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IMPLICIT FUNCTION METHOD FOR THE LIOUVILLE EQUATION*

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A number of problems may be named for which the solution of the corresponding Liouville equation might be useful. For example the interaction of an electron flow with an ungridded gap may be approached from this direction. Another example is obtaining the distribution function of a particle bunch at the exit of a transport line containing nonlinear elements should the distribution functions at the entrance is known. List of such problems may be extended.

In the present short note I derive a rather general method of solving the Liouville equation. No discussion will be presented on how fast the solution can be found or how convenient the method is in practical applications. Nor any numerical calculations will be done. All these tasks can be subjects of further studies should the method become useful later.

For the sake of simplicity and to shorten derivation I start from the case of a one-dimensional nonrelativistic motion. In addition I assume that the force acting on a particle does not depend on its velocity. Next section contains the results for general 3-D case and the force depending on velocity as well.

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1. One–Dimensional Motion

Let us assume the following equations of a particle motion

$$\frac{dx}{dt} = v \tag{1.1}$$

$$\frac{dv}{dt} = F(x,t) \tag{1.2}$$

1.1 THE LIOUVILLE EQUATION

In the phase space (x,v) the particle motion may be equivalently described by the corresponding Liouville equation for the distribution function $\psi = \psi(x,v,t)$:

$$\mathcal{L}\,\psi \equiv \frac{\partial\psi}{\partial t} + v\,\frac{\partial\psi}{\partial x} + F(x,t)\,\frac{\partial\psi}{\partial v} = 0 \tag{1.3}$$

Note that unlike (1.1) and (1.2) where $x = x(t)$ and $v = v(t)$, in Eq. (1.3) x and v are independent variables and they do not depend on t.

1.2 THE CHARACTERISTIC FUNCTION

The solution of (1.3) may be found in terms of a certain characteristic function $X(x,v,t,t')$ of two time variables t and t' which is defined implicitly by the following equation:

$$G \equiv X - x + v\cdot(t-t') - \int_{t'}^{t}(t''-t')F(X,t'')\,dt'' = 0 \tag{1.4}$$

Given the function $F(x,t)$, the characteristic function $X(x,v,t,t')$ can in principle be found. I will assume in what follows that $X(x,v,t,t')$ is known.

From the definition (1.4) immediately follows that at $t' = t$

$$X(x,v,t,t) = x \tag{1.5}$$

Remarkable property of $X(x,v,t,t')$ is that it satisfies the Liouville equation (1.3) for first time variable and for any value of the second variable t'. To see that calculate $\mathcal{L}G = 0$ (since $G = 0$):

$$\mathcal{L}G = \mathcal{L}X - \int_{t'}^{t} dt'' \cdot (t'' - t') \cdot \frac{\partial F}{\partial X} \cdot \mathcal{L}X = 0 \tag{1.6}$$

Let us denote for the time being

$$\mathcal{L}X \equiv f(x, v, t, t') \tag{1.7}$$

$$(t' - t) \cdot \frac{\partial F}{\partial x} \equiv g(x, v, t, t') \tag{1.8}$$

Then for f we have the integral equation

$$f(x, v, t, \tau) = \int_{\tau}^{t} dt' \; g(x, v, t, t') f(x, v, t, t') \tag{1.9}$$

Under certain conditions (1.9) has only the trivial solution $f(x, v, t, \tau) = 0$ for any τ. One notes first of all that $f(x, v, t, t) = 0$.

Expand now all the functions in (1.9) in the Taylor series in $u = t - \tau$ and perform integration:

$$-f_0' \cdot u + \frac{f_0''}{2} u^2 - \ldots = g_0 f_0' \frac{u^2}{2} - \frac{g_0 f_0''}{6} u^3 - \frac{g_0' f_0'}{3} u^3 + \ldots$$

All quantities with the subscript 0 here denote the value of corresponding function at $u = 0$ (or $\tau = t$).

Comparing now terms with equal powers in u one finds $f_0' = 0$, $f_0'' = 0$ and so on. Any function f which allows the Taylor expansion and has all its derivatives zero is zero itself. This means that X indeed satisfies the Liouville equation unless the function $F(x, t)$ has no Taylor expansion. We will not consider special cases like that.

1.3 Solution Of The Liouville Equation

Suppose now that at the beginning of the system $x = 0$ the distribution function ψ is known

$$\psi(x, v, t)\,|_{x=0} = \psi_0\,(v, t) \tag{1.10}$$

I will show that at any other place x of the system the solution of (1.3) which satisfies the boundary condition (1.10) can be written as follows:

$$\psi(x,v,t) = \psi_0(V(x,v,t),\ \Theta(x,v,t)), \tag{1.11}$$

where

$$V(x,v,t) = v - \int_{\Theta(x,v,t)}^{t} F(X(x,v,t,t'),t')\,dt' \tag{1.12}$$

and $\Theta(x,v,t)$ is a such a function implicity defined by the equation:

$$F \equiv x - v\cdot(t-\Theta) + \int_{\Theta}^{t} (t'-\Theta)\cdot F(X(x,v,t,t'),t')\,dt' = 0 \tag{1.13}$$

which goes into t for $x=0$:

$$\Theta(x,v,t)\,|_{x=0} \quad = t \tag{1.14}$$

In (1.12) and (1.13) the function $X(x,v,t,t')$ is the one defined in (1.4). To prove that (1.11) indeed satisfies (1.3) one first notes that

$$\mathcal{L}\psi = \frac{\partial\psi_0}{\partial V}\,\mathcal{L}V + \frac{\partial\psi_0}{\partial\Theta}\,\mathcal{L}\Theta \quad .$$

Next find $\mathcal{L}F = 0$:

$$\mathcal{L}F = -\int_{\Theta}^{t} dt'\,F(X,t')\cdot\mathcal{L}\Theta = 0 \tag{1.15}$$

Since (1.15) should be true for any value of x, v and t, it follows from (1.15)

$$\mathcal{L}\Theta = 0 \tag{1.16}$$

The calculation of $\mathcal{L}V$ gives

$$\mathcal{L}V = F(X,\Theta)\cdot\mathcal{L}\Theta = 0 \tag{1.17}$$

and hence $\mathcal{L}\psi = 0$.

Using (1.14) one finds $V(x,v,t)\,|_{x=0} = v$ and hence (1.10) is satisfied also.

2. General Case

Suppose now that the motion of a particle can be described by equations:

$$\frac{d\vec{r}}{dt} = \vec{v} \tag{2.1}$$

$$\frac{d\vec{v}}{dt} = \vec{F}\,(\vec{r}\,,\vec{v}\,,t) \tag{2.2}$$

The Liouville equation for the distribution function $\psi = \psi(\vec{r}\,,\vec{v}\,,t)$ in 6-D phase space is:

$$\mathcal{L}\psi \equiv \frac{\partial\psi}{\partial t} + \vec{v}\cdot\nabla_{\vec{r}}\psi + \vec{F}\,\nabla_{\vec{v}}\psi = 0 \tag{2.3}$$

To solve this equation by the same method one needs now two vector characteristic functions:

$$\vec{P} \equiv \vec{U} - \vec{v} + \int_{t'}^{t} dt''\,\vec{F}\,(\vec{S}\,(\vec{r}\,,\vec{v}\,,t,t''),\,\vec{U}\,(\vec{r}\,,\vec{v}\,,t,t''))\,dt'' = 0 \tag{2.4}$$

$$\vec{G} \equiv \vec{S} - \vec{r} + \vec{v}\cdot(t-t') - \int_{t'}^{t} dt''(t''-t')\,\vec{F}\,(\vec{S}\,,\vec{U})\,dt'' = 0 \tag{2.5}$$

Functions $\vec{S}$ and $\vec{U}$ have the following properties:

$$\vec{S}\,(\vec{r}\,,\vec{v}\,,t,t) = \vec{r} \tag{2.6}$$

$$\vec{U}\,(\vec{r}\,,\vec{v}\,,t,t) = \vec{v} \tag{2.7}$$

$$\mathcal{L}\vec{S} = 0 \tag{2.8}$$

$$\mathcal{L}\vec{U} = 0 \tag{2.9}$$

The proof of these properties is a multi-dimensional generalization of the proof given in Section 1.

Solution of the Liouville equation can be expressed in terms of the characteristic functions $\vec{S}$ and $\vec{U}$.

2.1 The Initial Value Problem

If ψ is known at $t = t_0$:

$$\psi(\vec{r}\,,\vec{v}\,,t)\,|_{t=t_0} \;=\; \psi_0\,(\vec{r}\,,\vec{v})\;, \tag{2.10}$$

then for any subsequent time moment:

$$\psi\,(\vec{r}\,,\vec{v}\,,t) \;=\; \psi_0(\vec{R}\,,\vec{V}) \quad , \tag{2.11}$$

where

$$\vec{V} \;=\; \vec{v} \;-\; \int_{t_0}^{t} dt'\;\vec{F}\,(\vec{S}\,,\vec{U}\,,t') \quad , \tag{2.12}$$

$$\vec{R} \;=\; \vec{r} \;-\; \vec{v}\cdot(t-t_0) + \int_{t_0}^{t} dt'\cdot(t'-t_0)\cdot\vec{F}\,(\vec{S}\,,\vec{U}\,,t')\;, \tag{2.13}$$

where $\vec{S}$ and $\vec{U}$ are implicitly defined in (2.4) and (2.5).

2.2 The Boundary Value Problem

If ψ is known at some point $\vec{r} = 0$

$$\psi\,(\vec{r}\,,\vec{v}\,,t)\,|_{\vec{r}=0} \;=\; \psi_0\,(\vec{v}\,,t) \quad , \tag{2.14}$$

then at any other place:

$$\psi = \psi_0\,(\vec{V}\,,\Theta) \quad , \tag{2.15}$$

where

$$\vec{V} = \vec{v} \;-\; \int_{\Theta}^{t} dt'\;\vec{F}\,(\vec{S}\,,\vec{U}\,,t') \tag{2.16}$$

and function Θ is implicitly defined as such a solution of the equation

$$\vec{r} \;-\; \vec{v}\cdot(t-\Theta) \;+\; \int_{\Theta}^{t} dt'\cdot(t'-\Theta)\;\vec{F}\,(\vec{S}\,,\vec{U}\,,t') = 0 \tag{2.17}$$

which satisfies the condition:

$$\Theta\,(\vec{r}\,,\vec{v}\,,t)\,|_{\vec{r}=0} \;=\; t \quad . \tag{2.18}$$

3. Conclusion

The method of solving the Liouville equation described here rests heavily on the existence of two implicit functions $\vec{U}$ and $\vec{S}$. Provided these functions are known many interesting problems can be solved by means of integration. For example, any bunch characteristics at the exit of a system (rms values, invariant curves, Poincaré sections of different kind, etc.) can be found from the known distribution function.

The computation of the characteristic functions $\vec{U}$ and $\vec{S}$ is transformed here to a (nonlinear) integral equation. There are no known methods of solving such equations other than a numerical one. At present time one can only guess which one of numerical solutions – direct solution of the differential equation (2.3) or solution of an integral equation – is faster and less expensive. The only general clue to this question is that there are cases when an integration is stable numerically while a differentiation is unstable.

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