. Figure 9.
obtained with PROGON, demonstrates that very clearly. The kick was placed in the 100-th cell of a 180 periodic array of cavities. Small ripples upstream are due to the imperfect matching by the output coupler.

If the radius $b$ is changed in all cavities simultaneously, the propagating constant at the synchronous frequency changes, since

$$q(k, z) = k + \frac{\Delta b}{b} \left( \frac{\partial q}{\partial b} \right).$$

(38)

The voltage in this case

$$V(z)e^{-ikz} \propto \cos \left( \frac{\Delta b}{b} \left( \frac{\partial q}{\partial b} \right) z \right)$$

oscillates along the structure (Fig. 10) produced by PROGON. This allows us to define the period of oscillations $N_{\Delta b}$ to be

$$N_{\Delta b} \frac{\Delta b}{b} \left( \frac{\partial q}{\partial b} \right) = 2\pi. $$

(39)

PROGON gives $N_{\Delta b} = 26$ for $(\Delta b/b)_0 = 0.0075$ for the 180-cell constant impedance structure.

For the adiabatic variation of the boundary with period $N_b$,

$$\frac{\Delta b}{b} = \left( \frac{\Delta b}{b} \right)_0 \cos \left( \frac{2\pi N}{N_b} \right).$$

(40)

It is natural to assume that the propagation constant varies according to a local variation of the cell radius

$$q(k, z) = k + \frac{2\pi}{N_{\Delta b} d} \cos \left( \frac{2\pi z}{N_b d} \right)$$

(41)

where $N_{\Delta b}$ is found by the simultaneous variation of $b$ by $(\Delta b/b)_0$. 
The expression of the energy gain Eq. 20 is modified by the periodic variation of $b$:

$$\Delta E \propto \int_0^{N_c d} \frac{dz}{\sqrt{\beta_g}} e^{\frac{2\pi i}{\Delta b} \int_0^{\frac{\pi}{\Delta b}} d\zeta \cos\left(\frac{2\pi \zeta}{\Delta b}\right)}$$  \hfill (42)

or

$$\Delta E \propto \frac{N_c d}{\sqrt{\beta_g}} J_0\left(\frac{N_b}{N_{\Delta b}}\right).$$ \hfill (43)

Similarly, the beam loading Eq. 27 takes the form

$$B_L \propto \left(\frac{N_c d}{\sqrt{\beta_g}}\right)^2 J_0^2\left(\frac{N_b}{N_{\Delta b}}\right) \propto (\Delta \epsilon)^2.$$ \hfill (44)

Hence, the energy gain and square root of the beam loading are modified by the same factor. This factor is equal to 0.88 for $N_b = 18$ (10 periods of the $b$-variation along the structure), and is 0.97 for $N_b = 9$ (20 periods). Calculation with PROGON confirms this conclusion. The energy gain and beam loading roll off in a similar way (Fig. 11). Quantitative comparison with Eqs. 43 and 44, however, is difficult because the wave is mismatched along the structure, and the change of the field pattern can not be compensated by the end couplers.

**Random Errors**

Equation 38 can be used to estimate the effect of random errors of the cavity radii on the energy gain. If errors for different cavities are not correlated, and

$$\sigma_1^2 = \left< \frac{\Delta b}{b} \right>^2$$

is the rms relative error for a single cavity, then

$$\left< \left(\frac{\Delta b}{b}\right)_n \left(\frac{\Delta b}{b}\right)_m \right> = \delta_{nm} \sigma_1^2,$$

or

$$\left< \left(\frac{\Delta b}{b}\right)_x \left(\frac{\Delta b}{b}\right)_{x'} \right> = \delta(x - x') \sigma_1^2.$$

\hfill (45)
The energy gain is

\[ \Delta E = \int_0^{N_{cd}} V(z) e^{-ikz} = \frac{\Delta E_0}{N_{cd}} \int_0^{N_{cd}} dz < e^{iq' \int_0^z d\zeta (\Delta b/b)\zeta >}. \]

where \( \Delta E_0 \) is the energy gain without errors, and \( q' = \partial q / \partial (\Delta b/b) \). The average in the integrand can be estimated as

\[ < e^{iq' \int_0^z d\zeta (\Delta b/b)\zeta >} \simeq 1 - \frac{q'^2}{2} \int_0^z d\zeta \sigma_1^2 d \simeq = e^{-\frac{q'^2}{2} \sigma_1^2 z}. \]

Hence,

\[ \Delta E = \frac{\Delta E_0}{N_{cd}} \int_0^{N_{cd}} dz e^{-(q' \sigma_1)^2 z}. \]

The effect of the errors is small provided

\[ \frac{N_{cd}^2}{2} \left[ \frac{\partial q}{\partial (\Delta b)} \right]^2 \sigma_1^2 \ll 1. \]

Using numerical parameters which follow Eq. 39, we see that random errors do not change the energy gain if rms error

\[ \sigma_1 \ll 3.0 \times 10^{-3}. \]

For large errors, the energy gain decreases proportionally by the ratio

\[ \Delta E \simeq \left[ \frac{\Delta b/b}{\sigma_1} \right]^2. \]
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Appendix

Circuit Model for the Monopole Mode

Consider one period of a periodic accelerating structure: an iris and an adjacent cavity. The iris is assumed to be negligibly thin, and the opening small enough to couple fundamental modes of the adjacent cavities without significant coupling to other modes of a cavity.

Following the standard procedure\(^{[a]}\), the field in a cavity is given as a superposition of the eigenmodes of a closed pill-box cavity \(\vec{e}_m, \vec{h}_m\):

\[
\vec{E} = \sum_m \vec{e}_m V_m, \quad \vec{H} = \sum_m \vec{h}_m I_m. \tag{A1}
\]

The eigen modes satisfy Maxwell's equations

\[
\nabla \times \vec{e}_m = i k_m \vec{h}_m, \quad \nabla \times \vec{h}_m = -i k_m \vec{e}_m.
\]

\[
(\nabla^2 + k_n^2)\vec{e}_m = 0, \quad (\nabla^2 + k_n^2)\vec{h}_m = 0
\]

with the boundary conditions on the metallic walls:

\[
\vec{n} \times \vec{e}_m = 0, \quad \vec{n} \vec{h}_m = 0
\]

They are normalized and orthogonal:

\[
\int dV \vec{e}_m^* \vec{e}_l = \delta_{m,l}, \quad \int dV \vec{h}_m^* \vec{h}_l = \delta_{m,l}.
\]

Multiplying Maxwell's equations for the fields \(\vec{E}\) and \(\vec{H}\) by \(\vec{e}_m\) and \(\vec{h}_m\) and integrating
over the cavity volume gives

\[ kI_m - k_m V_m + i \int dS \vec{n} (\vec{h}_m^* \times \vec{E}), \]

\[ kV_m = k_m I_m - iZ_0 \int dV (\vec{e}_m^* \vec{j}). \]  \quad \text{(A2)}

The surface integral is taken over the opening of the irises, \( \vec{n} \) is the external normal to the volume of a cell, and \( \vec{j} \) is a Fourier harmonic of a current along the axis \( z \) of a structure.

Parameter \( s \) is the initial position of a particle at \( t = 0 \).

The surface integrals depend on the field component \( E_r \) which vanishes for a closed cavity. It is clear that, for a thin iris, \( E_r \) is a linear function of the voltage on both sides of the iris. The coefficients, the coupling coefficients of the circuit model, can be found by a perturbation theory. In the zero-th approximation, \( E_r = 0 \), the normal component \( E_z \) in a cavity for the first longitudinal pass-band has the same structure as the normal component \( E_z = V_1 e_{1r}^z \) of a closed cavity. Within an iris is a superposition of the radial modes in the iris

\[ e_{z m}^r = \left( \frac{\nu_m}{a} \right)^2 J_0 \left( \frac{\nu_m r}{a} \right) \left[ c_m \cos(\mu_m z) + d_m \sin(\mu_m z) \right]. \] \quad \text{(A3)}

where \( \mu_m = \sqrt{k^2 - (\nu_m/a)^2} \). \( E_z \) has to be continuous at the opening. This defines the coefficients \( c_m, d_m \) of the superposition. The radial field \( E_r \) in the iris is defined by the same coefficients. The continuity condition gives \( E_r \) at the opening (cf. Eq. A2) in terms of the voltage \( V \) in the cavities on both sides of the iris.

This procedure gives the equation which couples the voltages \( V^N \) of the first cavity mode in the \( N \)-th and \( N \pm 1 \) cavities:

\[ (k^2 - k_{1r}^2)V^N - (\Delta^N + 1 + \Delta^N)V^{N+1} + \xi^N V^N + \xi^{N+1} V^{N+1} + \xi^N V^{N-1} \]

\[ = -\frac{2\pi e}{\sigma} \frac{\nu_1}{b^2} \frac{\sin(kg/2)}{J_1(\nu_1)} e^{i(gZ_N + g/2 - s)}. \] \quad \text{(A4)}

Here \( Z_N = Nd \) is the coordinate of the beginning of the \( N \)-th cavity along the
structure. Parameters defining the frequency shift due the coupling, and $\xi^N$ are

coupling coefficients:

$$\Delta^{N+1} = \left(\frac{a_{N+1}}{b_N}\right)^4 \sum_m \frac{\zeta_m}{\tan \alpha_m^{N+1}},$$

$$\Delta^N = \left(\frac{a_N}{b_N}\right)^4 \sum_m \frac{\zeta_m}{\tan \alpha_m^N},$$

$$\xi^{N+1} = \left(\frac{a_{N+1}^4}{b_N^2 b_{N+1}^2}\right) \sum_m \frac{\zeta_m}{\sin \alpha_m^{N+1}},$$

$$\xi^N = \left(\frac{a_N^4}{b_N^2 b_{N+1}^2}\right) \sum_m \frac{\zeta_m}{\sin \alpha_m^N}. \quad (A5)$$

The sum is over the radial modes in an iris, where

$$\alpha_m = l \mu_m, \quad \mu_m^N = \sqrt{k^2 - (\nu_m/a_N)^2}.$$ 

and $\nu_m$ are roots of the Bessel function $J_0(\nu_m) = 0$. The sum in $\xi^N$ is dominated by
the $m = 1$ term because all fields within an iris are evanescent fields. Finally, the
factor $\zeta_m$ is

$$\zeta_m = \frac{2 \nu_1^2}{\pi g} \frac{J_1^2(\nu_1 p)}{J_1^2(\nu_1)} \frac{\mu_m^N}{(\nu_1^2 - \nu_1^2 p^2)^2}. \quad (A6)$$

Figure Captions

Fig. 1 Variation of the radii $a$ (dashed) and $b$ (solid) in cm along the detuned
structure. Total number of cells is 204. $d=0.875$ cm, $l=0.146$ cm

Fig. 2 RF voltage (arbitrary units) along the detuned structure for two different
reflection coefficients: 0.056 and 0.033.

Fig. 3 Variation of the group velocity, stored energy, and power flow along the
lossless detuned structure. Dashed curves are calculated with a propagating mode
only found by exact calculations with PROGON.

Fig. 4 Dependence of the group velocity on the geometry of the structure.
Fig. 5 Scaling of the voltage with the geometry. The best scaling is given by Eq. 17, although the dominant factor for the detuned structure is \((a/b)^4\).

Fig. 6 Dependence of the total energy gain on the detuning from the synchronous frequency of the detuned structure. The wave impedance is defined as \(\sqrt{\Delta E^2/P}\).

Fig. 7 Voltage \(v(N)\) along the constant impedance structure at the synchronous frequency.

Fig. 8 The losses along the detuned structure at the synchronous frequency.

Fig. 9 Reflection from a localized variation of a cell radius in a constant impedance structure.

Fig. 10 Variation of the voltage along the constant impedance structure due to simultaneous change of all cell radii by \(\Delta b/b = 0.0075\).

Fig. 11 Variations of the energy gain (upper) and beam loading (middle) with periodic variation of the cell radii \(\Delta b/b\). The two curves correspond to the variation with period 10 and 20 cells. The lower figure shows that the change of the reflection coefficient is small.

REFERENCES


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Fig. 1

$a(n)$, $b(n)$ vs $n$ for the detuned structure
detuned 20°C CAV, 24.30 W MCDE
abs v(n) for rf wave. REF=0.0565

REF=0.0335

Fig. 2
Fig. 3
$F = 0.1(f/\text{Hz})(a/\text{cm})$ vs $v_g/c$. Dots are from the Helm's curve

Fig. 4
204 CAV, detuned, 24±30 W MODES

abs $v(n)$. Variation 28.84%

abs $v(n)$ scaling; 1.9%

abs $v(n)(a/b)^{1.50}$; 11.36%

abs $v(n)(a/b)^{2.0}$; 5.31%

Fig. 5
204 CAV. DETUNED, NON PERT

Cplrs Param (cm):
A(2) = 0.691337
B(2) = 1.115127
G(2) = 0.902231
A(NCAVO) = 0.53351
B(NCAVO-1) = 1.06081
G(NCAVO-1) = 0.900323

24 C MODES
30 W MODES

Fig. 6
RF. 184 CAV, PERIODIC, 30 W MODES

**Real v(n)**

**Imag V(n)**

Fig. 7
Current: 184 CAV, PERIODIC, 30 W MODES

Fig 8
184 CAV, PERIODIC, a kick in 100-th cell

Fig. 9
184 cells, kick in all cells, $db/b=0.0025$

Fig. 10
Fig. 11

Fig. 22 REL AMPL B VARIATION

IPER-20 (PLUS)
IPER-10 (CROSS)
SQRT(WAVE IMPEDANCE) (%

FREQUENCY 11.434 GHz
20 C MODES
30 W MODES
BEAMLOAD(%) ABS REF COEF