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VIBRATIONAL MODES OF THE PEDESTAL SUPPORT SYSTEM FOR THE SLC ARC MAGNETS*

W. T. WENG AND A. W. CHAO

Stanford Linear Accelerator Center

Stanford University, Stanford CA 94305

1. Introduction

The magnet support system for the SLC Arcs will be a long series of pedestals with each pedestal supporting the ends of two adjacent magnets. It has been pointed out by several authors^{1,2,3} that random magnet vibrations in the Arc with amplitudes larger than $0.1 \mu\text{m}$ rms are potentially harmful for the SLC operation. In order to assess the vibrational behavior of the Arc magnet system, we need to understand: (1) the sources and characteristics of the ground disturbances, (2) the coupled vibrational modes of the composite pedestal-magnet system and, (3) the response of the system to ground disturbance.

A review of the sources and characteristics of the ground vibration, *i.e.*, item (1), above can be found in Ref. 4. This note is an attempt to study item (2). The coupled vibrational modes of the pedestal-magnet system are calculated theoretically. The actual response of the system to ground vibration has been measured on a nine-magnet mock-up and will be discussed in a separate report.

The strength of the microseismic vibration typically peaks at 0.15 Hz and generally drops to negligible amplitude beyond 30 Hz. For an assumed sound velocity of 2000 ft/sec in the ground, the whole SLC site will move up and down coherently below 1 Hz; therefore, any vibration below 1 Hz should not be of

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concern to beam stability at SLC.⁴ In general the amplitude of vibrations between 1 and 30 Hz are not strong enough to affect the SLC operation. In case there are any unexpected harmful vibrations between 1 and 30 Hz, a dynamic feedback system at FFS region will be activated to correct the orbit. Consequently, the microseismic vibration would not be a serious problem for the magnet system. However, the cultural disturbances⁵ tend to peak between 15 to 35 Hz and are of sufficient magnitude to occasionally affect the luminosity. In the following effort to identify the normal modes of the magnet-pedestal support system, we will pay particular attention to the frequency range from 1 to 40 Hz.

The building blocks of the system are pedestals and magnets. To find the coupled vibrational modes, we have first to know the vibrational properties of each unit separately. The pedestal is a hollow steel column with a height to support the magnet to 42 inches from the floor. Its stiffness constants have been estimated to be⁶

$$k_{p,H} = 6.77 \times 10^5 \text{ lb/in} \quad (1a)$$

$$k_{p,V} = 1.34 \times 10^7 \text{ lb/in} \quad (1b)$$

where H and V stand for horizontal and vertical displacements. Consequently, a single pedestal loaded by a single magnet ($W = 1200 \text{ lb}$) will vibrate at a frequency of

$$f_{p,H} = \frac{1}{2\pi} \sqrt{\frac{k_{p,H}}{M}} = 74.3 \text{ Hz} \quad (2a)$$

$$f_{p,V} = \frac{1}{2\pi} \sqrt{\frac{k_{p,V}}{M}} = 330 \text{ Hz} \quad (2b)$$

Since the magnet is made of laminated iron plates, it is not possible to calculate its spring constant from the knowledge of the material alone. Therefore, an

empirical way of determining the frequency is employed. For example one finds that a welded Arc magnet (model number EM 4004, which is 2.5 m long and weighs 1200 lb), when simply supported rigidly at both ends, sags under its own weight by the following amount⁷

$$\delta_H = 12.9 \text{ mils}$$

$$\delta_V = 9.3 \text{ mils}$$

Therefore, the transverse flexural vibrational frequencies of the magnet can be estimated to be

$$f_{M,H} = \frac{1}{2\pi} \sqrt{\frac{g}{\delta_H}} \left(\frac{5\pi^4}{384} \right)^{1/2} = 31.02 \text{ Hz} \quad (3a)$$

$$f_{M,V} = \frac{1}{2\pi} \sqrt{\frac{g}{\delta_V}} \left(\frac{5\pi^4}{384} \right)^{1/2} = 36.54 \text{ Hz} \quad (3b)$$

where g is the gravitational constant. The above relationship will be derived in the next section.

Knowing the vibrational properties of a single pedestal and a single magnet, we can start to solve the coupled pedestal-magnet system. Before doing so, however, we would like to digress a little to discuss the transverse vibrational properties of a uniform beam and justify the treatment of the Arc magnet in that model. To simplify the problem we do not consider the torsional vibration in this calculation.

2. Magnet as a Simply Supported Uniform Beam

Although the vibrational behavior of a uniform beam is a well-known subject,^{8,9} we will review briefly the Euler equation for beams to introduce the basic concept and notations.

To determine the differential equation for the transverse vibration of beams, consider the force and bending moment of the beam shown in Fig. 1.

Then, the equations of motion are

$$\begin{aligned} F(x+dx) - F(x) &= -\rho \ddot{y} dx \\ M(x+dx) - M(x) &= F(x) dx \end{aligned} \quad (4)$$

where $F(x)$ and $M(x)$ are the force and moment acting on the element at position x , $y(x, t)$ is the magnet deflection at position x and time t , and ρ is the line mass density of the beam,

Fig. 1. An element of beam under force and bend.

$$\rho = \frac{M}{\ell} \quad (5)$$

Combining the two relations in Eq. (4), we obtain

$$\frac{d^2 M}{dx^2} = \frac{dF}{dx} = -\rho \ddot{y} \quad (6)$$

But the bending moment M is related to the curvature R of the beam through

$$M = EI \frac{1}{R} = EI \frac{\partial^2 Y}{\partial x^2} \quad (7)$$

where E is the Young's modulus of the material, and I is the moment of inertia of the beam cross section about the beam centerline, and EI is usually combined together to be called the flexural rigidity of the beam. Combining Eqs. (6) and (7) yields a fourth order differential equation, the Euler equation, governing the deflection of the beam,

$$EI \frac{\partial^4 y}{\partial x^4} = -\rho \frac{\partial^2 y}{\partial t^2} \quad (8)$$

Note that we have assumed the cross section of the beam remains rigid under deflection. The treatment therefore excludes those modes in which the

magnet cross section deforms. Those modes presumably are not strongly driven by ground disturbances.

Consider the whole beam vibrating at mode frequency ω , then Eq. (8) becomes

$$EI \frac{d^4 y}{dx^4} - \rho \omega^2 y = 0$$

or

$$\frac{d^4 y}{dx^4} - \beta^4 y = 0 \quad , \quad \beta^4 = \frac{\rho \omega^2}{EI} \quad . \quad (9)$$

The general solution of Eq. (9) can be expressed in the following form:

$$y(x) = A \cosh \beta x + B \sinh \beta x + C \cos \beta x + D \sin \beta x \quad . \quad (10)$$

The vibrational frequency ω is related to the constant β , and the coefficients A, B, C and D are determined by the prescribed boundary conditions. For example,

(a) Clamped ends: both deflection and slope are zero, i.e.,

$$y = 0 \quad , \quad \frac{dy}{dx} = 0 \quad ,$$

(b) Free ends: both force and torque are zero, i.e.,

$$\frac{d^2 y}{dx^2} = 0 \quad , \quad \frac{d^3 y}{dx^3} = 0 \quad ,$$

(c) Simply-supported ends: both deflection and torque are zero, i.e.,

$$y = 0 \quad , \quad \frac{d^2 y}{dx^2} = 0 \quad .$$

The fact that force is basically $d^3 y/dx^3$ and torque is related to $d^2 y/dx^2$ can be seen from Eq. (6).

To show how the boundary conditions (*BC*) determine the mode frequency and coefficients, let us consider the case of a simply supported beam of length ℓ . The *BC* requires that

$$\begin{aligned} y(0) &= 0 \quad , \quad \frac{d^2 y}{dx^2}(0) = 0 \\ y(\ell) &= 0 \quad , \quad \frac{d^2 y}{dx^2}(\ell) = 0 \end{aligned} \tag{11}$$

which gives

$$\begin{cases} A + C = 0 \\ A \cosh \beta \ell + B \sinh \beta \ell + C \cos \beta \ell + D \sin \beta \ell = 0 \\ A - C = 0 \\ A \cosh \beta \ell + B \sinh \beta \ell - C \cos \beta \ell - D \sin \beta \ell = 0 \end{cases} . \tag{12}$$

The only possible solution is

$$A = B = C = 0 \quad \text{and} \quad \sin \beta \ell = 0 . \tag{13}$$

Therefore, the normal modes are given by

$$y(x) = D \sin \beta_n x . \tag{14}$$

where

$$\beta_n \ell = n\pi \tag{15}$$

or

$$\omega_n = \sqrt{\frac{EI}{\rho}} \frac{n^2 \pi^2}{\ell^2}$$

It is important to be aware of the fact that the frequencies are proportional to n^2 . This is very different from the modes of a string under tension for which ω_n is proportional to the mode number n . This quadratic behavior of ω_n on n comes from the fact that the equation of motion, Eq. (8), is fourth order.

Similarly, it can be shown that for a beam with free supports at both ends, the normal modes are determined by

$$\cos \beta_n \ell \cdot \cosh \beta_n \ell = 1 \quad . \quad (16)$$

Now we are in a position to find the relationship between the vibration frequency and the sag of the magnet. For a magnet under its own weight, the Euler's equation (8) becomes

$$EI \frac{\partial^4 y}{\partial x^4} = \rho g \quad (17)$$

where g is the gravitation constant. Now the general solution can be put in the form:

$$y = A + Bx + Cx^2 + Dx^3 + \frac{1}{24} \frac{\rho g}{EI} x^4 \quad . \quad (18)$$

Applying the simply supported BC again, we obtain

$$\begin{aligned} A &= C = 0 \\ D &= -\frac{1}{12} \frac{\rho g}{EI} \ell \\ B &= \frac{1}{24} \frac{\rho g}{EI} \ell^3 \end{aligned}$$

giving

$$y = \frac{1}{24} \frac{\rho g}{EI} \ell^4 \left(\frac{x}{\ell} \right) \left[1 - \left(\frac{x}{\ell} \right) \right] \left[1 + \left(\frac{x}{\ell} \right) - \left(\frac{x}{\ell} \right)^2 \right] \quad (19)$$

which gives the sag at the middle of the magnet:

$$\delta = y \left(\frac{\ell}{2} \right) = \frac{5}{384} \frac{\rho g}{EI} \ell^4 \quad . \quad (20)$$

Combining Eqs. (20) and (15), we can relate the lowest mode vibrational frequency to the sag as:

$$f = \frac{1}{2\pi} \sqrt{\frac{g}{\delta}} \left(\frac{5\pi^4}{384} \right)^{1/2} \quad . \quad (21)$$

This is the relation we used for Eqs. (3a) and (3b).

The frequencies derived from Eq. (21) using the sag as measured agree with the measured frequencies to about 10%.⁶

3. A Single Magnet on Two Pedestal

So far we have discussed the transverse vibrational properties of magnets on rigid supports. But, as mentioned in the introduction, the pedestal itself could vibrate with its own frequency. The problem now is to calculate the modes of the combined system.

A magnet supported by two pedestals can be analyzed by the model as shown in Fig. 2. The pedestals are represented as springs with spring constant k .

Fig. 2. A magnet supported by two pedestals.

The solution can still be expressed as

$$y(x) = A \cosh \beta x + B \sinh \beta x + C \cos \beta x + D \sin \beta x \quad (10)$$

but now the boundary condition becomes

$$\begin{cases} M(0) = 0, \text{ and } M(\ell) = 0 \\ F(0) = ky(0) = -EI y'''(0) \\ F(\ell) = ky(\ell) = EI y'''(\ell) \end{cases} \quad (22)$$

which implies that

$$\begin{cases} A - C = 0 \\ A \cosh \beta \ell + B \sinh \beta \ell - C \cos \beta \ell - D \sin \beta \ell = 0 \\ A + C = -\alpha(B - D) \\ A \cosh \beta \ell + B \sinh \beta \ell + C \cos \beta \ell + D \sin \beta \ell \\ \quad = \alpha(A \sinh \beta \ell + B \cosh \beta \ell + C \sin \beta \ell - D \cos \beta \ell) \end{cases} \quad (23)$$

where

$$\alpha = \frac{EI}{k} \beta^3 \quad . \quad (24)$$

We first express A and C in terms of B and D , then for the system to have a solution the determinant formed by the coefficients for B and D should be zero. It is then found that the normal mode frequency satisfies

$$\begin{aligned} \det = & 2 \sin \beta \ell \sinh \beta \ell - 2\alpha (\sin \beta \ell \cosh \beta \ell - \cos \beta \ell \sinh \beta \ell) \\ & - \alpha^2 (\cos \beta \ell \cosh \beta \ell - 1) = 0 \quad . \end{aligned} \quad (25)$$

It is interesting to see that Eq. (25) agrees with Eq. (13) for the rigid support case ($\alpha = 0$) and with Eq. (16) for the free support case ($\alpha \rightarrow \infty$).

4. The M-Magnet and $(M + 1)$ Pedestal System

Now let us consider the M-magnet- $(M+1)$ -pedestal system as shown in Fig. 3.

Let the general solution to the i^{th} magnet be

Fig. 3. M-magnet- $(M + 1)$ -pedestal system.

$$y_i(x) = A_i \cosh \beta x + B_i \sinh \beta x + C_i \cos \beta x + D_i \sin \beta x \quad (26)$$

where x is measured from the left end of each magnet, and y stands for the transverse deflection of the magnet at position x . Then the boundary conditions to be satisfied for simply supported ends on pedestals are:

$$\left\{ \begin{array}{ll} y_1(\ell) = y_2(0) \quad , & ky_2(0) = EI[y_1'''(\ell) - y_2'''(0)] \\ y_2(\ell) = y_3(0) \quad , & ky_3(0) = EI[y_2'''(\ell) - y_3'''(0)] \\ \vdots & \\ y_{M-1}(\ell) = y_M(0) \quad , & ky_M(0) = EI[y_{M-1}'''(\ell) - y_M'''(0)] \end{array} \right.$$

and

$$\begin{cases} y_1''(0) = y_1''(\ell) = 0 & , & ky_1(0) = -EI y_1'''(0) \\ y_2''(0) = y_2''(\ell) = 0 & , & ky_M(\ell) = EI y_M'''(0) \\ \vdots & & \\ y_M''(0) = y_M''(\ell) = 0 & . \end{cases} \quad (27)$$

There are $4M$ conditions as there should be. Again, the A_i 's and C_i 's can be expressed in terms of the B_i 's and D_i 's, as

$$A_i = C_i = \frac{D_i \sin \beta \ell - B_i \sinh \beta \ell}{\cosh \beta \ell - \cos \beta \ell} , \quad i = 1, 2, \dots, M \quad (28)$$

and the equations for the B_i 's and D_i 's can be summarized in the matrix form:

$$K \bar{Y} = 0 \quad (29)$$

where $\bar{Y}$ is a state vector, and when transposed is

$$\bar{Y}^T = (B_1, D_1, B_2, D_2, \dots, B_M, D_M) \quad (30)$$

and K is the $2M \times 2M$ coefficient matrix,

$$K = \begin{pmatrix} a_1 & -a_2 & 0 & 0 & & & & & & \\ -a_7 & -a_8 & a_1 & a_2 & 0 & 0 & & & & \\ -a_3 & a_4 & a_5 & -a_6 & 0 & 0 & & & & \\ 0 & 0 & -a_7 & -a_8 & a_1 & a_2 & & & & \\ & & -a_3 & a_4 & a_5 & -a_6 & & & & \\ & & & & & & 0 & 0 & & \\ & & & & & & -a_7 & -a_8 & a_1 & a_2 \\ & & & & & & -a_3 & a_4 & a_5 & -a_6 \\ & & & & & & 0 & 0 & a_9 & a_{10} \end{pmatrix} \quad (31)$$

where

$$\left\{ \begin{array}{l} a_1 = -2 \sinh \beta \ell + \alpha \cosh \beta \ell - \alpha \cos \beta \ell \\ a_2 = 2 \sinh \beta \ell - \alpha \cosh \beta \ell + \alpha \cos \beta \ell \\ a_3 = \sinh \beta \ell \cos \beta \ell \\ a_4 = \sin \beta \ell \cosh \beta \ell \\ a_5 = \sinh \beta \ell \\ a_6 = \sin \beta \ell \\ a_7 = \alpha (1 - \cosh \beta \ell \cos \beta \ell - \sinh \beta \ell \sin \beta \ell) \\ a_8 = \alpha (1 - \cosh \beta \ell \cos \beta \ell + \sinh \beta \ell \sin \beta \ell) \\ a_9 = a_7 + 2 a_3 \\ a_{10} = a_8 - 2 a_4 \end{array} \right. \quad (32)$$

In order to have non-trivial solutions to Eq. (29), the determinant of the matrix K should be equal to zero, i.e.,

$$\det K = 0 \quad . \quad (33)$$

The discrete values of ω_n which satisfy Eq.(33) are the normal mode frequencies of the composite system.

In the limiting case for one magnet, $M = 1$, Eq. (33) becomes

$$\det \begin{vmatrix} a_1 & a_2 \\ a_9 & a_{10} \end{vmatrix} = 0 \quad (34)$$

which can be shown to be identical to Eq. (25) as it should be.

5. Solutions of the Normal Modes

To find the proper value ω to satisfy Eq. (33), we have to know all the constants needed in the matrix K . From Eq. (32) we know that the basic input parameters required are the constants α and β . From Eq. (9), β is related to ω by

$$\beta = \left(\frac{\rho}{EI} \right)^{1/4} \omega^{1/2} \quad (9)$$

and α is related to ω through Eq.(24),

$$\alpha = \frac{EI}{k} \beta^3 = \left(\frac{EI}{\rho} \right) \frac{\rho}{k} \beta^3 \quad (24)$$

For the model magnet EM 4004, we have $\delta_h = 12.9$ mils, $\rho = 0.03182$ lb/in, and $\ell = 97.6$ in, which give

$$\beta \ell = 0.2251 \omega^{1/2} \quad (35)$$

$$\alpha = 0.00002039 \omega^{3/2} \quad (36)$$

After the K matrix is constructed, we use the drive routine DGEFDI of LINPACK to find the determinant of the matrix K . We then numerically look for the zeros of the determinant by scanning ω in order to find the normal mode frequencies.

For example, let us look at the case of a single magnet. For a magnet simply-supported on rigid pedestals, the frequencies of the lowest two modes derived from the sag by Eqs. (3a) and (15) are $\omega_1 = 194.899$ (31.02 Hz) and $\omega_2 = 779.596$ (124.08 Hz). In comparison, for the rigid support the condition of determinants equal to zero by Eq. (25) gives $\omega_1 = 194.782$ (31.00 Hz) and $\omega_2 = 779.128$ (124.00 Hz), almost identical to the analytic calculation. It is especially interesting to see that the n^2 dependence of the frequency is correctly predicted. If we use the realistic stiffness constant of the pedestal given in Eq. (1), the frequencies are shifted to $\omega_1 = 188.155$ (29.95 Hz) and $\omega_2 = 676.036$ (107.59 Hz). This is no longer exactly $4 \times \omega_1$.

With ω known, the expansion coefficients A , B , C and D for the beam deflection can be found through Eq. (23), and the corresponding patterns are plotted in Fig. 4 for the first three modes. It is worth emphasizing that there are $(n - 1)$ nodes in the n^{th} mode.

Fig. 4 The deflection pattern of the first three modes of a magnet.

Fig. 5 First three deflection patterns of the first mode of the 10-magnet system.

Next let us look at the case of ten magnets on eleven pedestals as an example to illustrate the behavior of the composite magnet and pedestal system. Assuming that the magnets are all identical to each other, then the coefficient matrix can be set up according to Eq. (31). The requirements that the determinant equals zero gives ten ω 's clustered around the fundamental mode $\omega_1 = 188.155$ and another ten ω 's clustered around $\omega_2 = 676.036$. Specifically, the ten fundamental modes now range from 182.23 to 194.49. Again, the first three cases of the fundamental mode are plotted in Fig. 5.

For comparison, we list in Table 1 the results of the calculation of the first two modes of a single magnet and a coupled magnet-pedestal system.

Table 1. Summary of Normal Mode Frequencies

Mode	Single Magnet ($M = 1$)			Magnet-Pedestal System ($M = 10$)	
	Rigid Support		On Pedestal		
	Formula Eq. (3a)	Numerical Eq. (25) $\alpha = 0$	Numerical Eq. (23) $\alpha \neq 0$	Numerical Eq. (33)	
$n = 1$	31.02 Hz	31.00	29.95	29.00	30.07
				29.11	30.36
				29.28	30.61
				29.51	30.82
				29.78	30.95
$n = 2$	124.08 Hz	124.00	107.59	94.96	107.75
				96.16	119.2
				98.07	116.15
				100.71	120.07
				103.96	122.92

Since each magnet is 97.6 inches long, the pedestals are located at 97.6 inches intervals. It is interesting to see that the deflection of the pedestals form a pattern like that of the magnets, i.e., there are $(\ell - 1)$ nodes in the ℓ^{th} case of the

$n = 1$ mode. However, the deflection of the magnets themselves in the $n = 1$ mode is always one with half sine waveform. The pattern formed by the pedestal will be of importance when the response of the system under the ground vibration is to be estimated.

6. Conclusions and Discussions

Using the sag of an arc magnet under its own weight and the stiffness constant of the supporting pedestal, we have derived the normal modes of vibration of a string of coupled magnet-pedestal systems. The normal mode frequencies and mode patterns are evaluated. In an M -magnet system, M solutions cluster around the corresponding mode of a single magnet. In other words, suppose the mode frequencies for a single magnet are $\omega_1, \omega_2 \dots, \omega_n$, then for the M -magnet system there are M modes cluster around each ω_n . For the n^{th} mode, each magnet will assume the deflection pattern of the n^{th} single magnet pattern, but the pattern of the pedestal itself ranges from the first to the M^{th} mode.

As explained in the Introduction, we are basically concerned with the perturbing vibration in the frequency range of 1 to 40 Hz. The finding that the coupled systems vibrate at frequencies around that of a single magnet has two practical implications in the consideration of the stability of the system. *First, if the pedestals are stiff enough, the coupled system does not vibrate at any lower frequency than the single magnet frequency; therefore, we only have to concentrate on the lowest mode without worrying about the lowering of the vibration frequencies from the higher order modes through coupling. However, if the fundamental frequency of the pedestal itself is lower than that of the magnet, the lowest mode of the coupled system will correspond to that of the pedestal. This should be avoided by all means.* Second, any method to stiffen a magnet to raise the vibrational frequency beyond 40 Hz will make the coupled system vibrate at a higher frequency, as well.

So far, we have only addressed the free oscillation problem. In order to predict the response of the system to the ground vibration, we have to know the behavior of the ground vibration and work with a more realistic model for the magnet-pedestal system. In particular, damping of the vibration needs to be included in order to predict the amplitude of vibration, given the ground motion. Another possible improvement is to include the torsional vibration in the calculation if that is proved to be important.

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