

NOTE ON THE FIELD OF A FAST, CHARGED PARTICLE^{*}

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As the velocity of a charged particle increases, its electric flux lines become more and more concentrated about the transverse plane. This picture is sometimes construed to mean that the electric field is zero in front of a very fast particle. For many purposes, this is a valid approximation, but we must remember that it is still an approximation. No matter what the velocity of the particle, the field ahead of it never is zero. This becomes apparent if the field component along the direction of motion is integrated over the transverse plane. Here, Gauss' Theorem constrains the longitudinal field component to be finite, and the approximation that sets it equal to zero breaks down.

To see this, let D be the displacement vector along the direction of motion. The integral of D over the plane transverse to the particle motion is a relativistic invariant; it has the same value in the lab system and in the rest system, for D is the same in both systems, and so are the transverse coordinates appearing in the integral. In both the rest system and the lab system, the integral has the MKS value $Q/2$ in front of the particle, and the same value in back. Thus, in both systems, D must be different from zero somewhere in the transverse plane, and in such a way that its integral over the transverse plane is $Q/2$.

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To illustrate these remarks, expressions for D are derived below for: a particle in free space (A), and a particle in a conducting tube (B). We find that, in a calculation carried out to a limited order of accuracy, the field can be neglected in a certain region in front of the particle; however, setting the field equal to zero everywhere in front of the particle is impermissible, since it violates Gauss' Theorem. These remarks apply when the field point is a finite distance from the particle; in contrast, as the field point approaches infinitesimally close to the particle, the field acquires a singularity corresponding to the derivative of a delta function.

(A) Particle in Free Space.

In the rest system,

$$D_{z'} = \frac{Q_{z'}}{4\pi[z'^2 + r^2]^{3/2}} , \quad (1)$$

where we have introduced a cylindrical coordinate system with the z' -axis along the direction of particle motion; the field point is at (r, z') relative to the charged particle, which is located at the origin.

In the lab system,

$$D_z = \frac{Q\gamma z}{4\pi[(\gamma z)^2 + r^2]^{3/2}} , \quad (2)$$

again using a cylindrical coordinate system; the field point is at (r, z) relative to the charged particle, where $z' = \gamma z$.

In either case, we find that

$$\int_0^\infty dr 2\pi r D_z = \frac{Q}{2} , \quad (3)$$

as required by Gauss' Theorem. Note that this result is independent of z (or z').

Suppose we integrate only between $r = 0$ and $r = \epsilon\gamma z$, where $\epsilon \ll 1$:

$$\int_0^{\epsilon\gamma z} dr \, 2\pi r \, D_z = \frac{Q}{2} \left\{ 1 - \frac{1}{\sqrt{1 + \epsilon^2}} \right\} \quad (4)$$

$$\sim \frac{Q}{2} \left(\frac{\epsilon^2}{2} \right) .$$

We now see that if we are calculating to order ϵ only, the field in the region from $r = 0$ to $r = \epsilon\gamma z$ may be disregarded, since its contribution to the transverse integral is of second order. If this is done, the remaining integration

$$\int_{\epsilon\gamma z}^{\infty} dr \, 2\pi r \, D_z = \frac{Q}{2} \frac{1}{\sqrt{1 + \epsilon^2}} \sim \frac{Q}{2} , \quad (5)$$

satisfies Gauss' Theorem to the same order of approximation. Note that $D_z \neq 0$ in the region from $r = \epsilon\gamma z$ to ∞ .

(B) Particle in a Conducting Tube.

First, an identity. In the region from $r = 0$ to $r = 1$,

$$\frac{\delta(r - r')}{r} = \sum_{\alpha_n} \frac{J_0(\alpha_n r) J_0(\alpha_n r')}{\frac{1}{2} J_1^2(\alpha_n)} , \quad (6)$$

where the α_n 's are the roots of $J_0(\alpha_n) = 0$.

Therefore, for a particle located at the origin of the rest system, in a conducting tube of unit radius, the potential function is

$$\epsilon_0 \phi(r, z') = -\frac{1}{2\pi} \sum_{\alpha_n} \frac{J_0(\alpha_n r) e^{-\alpha_n |z'|}}{\alpha_n J_1^2(\alpha_n)} . \quad (7)$$

The components of the dielectric displacement vector are obtained from (7) by taking the negative gradient of (7); if one then applies the usual relativistic

transformation formulae ($D_{\parallel} = D'_{\parallel}$; $D_{\perp} = \gamma D'_{\perp}$; $z' = \gamma z$), the displacement vectors in the lab system are obtained. For $z > 0$, these are:

$$D_r(r, z) = \frac{Q}{2\pi} \sum_{\alpha_n} \frac{\gamma J_1(\alpha_n r) e^{-\alpha_n \gamma z}}{J_1^2(\alpha_n)}$$

$$D_z(r, z) = \frac{Q}{2\pi} \sum_{\alpha_n} \frac{J_0(\alpha_n r) e^{-\alpha_n \gamma z}}{J_1^2(\alpha_n)} .$$
(8)

To verify Gauss' Theorem, we integrate first along the surface of the tube to a fixed value of z , and then across the transverse plane:

$$\int_0^z dz 2\pi D_r = Q \sum_{\alpha_n} \frac{1 - e^{-\alpha_n \gamma z}}{\alpha_n J_1(\alpha_n)}$$

$$\int_0^1 dr 2\pi r D_z = Q \sum_{\alpha_n} \frac{e^{-\alpha_n \gamma z}}{\alpha_n J_1(\alpha_n)}$$
(9)

where the last integral is easily done by recalling that

$$\alpha_n r J_0(\alpha_n r) = \frac{d}{dr} (r J_1(\alpha_n r)) .$$
(10)

The sum of the two integrals in (9) is

$$\sum_{\alpha_n} \frac{Q}{\alpha_n J_1(\alpha_n)} .$$
(11)

This formidable-looking expression sums to just $Q/2$, as can be verified by multiplying (6) by r and integrating over r from zero to unity. Thus, as expected, Gauss' Theorem is again confirmed, provided we keep the exact expression for the fields.

If we set $z = 0$, and integrate radially only from zero to the small quantity ϵ , we find the result

$$Q \sum_{\alpha_n} \frac{\epsilon J_1(\alpha_n \epsilon)}{\alpha_n J_1^2(\alpha_n)} . \quad (12)$$

For small ϵ , this reduces to

$$Q \sum_{\alpha_n} \frac{\epsilon^2}{2J_1^2(\alpha_n)} \sim \epsilon^2 . \quad (13)$$

As in (A), we again conclude that, to order ϵ , the field in front of the particle, in the region from $r = 0$ to $r = \epsilon$, can be neglected, but not outside this region, if Gauss' Theorem is to be satisfied.

So far, the distance between the field point and the particle has implicitly been assumed finite. If, instead, it is allowed to approach zero, the right sides of (8) and (6) become essentially the same ($r' = 0$). D_z then is proportional to the left side of (6), which is equivalent to the derivative of the delta function, and which characterizes the singularity of the field as z approaches zero.

A similar result is obtained in case (A), since, as z approaches zero, the presence or absence of a conducting tube of unit radius is immaterial.