MAGNETIC SYSTEMS FOR LINEAR ACCELERATOR BEAM INJECTION

by

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I. INTRODUCTION

The problem of multiple-beam operation in a linear electron accelerator is greatly simplified if the electrons can be injected into the machine at some angle with respect to the axis of the accelerator. In this case, if the electron beam is bunched before entering the accelerator, it would not be necessary to place the guns and the bunching devices in the main tunnel, thus making it possible to replace or repair these elements without stopping the machine.

It is necessary for the deflecting system to preserve the parallel characteristic of the beam and to ensure that no appreciable debunching occurs in a polyenergetic beam having a momentum spread of the order of 10%.

The first condition is achieved by systems having zero transverse dispersion for object and image points at infinite distance (i.e., achromatic). These systems do not introduce the time-of-flight dispersion that is the cause of debunching (or "longitudinal dispersion") for a monoenergetic beam. The path length is only a function of energy, and the choice of a suitable system for injection, therefore, will be determined by the debunching coefficient of the system and the energy spread of the beam. It is possible to imagine some isochronous systems where the longitudinal dispersion is only a second-order effect.

In this report we give some general results in a matrix form necessary to study multicomponent systems, and we analyze some of the simplest systems that seem available for injection purposes.

We have limited our study to the first order, which is sufficient to choose an appropriate deflecting system.
1. Matrix Notations

The position of a ray in the beam, in a transverse plane, is given by two distances and two angles obtained by projecting the trajectory on the horizontal and vertical planes. Generally the central trajectory is a planar curve, and this plane is defined here as the horizontal plane (Fig. 1).

\[ MU = \text{central trajectory} \]
\[ NV = \text{projection of a trajectory on the horizontal plane} \]
\[ MN = x \]
\[ PW = \text{projection of a trajectory on the vertical plane tangent to the central trajectory} \]
\[ MP = y \]

Fig. 1


If \( t \) is the running variable along the central trajectory, the first-order motion is given by the following relations:

\[
\begin{bmatrix}
  x(t) \\
  \theta(t) \\
  \Delta p/p_0
\end{bmatrix} = M_H(t) \begin{bmatrix}
  x(0) \\
  \theta(0) \\
  \Delta p/p_0
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
  y(t) \\
  \phi(t) \\
  \Delta \phi/p_0
\end{bmatrix} = M_V(t) \begin{bmatrix}
  y(0) \\
  \phi(0) \\
  \Delta \phi/p_0
\end{bmatrix}
\]

Often it is more convenient to know the \( M_H \) and \( M_V \) matrices for each elementary magnet and obtain the total transfer matrix by multiplication. This report will use this technique.

2. **Tabulation of the Different Matrices**

2.1 Uniform field bending magnet (Fig. 2).

![Diagram](image)

O = center of curvature of the central trajectory

\( F_1 \) and \( F_2 \): poles faces

Fig. 2
\[
M_h(\alpha_1, \beta_1, \beta_2) = \begin{pmatrix}
\frac{\cos(\alpha - \beta_1)}{\cos \beta_1} & \rho \sin \alpha & \rho(1 - \cos \alpha) \\
\frac{-(1 - \tan \beta_1 \tan \beta_2) \sin(\alpha - \beta_1 - \beta_2)}{\rho \cos(\beta_1 + \beta_2)} & \frac{\cos(\alpha - \beta_2)}{\cos \beta_2} & \sin \alpha + (1 - \cos \alpha) \tan \beta_2 \\
0 & 0 & 1
\end{pmatrix}
\]

(1)

\[
M_v(\alpha_1, \beta_1, \beta_2) = \begin{pmatrix}
\frac{1 - \alpha \tan \beta_1}{\rho} & \rho \alpha & 0 \\
\frac{-\alpha \tan \beta_1 + \tan \beta_2 + \alpha \tan \beta_1 \tan \beta_2}{\rho} & \frac{1 - \alpha \tan \beta_2}{\rho} & 1 - \alpha \tan \beta_2 \\
0 & 0 & 1
\end{pmatrix}
\]

(2)
If the magnet bends the beam in the opposite direction (inverted magnet), the horizontal matrix is derived from the normal matrix by $M_I = IMI$ with

$$I = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.2 "n-type" magnet with perpendicular input and output.

$$M_H(\alpha,n) = \begin{pmatrix} \cos \sqrt[1-n] \alpha & \rho \sin \sqrt[1-n] \alpha & \rho \frac{1 - \cos \sqrt[1-n] \alpha}{1-n} \\ -\sqrt[1-n] \sin \sqrt[1-n] \alpha & \cos \sqrt[1-n] \alpha & \frac{\sin \sqrt[1-n] \alpha}{\sqrt[1-n]} \\ 0 & 0 & 1 \end{pmatrix}$$

(3)

$$M_V(\alpha,n) = \begin{pmatrix} \cos \sqrt{n} \alpha & \rho \frac{\sin \sqrt{n} \alpha}{n} & 0 \\ -\sqrt{n} \sin \sqrt{n} \alpha & \cos \sqrt{n} \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(4)

If $n = \frac{1}{2}$, both matrices are identical for a monoenergetic beam.
2.3 Quadrupole magnet.

If we assume the quadrupole magnet to be a thin lens, the matrices are very simple:

\[
M_H = \begin{pmatrix}
1 & 0 & 0 \\
-\frac{1}{r} & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\quad M_V = \begin{pmatrix}
1 & 0 & 0 \\
\frac{1}{r} & 1 & 0 \\
\rho & 0 & 1
\end{pmatrix}
\]

with

\[
\frac{1}{r} = \frac{e \delta B}{P \delta x}
\]

2.4 Drift space.

The transfer through a space without a magnetic field is given by the following matrices:

\[
M_H = M_V = \begin{pmatrix}
1 & \ell & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

where \( \ell \) is the distance between the two reference planes.

3. Perturbation Matrices for a Displaced Magnet

3.1 Uniform field bending magnet.

In a system that involves several magnets, it is generally possible to define the theoretical position of an element by a point \( \omega \) and three directions \( (\omega u, \omega v, \omega w) \). The most convenient choice of these four elements is shown in Fig. 3.
The real magnet generally is displaced with respect to its theoretical position. This displacement can be represented by a translation $\mathbf{w}$ and rotations around $\omega_u$, $\omega_v$, $\omega_w$. We shall assume that these displacements are small enough to neglect second-order terms.

The position of the real magnet is given by six quantities:

- $a, b, c$ are the projections of $\mathbf{w}$ on $\omega_u$, $\omega_v$, $\omega_w$
- $\gamma, \delta, \beta$ are the angles of rotation around $\omega_u$, $\omega_v$, $\omega_w$
The matrices written in 2.1 give the structure of the beam, the distances and angles being measured with respect to the theoretical central orbit of the real magnet (defined by \( \omega', \omega'u', \omega'v', \omega'w' \)). It is interesting to have the structure of the beam and the distances and angles measured with regard to the central orbit of the theoretical magnet (defined by \( \omega, \omega_u, \omega_v, \omega_w \)).

To first order, the displacement of a magnet introduces a displacement of the output beam without any modification of the internal structure of the beam. The position of the new central orbit at the output will be defined by two distances \( D_H, D_v \) and two angles \( \Delta_H, \Delta_v \). To be able to use the matrix formalism we shall introduce the four-component vectors \( x, \theta, \Delta \rho/\rho_0, 1 \) and \( y, \varphi, \Delta \varphi/\rho_0, 1 \).

The transfer matrices for a displaced magnet will be:

\[
\begin{pmatrix}
\begin{bmatrix}
M(\alpha_1 \beta_2)
\end{bmatrix} & D \\
(3 \times 3) & \Delta
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}
\]

with

\[
D_H = a \sin \alpha \left[ \sin \frac{\alpha}{2} \tan \beta_1 \cos \frac{\alpha}{2} + b \sin \alpha \left[ \cos \frac{\alpha}{2} \tan \beta_1 \sin \frac{\alpha}{2} \right] \right]
- \beta \sin \alpha \tan \frac{\alpha}{2} \tan \beta_1
\]

\[
\Delta_H = a \left[ \sin \alpha \cos \frac{\alpha}{2} - \cos \alpha \cos \frac{\alpha}{2} \left( \tan \beta_1 + \tan \beta_2 \right) + \sin \alpha \cos \frac{\alpha}{2} \tan \beta_1 \tan \beta_2 \right]
+ b \left[ - \sin \alpha \sin \frac{\alpha}{2} + \cos \alpha \sin \frac{\alpha}{2} \left( \tan \beta_1 + \tan \beta_2 \right) + \sin \alpha \sin \frac{\alpha}{2} \tan \beta_1 \tan \beta_2 \right]
+ \beta \left[ \cos \alpha \tan \frac{\alpha}{2} \tan \beta_1 - \sin \alpha \tan \frac{\alpha}{2} \tan \beta_1 \tan \beta_2 \right]
\]
$D_V = c \alpha \tan \beta_1 + \gamma \left[ -2 \sin \frac{\alpha}{2} + \alpha \cos \frac{\alpha}{2} + \alpha \sin \frac{\alpha}{2} \tan \beta_2 \right]$

\[- \delta \left[ \alpha \sin \frac{\alpha}{2} + \alpha \frac{\sin^2 \alpha/2}{\cos \alpha/2} \tan \beta_1 \right] \]

$\Delta_V = c \left( \tan \beta_1 + \tan \beta_2 \right) - \gamma \left[ \sin \alpha \tan \beta_1 + \left( \sin \frac{\alpha}{2} + \alpha \cos \frac{\alpha}{2} \right) \tan \beta_2 \right]$

\[- \delta \left[ 2 \sin \frac{\alpha}{2} + \frac{\sin^2 \alpha/2}{\cos \alpha/2} \left( \tan \beta_1 + \tan \beta_2 \right) - \alpha \sin \frac{\alpha}{2} \tan \beta_2 \right] \]

3.2 Quadrupole magnet.

We shall follow the same procedure as was used for bending magnets. Here $w$ is the center of the quadrupole lens.

$D_H = a \left( 1 - \cos kL \right) + \beta \left[ \frac{L}{2} \left( 1 + \cos kL \right) - \frac{\sin kL}{k} \right]$  

$\Delta_H = a k \sin kL + \beta \left[ -\frac{L k}{2} \sin kL + 1 - \cos kL \right]$  

$D_V = c \left( 1 - \cosh kL \right) + \gamma \left[ \frac{L}{2} \left( 1 + \cosh kL \right) - \frac{\sinh kL}{k} \right]$  

$\Delta_V = -ck \sinh kL + \gamma \left[ 1 - \cosh kL + \frac{kL}{2} \sinh kL \right]$  

In the thin-lens approximation these expressions are

$D_H = 0 \quad D_V = 0$  

$\Delta_H = \frac{a}{r} \quad \Delta_V = -\frac{c}{r}$  

which it is easy to verify directly.

---

*In this formula $a, \delta, c$ are normalized to the radius of curvature of the central trajectory.*
4. Path Length

It is important to know the path length as a function of input conditions \((x_o, \theta_o, \Delta p/p_o, y_o, \phi_o)\) in order to calculate the longitudinal dispersion. In Fig. 4, MM' is the central trajectory and NN' another trajectory in the horizontal plane.

\[
\begin{align*}
MM' &= dt \\
NN' &= ds
\end{align*}
\]

\[
\rho = \rho_o + x
\]

\[
ds = \sqrt{\rho^2 + \left(\frac{d\rho}{d\alpha}\right)^2} \, d\alpha
\]

Generally \(\rho_o\) is constant through a magnet. To first order, \(ds = \rho_o \, d\alpha(1 + x/\rho_o)\), and the path length is

\[
L = \int_{0}^{\alpha} \rho_o \, d\alpha + \int_{0}^{\alpha} x \, d\alpha = L_o + \int_{0}^{\alpha} x \, d\alpha
\]

(10)
4.1 For a uniform field magnet:

\[ x = \left[ \cos \alpha + \sin \alpha \tan \beta \right] x_0 + \rho_0 \sin \alpha \theta_0 + \rho_0 (1 - \cos \alpha) \frac{\Delta \rho}{\rho_0} \]

\[ L - L_0 = \left[ \sin \alpha + (1 - \cos \alpha) \tan \beta \right] x_0 + \rho_0 (1 - \cos \alpha) \theta_0 + \rho_0 (\alpha - \sin \alpha) \frac{\Delta \rho}{\rho_0} \]

(11)

4.2 For an n-type magnet with perpendicular input and output:

\[ x = \cos \sqrt{1 - n} \alpha \cdot x_0 + \rho_0 \frac{\sin \sqrt{1 - n} \alpha}{\sqrt{1 - n}} \theta_0 + \rho_0 \frac{1 - \cos \sqrt{1 - n} \alpha}{1 - n} \frac{\Delta \rho}{\rho_0} \]

(12)

and

\[ L - L_0 = \frac{\sin \sqrt{1 - n} \alpha}{\sqrt{1 - n}} x_0 + \rho_0 \frac{1 - \cos \sqrt{1 - n} \alpha}{1 - n} \theta_0 + \rho_0 \frac{\sqrt{1 - n} \alpha - \sin \sqrt{1 - n} \alpha}{(1 - n)^{3/2}} \frac{\Delta \rho}{\rho_0} \]

(13)

From Eq. (10) we can see that the first-order path length difference is zero when there is no curvature of the central orbit. This happens in a drift space, in the vertical motion of a bending magnet, and in a quadrupole lens. It is convenient for calculation to introduce the path length in the matrix formalism of the horizontal motion by adding another row and another column to the matrix:

\[
\begin{pmatrix}
  x \\
  \theta \\
  \Delta \rho \\
  L
\end{pmatrix}
= \begin{pmatrix}
  M_H \\
  (3 \times 3)
\end{pmatrix}
\begin{pmatrix}
  0 \\
  0 \\
  0 \\
  (L|x_0|, L|\theta_0|, L|\Delta \rho|)$\rho_0$
\end{pmatrix}
\begin{pmatrix}
  0 \\
  \theta_0 \\
  \Delta \rho \\
  L
\end{pmatrix}
\]

(14)
4.3 Normalization of the matrices.

It is possible to simplify the $4 \times 4$ matrices for a bending magnet with perpendicular input and output by using new variables:

$$\begin{pmatrix}
\frac{x \sqrt{1 - n}}{\rho_o} \\
\frac{\theta}{\rho_o} \\
\frac{\Delta \varphi}{\rho_o} \cdot \frac{1}{\sqrt{1 - n}} \\
\frac{L (1 - n)}{\rho_o}
\end{pmatrix} =
\begin{pmatrix}
\cos \alpha' & \sin \alpha' & 1 - \cos \alpha' & 0 \\
- \sin \alpha' & \cos \alpha' & \sin \alpha' & 0 \\
0 & 0 & 1 & 0 \\
\sin \alpha' & 1 - \cos \alpha' & \alpha' - \sin \alpha' & 1
\end{pmatrix}
\begin{pmatrix}
\frac{x_0 \sqrt{1 - n}}{\rho_o} \\
\frac{\theta_0}{\rho_o} \\
\frac{\Delta \varphi}{\rho_o} \cdot \frac{1}{\sqrt{1 - n}} \\
\frac{L_0 (1 - n)}{\rho_o}
\end{pmatrix}
$$

(15)

$$\begin{pmatrix}
\frac{y_0 \sqrt{1 - n}}{\rho_o} \\
\frac{\varphi}{\rho_o} \\
\frac{\Delta \varphi}{\rho_o} \\
L
\end{pmatrix} =
\begin{pmatrix}
\cos \alpha'' & \sin \alpha'' & 0 & 0 \\
- \sin \alpha'' & \cos \alpha'' & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{y_0 \sqrt{1 - n}}{\rho_o} \\
\varphi_0 \\
\frac{\Delta \varphi}{\rho_o} \\
L
\end{pmatrix}
$$

(16)

with $\alpha' = \sqrt{1 - n} \alpha$ and $\alpha'' = \sqrt{n} \alpha$. 
With these variables the matrices for a quadrupole lens and a
Drift space are

\[
M_{QH} = \begin{pmatrix}
1 & 1 & 0 & 0  \\
-\frac{\rho_0}{f \sqrt{1 - n}} & 1 & 0 & 0  \\
0 & 0 & 1 & 0  \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
M_{QV} = \begin{pmatrix}
1 & 0 & 0 & 0  \\
\frac{\rho_0}{f \sqrt{n}} & 1 & 0 & 0  \\
0 & 0 & 1 & 0  \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
M_H = \begin{pmatrix}
1 & \frac{\sqrt{1 - n}}{\rho_0} & 0 & 0  \\
0 & 1 & 0 & 0  \\
0 & 0 & 1 & 0  \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
M_V = \begin{pmatrix}
1 & \frac{\sqrt{n}}{\rho_0} & 0 & 0  \\
0 & 1 & 0 & 0  \\
0 & 0 & 1 & 0  \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

5. **General Considerations**

To simplify the writing we shall use the following notations:

\[
x = \sum_i (x|q_i)q_i  \\
\theta = \sum_i (\theta|q_i)q_i  \\
L = \sum_i (L|q_i)q_i
\]

where \(q_i\) are the input conditions. All the coefficients defined above
are functions of the running variable \( t \). If

\[
h_0 = -\frac{e}{\rho_0} B_y(x = 0, y = 0, t)
\]

then the coefficients for \( t = t_0 \) are

\[
\begin{align*}
\langle x | x_0 \rangle &= c(t_0) \\
\langle x | \theta_0 \rangle &= s(t_0) \\
\langle x | \frac{\Delta \Phi}{p_0} \rangle &= s(t_0) \int_0^{t_0} h_0 c(t) \, dt - c(t_0) \int_0^{t_0} h_0 s(t) \, dt \\
\langle \theta | \frac{\Delta \Phi}{p_0} \rangle &= s'(t_0) \int_0^{t_0} h_0 c(t) \, dt - c'(t_0) \int_0^{t_0} h_0 s(t) \, dt \\
\langle L | x_0 \rangle &= -\int_0^{t_0} h_0 c(t) \, dt \\
\langle L | \theta_0 \rangle &= -\int_0^{t_0} h_0 s(t) \, dt
\end{align*}
\]  

(17)*

The most important magnetic systems are those that have a plane of symmetry, as shown in Fig. 5.

*The fourth relation in Eq. (17) is true only if there is no discontinuity for \( t = t_0 \) (for example, a rotated pole face).
The conditions to transfer a parallel beam to a parallel beam without transverse dispersion are

\[
(\theta | x_0)_{C} = c'(t_0) = 0
\]

\[
(x \frac{\Delta p}{\rho_0})_{C} = s(t_0) \int_{0}^{t_0} h_0 c(t) \, dt - c(t_0) \int_{0}^{t_0} h_0 s(t) \, dt = 0 \quad (18)
\]

\[
(\theta \frac{\Delta p}{\rho_0})_{C} = s'(t_0) \int_{0}^{t_0} h_0 c(t) \, dt - c'(t_0) \int_{0}^{t_0} h_0 s(t) \, dt = 0
\]
or

\[ c'(t_0) = 0 \]

\[ \int_0^{t_0} h_0 c(t) \, dt = 0 \quad \text{or} \quad (L | x_0)_c = 0 \]  \hspace{1cm} (19)

\[ \int_0^{t_0} h_0 s(t) \, dt = 0 \quad \text{or} \quad (L | \theta_0)_c = 0 \]

or

\[ (x | x_0)_B = c(t_1) = 0 \]  \hspace{1cm} (20)

\[ (\theta | \Delta \theta)_{B_0} = s'(t_1) \int_0^{t_1} h_0 c(t) \, dt - c'(t_1) \int_0^{t_1} h_0 s(t) \, dt = 0 \]

These three sets of conditions are equivalent, and the total horizontal transfer matrix is

\[
M_H = \begin{pmatrix}
-1 & s(t_0) & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & (L | \Delta \theta)_{P_0} & 1
\end{pmatrix}
\]  \hspace{1cm} (21)
If the system involves some vertical focusing, it is necessary to add another condition,

\[
\left(\varphi | y_0 \right)_C = 0
\]

that involves \( \left( y | y_0 \right)_B = 0 \) or \( \left( \varphi | y_0 \right)_B = 0 \).*

Generally the necessary conditions for transferring a parallel beam to a parallel beam without transverse and longitudinal dispersion are represented by four equations written in the plane of symmetry:

\[
\begin{align*}
(x | x_0) &= 0 \\
(y | y_0) &= 0 \quad [\text{or } (\varphi | y_0) = 0] \\
(\theta | \Delta \varphi \, p_0) &= 0 \\
(L | \Delta \varphi \, p_0) &= 0
\end{align*}
\]

(22)

To calculate the non-zero terms \( (x | \theta_0) \) and \( (y | \varphi_0) \) we can use 2 \times 2 matrices:

\[
\begin{pmatrix}
c(t) & s(t) \\
c'(t) & s'(t)
\end{pmatrix}
\]

* Generally the system is simpler to solve when \( (y | y_0)_B = 0 \). For example, if n-type magnets are used to achieve a vertical focusing, \( n = \frac{1}{3} \) will give an easy solution; the quadrupole lens must be located at the cross-over. In these conditions the motion is the same in both planes. To solve the system with \( (\varphi | y_0)_B = 0 \), we must keep \( n \) as a parameter and the equations are more complicated. For the n-type magnet systems described in this report, where we have made \( n = \frac{1}{2} \), it is possible to imagine the equivalent system with \( n \neq \frac{1}{2} \) that achieves \( (\varphi | y_0) = 0 \). In this case the system is more compact, the dispersion is less, and the beam on the plane of symmetry is focused only in the horizontal plane.
III. PROPERTIES OF THE BEAM

Before starting the study of the different systems available, it is necessary to know the properties required for injecting the beam into the machine. As specified for the Stanford two-mile accelerator, the properties of the beam are as follows:

1. **Optical Properties**
   - Beam diameter < 1 cm
   - Beam angular divergence < 0.1 radian

   These conditions are met to first order by Eq. (22). It is important to calculate the tolerances on the alignment of the components and to calculate the second-order effects in order to maintain these conditions.

2. **Bunching Properties**
   - Phase spread extension = 5°
   - Phase coherence = ± 5°

3. **Energy Spread**: Two important phenomena may increase the energy spread of the beam.
   3.1 **Beam loading.** With one ten-foot accelerator section (with normal beam loading of 10%) before the deflecting system, the contribution of the beam loading to the energy spread can be reduced to 5%.
   3.2 **Incorrect "phasing" of the injection section.**

   After three or four wavelengths, the electron bunches reach an asymptotic phase. If the field has not the correct value this phase will be different from 90°. For example, if $\alpha \approx 2.8$, the electrons

---


** Here $\alpha$ is the parameter generally used to describe an accelerator section: the energy gain per wavelength with respect to the rest energy of electrons is

$$\alpha = \frac{eE_0\lambda_0}{m_0c^2}$$
are 20° off the crest of the wave and this will contribute about 3% to the energy spread. Under these conditions the momentum spread would be $\Delta p/p_0 \approx 8\%$; we should assume a 10% spread in the following calculations. If two accelerator sections are used before deflection the beam loading can be reduced and a proper phasing of the second section can cancel the effect of the first. Under these conditions 3% can be the expected value for $\Delta p/p_0$.

4. **Deflection Angle**

A more convenient value is 90°, but this seems impossible to achieve, as will be seen below, with simple nonisochronous systems for which the $(L|\Delta p/p_0)$ coefficient fixes the maximum angle of deviation.

5. **Radius of Curvature**

It would be convenient to pulse the deflecting systems in order to avoid perturbation to the other beams in the accelerator. This requires a low magnetic field and large radius of curvature. But $(L|\Delta p/p_0)$ is proportional to the radius of curvature, and a compromise between these two aspects is

$$ B = 5000 \text{ gauss} $$

and

$$ \rho_{cm} = \frac{10^4}{3} \frac{E}{5000} = \frac{2}{3} E $$

where $E$ is energy in Mev. For $E = 40$ Mev, which is the energy at the end of one accelerator section,

$$ \rho_0 = 26.6 \text{ cm} = 10.5 \text{ in.} $$
IV. THREE UNIFORM FIELD BENDING MAGNET SYSTEMS

1. Identical Magnet System (Fig. 6)

1.1 Conditions for zero transverse dispersion. The distance $l$ between magnets has to be

$$ l = \rho_o \frac{2 \cos \alpha - 1}{\sin \alpha} $$

and the total transfer matrix in the horizontal plane is

$$
\begin{pmatrix}
-1 & \rho_o \frac{2 \cos \alpha - 1}{\sin \alpha} \\
0 & -1
\end{pmatrix}
$$

The maximum deviation angle for this system is $D = \alpha = 60^\circ$, which corresponds to $l = 0$. 
1.2 Debunching. The path length of an arbitrary trajectory is

\[ L = L_0 + \rho_0 (3\alpha - 4 \sin \alpha) \frac{\Delta \varphi}{\rho_0} \]

and the corresponding debunching is

\[ \Delta \varphi = 360^\circ \frac{L - L_0}{\lambda_0} = -360^\circ (4 \sin \alpha - 3\alpha) \frac{\rho_0 \Delta \varphi}{\lambda_0 \rho_0} \]

<table>
<thead>
<tr>
<th>( \alpha ) (degrees)</th>
<th>( \Delta \varphi ) (degrees per % of ( \Delta \varphi )/( \rho_0 ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>30°</td>
<td>- 3.9</td>
</tr>
<tr>
<td>40°</td>
<td>- 4.3</td>
</tr>
<tr>
<td>50°</td>
<td>- 4.1</td>
</tr>
</tbody>
</table>

Debunching for \( \rho_0 = 26.6 \text{ cm} \)

Below \( \alpha = 30^\circ \) the debunching is proportional to \( \alpha \).

We see from the table above that this system is not acceptable for an injection device because of the large debunching, even for a very small angle of deflection.
2. Non-Identical Magnet Systems (Fig. 7)

The conditions (22) are

\[ (x|x_o) = \cos \alpha' \left[ \cos \alpha - \frac{l \sin \alpha}{\rho} \right] - \rho' \frac{\sin \alpha \sin \alpha'}{\rho} = 0 \]

\[ l = \frac{\rho}{\tan \alpha} - \rho' \tan \alpha' \quad (23) \]

\[ (e|\Delta P) = \sin \alpha' (1 - \cos \alpha) \frac{\rho}{\rho'} - \frac{l \sin \alpha \sin \alpha'}{\rho'} + \sin \alpha \cos \alpha' - \sin \alpha' = 0 \]

\[ (1 + \frac{\rho}{\rho'}) \sin 2\alpha' = 2 \sin \alpha \quad (24) \]
\[(L \Delta \rho / \rho_o) = - \sin \alpha' [\rho (1 - \cos \alpha) + \ell \sin \alpha] - \rho' \sin \alpha (1 - \cos \alpha')
+ \rho' (\alpha' - \sin \alpha') + \rho (\alpha - \sin \alpha) = 0
\]
\[
\left( \frac{2 \sin \alpha}{\sin 2 \alpha} - 1 \right) (\alpha - \sin \alpha - \sin \alpha') + (\alpha' - \sin \alpha' - \sin \alpha) + \frac{\sin \alpha}{\sin \alpha'} = 0
\]

We have taken \( \rho/\rho' = k \) as a parameter and plotted in Curve I the curves \( k = \text{constant} \), \( \ell = 0 \) and \( (L \Delta \rho / \rho_o) = 0 \). The curve \( \ell = 0 \) specifies in the plane the available region \((\ell, \theta)\) to have a zero transverse-dispersion system for a parallel beam. The part of the curve \( (L \Delta \rho / \rho_o) = 0 \) in this region gives all the isochronous systems that are physically possible. The angle of deviation is \( D = 2(\alpha - \alpha') \). As shown in Curve I, the maximum deviation for an isochronous system is \( D = 105^\circ \).

For \( D = 90^\circ \) the parameters are as follows:

\[ \alpha = 52^\circ 30' \quad \rho/\rho' = 5.13 \]
\[ 2\alpha' = 15' \quad \ell/\rho' = 3.81 \]

It is possible to use the curves drawn for the uniform field system to study the case of n-type magnets when \( n = \frac{1}{2}^* \) by using the variables defined in I.4.3. Then the diagram is plotted in terms \( \alpha/\sqrt{2} \) and \( \alpha'/\sqrt{2} \) with

\[ D = 2(\alpha - \alpha') = 2\sqrt{2} \left[ \frac{\alpha}{\sqrt{2}} - \frac{\alpha'}{\sqrt{2}} \right] \]

For \( D = 90^\circ \) the results are as follows:

\[ \alpha/\sqrt{2} = 34^\circ \quad \alpha = 48^\circ \]
\[ \alpha'/\sqrt{2} = 2^\circ \quad 2\alpha' = 6^\circ \]
\[ \rho/\rho' = 10 \]

The ratio of radii of curvature is too great, and this system is not practical.

*If \( n = \frac{1}{2} \) the motion in both planes is identical and the vertical motion does not introduce any supplementary conditions.
--- isochronous systems

\[ k = \frac{\rho}{\rho'} \]
V. TWO BENDING AND ONE QUADRUPOLE MAGNET SYSTEMS

1. Uniform Field Bending Magnets (Fig. 8)

Since we introduced vertical focusing in the quadrupole, we must also introduce vertical focusing in the bending magnets. This may be achieved by rotating the faces as shown in Fig. 8.

1.1 Conditions for zero transverse dispersion.

\[
\begin{align*}
(y|y_o)_B &= 1 - \frac{L}{\rho_o} \tan \beta \\
(x|x_o)_B &= \cos \alpha + \frac{L}{\rho_o} (\sin \alpha + \cos \alpha \tan \beta) = 0
\end{align*}
\]

\[
\left( \frac{\Delta p}{\rho_o} \right) = -\frac{\rho_o}{r} + \frac{\sin \alpha (1 - \cos \alpha)}{2 \cos \alpha}
\]

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These relations give the parameters

\[ I = \frac{2 \rho_0}{\tan \alpha}, \quad \tan \beta = \frac{\tan \alpha}{2}, \quad f = \rho_0 \frac{2 \cos \alpha}{\sin \alpha(1 + \cos \alpha)} \]  

(26)

The total transfer matrices for this system are

\[
M_H = \begin{pmatrix}
-1 & \frac{2 \rho_0 (1 - \cos \alpha)}{\sin \alpha} \\
0 & -1
\end{pmatrix}
\]

\[
M_V = \begin{pmatrix}
-1 & \rho_0 \left[ \frac{2 \cos \alpha}{\sin \alpha} \left(3 + \cos \alpha - 2x\right) \right] \\
0 & -1
\end{pmatrix}
\]

1.2 Path length and debunching.

As we have seen above in 11.5, the difference of path lengths depends only on \( \Delta p/p_0 \) and is twice the value between the input plane and the plane of symmetry for a ray \( x_0 = 0, \theta_0 = 0 \), and \( \Delta p/p_0 \neq 0 \). Then \( \Delta L = 2(\alpha - \sin \alpha) \rho_0 \Delta p/p_0 \), and the corresponding debunching is \( \Delta \varphi = 360^\circ \times \frac{2(\alpha - \sin \alpha)\rho_0 \Delta p}{\lambda_0 / p_0} \)

with \( \alpha = D/2 \).
Debunching for $\rho_0 = 26.6$ cm

For a 10% (± 5%) energy spread, the debunching becomes important at $60^\circ$ ($\phi \approx \pm 2^\circ$). If two accelerating sections are used, the results are about the same, the radius of curvature increasing by a factor of 2 and the energy spread by a factor of 1/3. $D = 60^\circ$ seems to be the upper limit of the deflection angle for this injection device and $D = 45^\circ$ a reasonable value. Then $\beta = 11^\circ 42'$.

1.3 Geometrical tolerances of positioning the different components.

The alignment of the system is defined by 18 quantities:

- $a_1b_1c_1\gamma_1\delta_1\beta_1$ for the first magnet
- $a_qb_qc_q\gamma_q\delta_q\beta_q$ for the quadrupole magnet
- $a_2b_2c_2\gamma_2\delta_2\beta_2$ for the second magnet

If we neglect the second-order terms we find for the displacement in the horizontal and the vertical planes

$$D_H = D_2H + \frac{1 + \cos^2 \alpha}{2 \cos \alpha} D_qH + 2 \frac{\Delta qH}{\sin \alpha} + \frac{\cos^2 \alpha - 2 \cos \alpha - 1}{2 \cos \alpha}$$

$$+ \frac{(1 - \cos \alpha)^2}{\sin \alpha} \Delta H$$
\[ \Delta_H = \Delta_{2H} - \frac{\sin \alpha}{2} D_{qH} - \frac{\sin \alpha}{2} D_{1H} - \cos \alpha \Delta_{1H} \]

\[ D_V = D_{2V} + \left(1 - \frac{\alpha \sin \alpha}{2 \cos \alpha}\right) D_{qV} + \frac{2 \cos \alpha}{\sin \alpha} \Delta_{qV} + \left(2 + \cos \alpha - \frac{\alpha \sin \alpha}{2 \cos \alpha}\right) D_{1V} \]

\[ + \left[ \frac{2 \cos \alpha}{\sin \alpha} (3 + \cos \alpha) - \alpha \right] \Delta_{1V} \]

\[ \Delta_V = \Delta_{2V} - \frac{\sin \alpha}{2 \cos \alpha} D_{qV} - \frac{\sin \alpha}{2 \cos \alpha} D_{1V} - \Delta_{1V} \]

These relations are normalized to \( \rho_0 = 1 \). If we assume that all the quantities \( |a_1|, |b_1|, \) and \( |c_1| \) are less than \( d \), and \( |\gamma_1|, |\varepsilon_1|, \) and \( |\beta_1| \) are less than \( \varepsilon \), we find (for \( \alpha \) small enough)

\[ |D_H| \leq 2 \alpha d \]

\[ |D_V| \leq 4d + \varepsilon(8 - 6\alpha) \]

\[ |\Delta_H| \leq \alpha d + \frac{\alpha \varepsilon}{2} \]

\[ |\Delta_V| \leq \frac{\alpha \varepsilon}{2} + 2\alpha \varepsilon \]
For $\alpha = 22^\circ 30'$ and $d = 10^{-3}$ (which correspond to positioning the centers $\omega$ to a precision of 0.5 mm for $\rho_0 = 26.6$ cm)

\[
D_H \leq 0.4 \text{ mm} \quad \text{and} \quad D_V \leq 2 \text{ mm}
\]

\[
\Delta H \leq 8 \times 10^{-4} + 0.28 \quad \text{and} \quad \Delta V \leq 4 \times 10^{-4} + 0.88
\]

For these conditions it follows that the tolerances on the alignment angles are of the same order of magnitude as the acceptable divergence of the beam.

1.4 Adjustment of the focal length of the quadrupole lens.

If we assume for this system

\[
\tan \beta = \frac{\tan \alpha}{2} \quad \ell = \frac{2\rho_0}{\tan \alpha}
\]

and use $f$ as a variable parameter, the total transfer matrix in the horizontal plane is

\[
M_H = \begin{pmatrix}
-1 & 4\rho_0 \left[ \frac{1 + \cos^2 \alpha}{\sin 2 \alpha} - \frac{\rho_0}{f \sin^2 \alpha} \right] & 2\rho_0 \left[ \frac{1 + \cos \alpha}{\cos \alpha} - \frac{2\rho_0}{f \sin \alpha} \right] \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

If the beam at the input is parallel, we have

\[
x_{\text{output}} = -x_{\text{input}} + 2\rho_0 \left[ \frac{1 + \cos \alpha}{\cos \alpha} - \frac{2\rho_0}{f \sin \alpha} \right] \frac{\Delta p}{\rho_0}
\]

The transverse dimension of a polyenergetic beam will increase by an amount given by:

\[
\Delta x = 2\rho_0 \left( \frac{1 + \cos \alpha}{\cos \alpha} - \frac{2\rho_0}{f \sin \alpha} \right) \frac{\Delta p}{\rho_0}
\]
If the focal length of the quadrupole is not properly adjusted, then

\[
\frac{f}{\rho_o} = \frac{2 \cos \alpha}{\sin \alpha (1 + \cos \alpha)} + \frac{\Delta f}{\rho_o} = \frac{f_0}{\rho_o} + \frac{\Delta f}{\rho_o}
\]

and

\[
\Delta x = \rho_o \frac{\Delta p}{f_0} \frac{\Delta f}{f_0} \cdot \frac{\Delta p f_0}{\rho_o f \sin \alpha} = \rho_o \frac{\Delta p}{\rho_o} \frac{1 + \cos \alpha}{2 \cos \alpha} \frac{\Delta f}{f}
\]

| \frac{\Delta f}{f_0} | \leq \frac{\Delta x_{\text{max}}}{\rho_o} \frac{2 \cos \alpha}{1 + \cos \alpha} \frac{1}{\Delta f / \rho_o}

and for the cases \( E = 40 \text{ Mev} \) and \( E = 80 \text{ Mev} \), which correspond respectively to the following parameters,

\[
\alpha = 22^\circ 30', \Delta x_{\text{max}} = 0.5 \text{ mm}, \rho_o = 266 \text{ mm}, \frac{\Delta p}{\rho_o} = \pm 5\%
\]

\[
\alpha = 22^\circ 30', \Delta x_{\text{max}} = 0.5 \text{ mm}, \rho_o = 532 \text{ mm}, \frac{\Delta p}{\rho_o} = \pm 1.5\%
\]

\( \Delta f / f_0 \) has respectively to be less than 4% and 6.5%. There is no supplementary condition for the vertical plane because there is no dispersion in this plane. From the above results we can see that the adjustment of the quadrupole focal length is not very critical. Physically, it is easy to see that the tolerance on one component is not very critical when this component is located in the plane of symmetry of the system.

2. Non-Uniform Field Bending Magnets: Modification of the Results

2.1 n-type bending magnets, \( n = \frac{1}{2} \).

It is possible to avoid the rotated pole faces by using n-type magnets. For \( n = \frac{1}{2} \) it follows, for the distance \( l \), the focal length \( f \),
and the difference of path length $\Delta L$, that

$$l = \frac{\rho_o \sqrt{2}}{\tan \alpha/\sqrt{2}}$$

$$\frac{1}{f} = \sqrt{2} \sin \alpha/\sqrt{2}$$

$$\Delta L = 2\rho_o 2^{3/2} \left[ \frac{\alpha}{\sqrt{2}} - \sin \frac{\alpha}{\sqrt{2}} \right] \frac{\Delta P}{\rho_o}$$

If $\alpha$ is small enough to neglect the $\alpha^3$ terms in $l$ and $\frac{1}{f}$, we can see by comparing Eqs. (26) and (27) that the results are exactly the same as for a uniform field magnet with rotated pole faces.

$$l = 2\rho_o / \alpha$$

$$\frac{1}{f} = \alpha / \rho_o$$

$$\Delta L \approx \rho_o \frac{\alpha^3}{3} \frac{\Delta P}{\rho_o}$$

2.2 n-type bending magnets, $n = \frac{1}{3}$.

We have seen in V.2.1 that for the two bending and one quadrupole system with $n = \frac{1}{2}$ the conditions for zero transverse dispersion are in the plane of symmetry:

$$(x | x_o) = 0$$

$$(y | y_o) = 0$$

$$(\theta | \Delta P) = 0$$

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If \( n \neq \frac{1}{2} \) it is impossible to fulfill simultaneously (29) and (30). But in this case it is possible to replace the condition (30) by

\[
(q|y_o) = 0
\]  
(32)

The interesting coefficients in the plane of symmetry are (with \( p_o = 1 \))

\[
(x|x_o) = \cos \sqrt{1 - n} \alpha - \sqrt{1 - n} \ell \sin \sqrt{1 - n} \alpha
\]

\[
(q|y_o) = \frac{\cos \sqrt{n'} \alpha}{2f} - \sqrt{n} \sin \sqrt{n'} \alpha \left(1 + \frac{\ell}{2f}\right)
\]

\[
(\theta|p_o) = \frac{\sin \sqrt{1 - n} \alpha}{\sqrt{1 - n}} - \frac{1}{2f(1 - n)} \left(1 - \cos \sqrt{1 - n} \alpha + \ell \sqrt{1 - n} \sin \sqrt{1 - n} \alpha\right)
\]

By writing the conditions (29), (31), and (32), we obtain a set of three equations which give the three parameters \( f, \ell, \) and \( n \):

\[
\ell = \frac{\cos \sqrt{1 - n} \alpha}{\sqrt{1 - n} \sin \sqrt{1 - n} \alpha}
\]

\[
\frac{1}{2f} = \sqrt{1 - n} \sin \sqrt{1 - n} \alpha
\]

and \( n \) must verify the following relation

\[
\frac{1}{2f} = \sqrt{1 - n} \sin \sqrt{1 - n} \alpha = \frac{\sqrt{n} \sin \sqrt{n} \alpha}{\cos \sqrt{n} \alpha - \ell \sqrt{n} \sin \sqrt{n} \alpha}
\]

After simplification we have

\[
\frac{\sin \left(\sqrt{1 - n} + \sqrt{n}\right) \alpha}{\sqrt{1 - n} + \sqrt{n}} + \frac{\sin \left(\sqrt{1 - n} - \sqrt{n}\right) \alpha}{\sqrt{1 - n} - \sqrt{n}} = \frac{2\sqrt{n}}{1 - 2n} \sin \sqrt{n} \alpha
\]

(36)
If $\alpha$ is not too great, an approximate solution is given by expanding the sinus, and then $n = \frac{1}{3}$. The path length of a particle through the system is

$$L = L_0 + 2 \frac{\alpha \sqrt{1 - n} \sin \alpha \sqrt{1 - n} \Delta p}{(1 - n)^{3/2}} \approx \frac{\alpha^3}{3} \frac{\Delta p}{p_0} \quad (37)$$

In regard to the debunching, this system is equivalent to the classical one ($n = \frac{1}{2}$ or rotated pole faces). The distance between the magnets and the center of the quadrupole is

$$l = \frac{1}{\sqrt{1 - n} \tan \sqrt{1 - n} \alpha} \approx \frac{1}{(1 - n) \alpha}$$

For $n = \frac{1}{2}$,

$$\frac{l}{\alpha} = \frac{2}{\alpha}$$

and for $n = \frac{1}{3}$,

$$\frac{l}{\alpha} = \frac{1.5}{\alpha}$$

This system is shorter than the classical one, but as we suppress the vertical focus in the quadrupole the adjustment of the focal length affects the vertical divergence of the beam. If we adjust the distance $l$ to be

$$l = \frac{1}{\sqrt{1 - n} \tan \sqrt{1 - n} \alpha}$$

at the output the following coefficients will not depend on $f$:

$$(x|x_o) = -1 \quad (\theta|x_o) = 0 \quad (\theta|\theta_o) = -1 \quad (\theta|\frac{\Delta p}{p_0}) = -1$$

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The only term for the horizontal motion that depends on $f$ is
\[
\left( \begin{array}{c} x \\ p_0 \\
\end{array} \right) = \frac{2}{1-n} - \frac{1}{f(1-n)^{3/2} \sin \sqrt{1-n} \alpha}
\]

and
\[
x = -x_0 + \frac{\Delta p}{p_0} \left[ \frac{2}{1-n} - \frac{1}{f(1-n)^{3/2} \sin \sqrt{1-n} \alpha} \right]
\]

For the vertical motion
\[
\phi = y_0 \left[ \sqrt{\frac{n}{1-n}} \frac{\sin n \alpha}{\tan \sqrt{1-n} \alpha} - \cos \sqrt{n} \alpha \right]
\]
\[
x \left[ 2 \sqrt{n} \sin \sqrt{n} \alpha + \frac{1}{f} \left( \frac{\sqrt{n}}{1-n} \frac{\sin \sqrt{n} \alpha}{\tan \sqrt{1-n} \alpha} - \cos \sqrt{n} \alpha \right) \right]
\]

If the focal length is $f = f_0 + \Delta f$, with $f_0$ the correct value, then the divergence $\phi$ in the vertical plane is
\[
\phi = 2 \sqrt{n} \sin \sqrt{n} \alpha \left[ \frac{n}{1-n} \frac{\sin \sqrt{n} \alpha}{\tan \sqrt{1-n} \alpha} - \cos \sqrt{n} \alpha \right] \frac{\Delta f}{f_0} y_0
\]

and to a good approximation
\[
\phi \approx 2 \pi \Delta f \left( \frac{2n-1}{1-n} \right) \frac{\Delta f}{f_0} y_0 = -\frac{\alpha \Delta f}{f_0} y_0
\]

If the specifications require $|\phi| \leq \phi_0$, then
\[
\left| \frac{\Delta f}{f_0} \right| \leq \frac{3\phi_0}{\alpha y_0}
\]

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where $y_0$ is equal to half the height of the beam normalized to the radius of curvature.

For $\alpha = 22^\circ 30' \ (\alpha = 0.4)$, and $y_0 = 2 \times 10^{-2}$,

$$\left| \frac{\Delta f}{f_0} \right| \leq \frac{3}{8 \times 10^{-3}} \Phi_0$$

We see from this formula that the requirement for the adjustment is not very critical. This adjustment reacts also on the vertical dimension of the beam:

$$y = y_0 \left[ A + \frac{B}{f} \right]$$

with

$$A = C''^2 - S''^2 - 2 \frac{\sqrt{n}}{\sqrt{1 - n}} \frac{S''C'C''}{S'}$$

$$B = \frac{S''C''}{\sqrt{n}} + \frac{C'C''^2}{\sqrt{1 - n} S'} - \frac{\sqrt{n}}{1 - n} \frac{S''C'C''^2}{S'^2} - \frac{S''C'}{\sqrt{1 - n} S'}$$

$$S' = \sin \sqrt{1 - n} \alpha \quad C' = \cos \sqrt{1 - n} \alpha$$

$$S'' = \sin \sqrt{n} \alpha \quad C'' = \cos \sqrt{n} \alpha$$

if

$$\frac{1}{f} = \frac{1}{f_0} + \Delta \left( \frac{1}{f} \right)$$

then

$$y = y_0 \left[ A + \frac{B}{f_0} + B \Delta \left( \frac{1}{f} \right) \right] = -y_0 + B y_0 \Delta \left( \frac{1}{f} \right)$$
The corresponding variation of the position of a ray will be

$$\Delta y = B y_o \Delta\left(\frac{1}{f}\right) = -\frac{B y_o \Delta f}{f_o^2}$$

$$\Delta y = -2 \frac{\Delta f}{f_o} y_o \sqrt{1 - n} \frac{S'}{S'} + \frac{C' C''}{\sqrt{1 - n} S'} - \frac{\sqrt{n}}{1 - n} \frac{S'' C'^2}{S'} - \frac{S''^2 C'}{\sqrt{1 - n} S'}$$

and to a good approximation

$$\Delta y \approx -2(1 - 2n) \left[ \alpha^2 + \frac{1}{1 - n} \right] \frac{\Delta f}{f_o} y_o$$

(38)

For $n = \frac{1}{3}$ and $\alpha = 22^\circ 30'$

$$\Delta y \approx -1.1 \frac{\Delta f}{f_o} y_o$$

$$\left| \frac{\Delta f}{f_o} \right| \leq \frac{\Delta y_{\text{max}}}{y_o}$$

The adjustment for this purpose also is not critical.
1. Four Bending and One Quadrupole Magnet System (Fig. 9)

Hypotheses and notations. We shall use n-type magnets with \( n = -\frac{1}{2} \) and assume the same radius of curvature for all magnets. We shall use the normalized notations described above in II.4.3.

In studying this system, it is convenient to use the following parameters:

\[
S = \frac{\alpha_0 + \alpha_0'}{\sqrt{2}} \quad \text{and} \quad D = \frac{\alpha_0 - \alpha_0'}{\sqrt{2}}
\]

If we fix \( D_0 = \alpha_0 - \alpha_0' \), the system depends on 4 variables, \( l_1, l_2, 1/f, \) and \( S \); and we have 3 conditions to fulfill:
\[(y|y_0) = (x|x_0) = 0\]

\[
\begin{align*}
\left( \theta \mid \frac{\Delta \rho}{\rho_0} \right) &= 0 \\
\left( L \mid \frac{\Delta \rho}{\rho_0} \right) &= 0
\end{align*}
\] (39)

The system then has one degree of freedom remaining. In the following relations we have changed the parameters to the following for convenience:

\[
L_1 = \frac{L_1}{\rho_0 \sqrt{2}} \quad L_2 = \frac{L_2}{\rho_0 \sqrt{2}} \quad F = \frac{f}{\rho_0 \sqrt{2}}
\]

\[
\alpha = \frac{\alpha_0}{\sqrt{2}} \quad \alpha' = \frac{\alpha'_0}{\sqrt{2}}
\]

With the above notations, Eq. (39) gives us the relations

\[
L_1 = \frac{S - 4 \sin \frac{S}{2} \cos \frac{D}{2} + \sin S}{\cos D - \cos S}
\]

\[
L_2 = \frac{\cos S - \frac{L_1}{2} (\sin S + \sin D)}{\sin S - \frac{L_1}{2} (\cos D - \cos S)}
\] (40)

\[
\frac{1}{F} = 2B \frac{B + 2 \sin \alpha'}{B + 2 \sin \alpha}
\]

with

\[
B' = - \sin (\alpha + \alpha') + L_1 \sin \alpha \sin \alpha'
\]

The results are given in Curve II, where we have plotted in the S-D plane the curves \( L_1 = \text{constant} \) and \( L_2 = \text{constant} \). The available region of this plane for which a physical solution exists is inside the curves \( L_1 = 0 \) and \( L_2 = 0 \).
\[ s = \frac{(\alpha_0 + \alpha'_0)}{\sqrt{2}} \]
\[ D = \frac{(\alpha_0 - \alpha'_0)}{\sqrt{2}} \]

\[ L_1 = \frac{l_1}{\rho_0 V^2} \]
\[ L_2 = \frac{l_2}{\rho_0 V^2} \]

\[ \alpha_0 - \alpha'_0 = 45^\circ \]

Curve II
The parameters for a $90^\circ$ deflection device will be given by

\[ D = \frac{D_0}{\sqrt{2}} = \frac{45^\circ}{\sqrt{2}} = 31^\circ 51' \]

It is necessary to find a compromise between $S$ as great as possible (which means $\alpha'$ great) and $L_1L_2$ not too small. This is achieved for

\[ l_1 = \frac{\sqrt{2}}{4} \rho_0 \quad l_2 = \frac{3}{4} \sqrt{2} \rho_0 \quad \alpha_0 = 55^\circ 20' \quad \alpha'_0 = 10^\circ 20' \]

The total transfer matrices in the vertical and horizontal planes for this isochronous system are given in Appendix B.

2. **Three Bending and Two Quadrupole Magnet Systems** (Fig. 10)

Fig. 10
We shall take the same radius of curvature and the same field index for all 3 magnets. It is easy to see that \( n = 1/2 \) cannot yield a zero dispersion system, because for this value the transfer matrices are the same for the bending magnets in the vertical and the horizontal planes and different for the quadrupole magnet. Therefore, the conditions \((x|y)_o = 0\) and \((y|x)_o = 0\) cannot be fulfilled simultaneously.* We must take \( n \neq 1/2 \) and use the 4 conditions

\[
(x|y)_o = 0 \quad (y|x)_o = 0 \quad (y|\Delta p/p_o)_B = 0 \quad (L|\Delta p/p_o)_B = 0
\]

\[
0 = \left(1 - \frac{l_2}{f}\right) \cos \alpha'_1 \cos \alpha'_2 - \left(1 - \frac{l_1}{f}\right) \sin \alpha'_1 \sin \alpha'_2
\]

\[
- \sqrt{1-n} \left(l_1 + l_2 - \frac{l_1 l_2}{f}\right) \sin \alpha'_1 \cos \alpha'_2 - \frac{1}{f \sqrt{1-n}} \cos \alpha'_1 \sin \alpha'_2
\]

\[
0 = - \left(1 - \frac{l_2}{f}\right) \sin \alpha'_1 \left(1 - \cos \alpha'_1\right) + \left(1 - \frac{l_1}{f}\right) \sin \alpha'_1 \cos \alpha'_2
\]

\[
- \sqrt{1-n} \left(l_1 + l_2 - \frac{l_1 l_2}{f}\right) \sin \alpha'_1 \sin \alpha'_2 - \frac{1}{f \sqrt{1-n}} \left(1 - \cos \alpha'_1\right) \cos \alpha'_2 - \sin \alpha'_2
\]

\[
0 = - \left(1 - \frac{l_2}{f}\right) \sin \alpha'_1 \left(1 - \cos \alpha'_1\right) - \left(1 - \frac{l_1}{f}\right) \sin \alpha'_1 \left(1 - \cos \alpha'_2\right)
\]

\[
- \sqrt{1-n} \left(l_1 + l_2 - \frac{l_1 l_2}{f}\right) \sin \alpha'_1 \sin \alpha'_2 + \frac{1}{f \sqrt{1-n}} \left(1 - \cos \alpha'_1\right) \left(1 - \cos \alpha'_2\right)
\]

\[
+ \alpha'_1 + \alpha'_2 - \left(\sin \alpha'_1 + \sin \alpha'_2\right)
\]

\[
0 = \left(1 + \frac{l_2}{f}\right) \cos \alpha''_1 \cos \alpha''_2 - \left(1 + \frac{l_2}{f}\right) \sin \alpha''_1 \sin \alpha''_2
\]

\[
- \sqrt{n} \left(l_1 + l_2 + \frac{l_1 l_2}{f}\right) \sin \alpha''_1 \cos \alpha''_2 + \frac{1}{f \sqrt{n}} \cos \alpha''_1 \sin \alpha''_2
\]  

*It is also possible to have \( n = 1/2 \) and replace \((y|y)_o = 0\) by \((\phi|y)_o = 0\). This system is more compact.

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where we have again taken $\rho_0 = 1$ and

\[
\alpha_1' = \sqrt{1 - n \alpha_1} \quad \alpha_1'' = \sqrt{n \alpha_1}
\]

\[
\alpha_2' = \sqrt{1 - n \alpha_2} \quad \alpha_2'' = \sqrt{n \alpha_2}
\]

If we fix the angle of deviation $D = 2 (\alpha_1 - \alpha_2)$, the five variables $\alpha_2$, $n$, $l_1$, $l_2$ and $1/f$ are bound by the four relations (41). The system has one degree of freedom ($\alpha_2$, for example). We have not solved for the numerical solutions of this system.

3. Two Bending and One Off-Center Quadrupole Magnet Systems

It is easy to see that to have an isochronous system it is necessary to change the sign of the angle of deviation along the central trajectory. To do that it is possible to use an off-center quadrupole.

3.1 Field equations in a quadrupole (Fig. 11).

\[ - \frac{e}{p} B_x = \alpha_2 y \]

\[ - \frac{e}{p} B_y = \alpha_2 x \]

\[ - \frac{e}{p} B_z = 0 \]
with
\[
\frac{e\,\partial B_x}{p\,\partial y} = -\frac{e\,\partial B_x}{p\,\partial x} = \alpha^2
\]
and \( p \) the momentum of the particle.

3.2 Equations of motion of a particle.

\[
\frac{d}{dt} \left( p \frac{dp}{ds} \right) = e \, v \, \frac{dp}{ds} \times \vec{B}
\]

\[
\frac{dp}{dt} = 0 \quad \frac{d}{dt} = v \, \frac{d}{ds}
\]

\[
\frac{d^2p}{ds^2} = -B \times \frac{e}{p} \frac{dp}{ds}
\]

By projecting on the three axes we have

\[
\frac{d^2x}{ds^2} = \alpha^2 \frac{dz}{ds}
\]

\[
\frac{d^2y}{ds^2} = -\alpha^2 \frac{dy}{ds}
\]

\[
\frac{d^2z}{ds^2} = \alpha^2 \left[ \frac{dy}{ds} - x \frac{dx}{ds} \right]
\]

\( y = 0 \) for the central trajectory and all the trajectories in the horizontal plane. Then for these trajectories,

\[
\frac{d^2x}{ds^2} = \alpha^2 \frac{dz}{ds}
\]

\[
\frac{d^2z}{ds^2} = -\alpha^2 \frac{dx}{ds}
\]
By using \((43)\) and the input conditions \(\frac{ds}{dz} = \cos \theta\) and \(x = \xi\) for \(z = -L\) (see Fig. 12), we obtain:

\[
\frac{dz}{ds} = -\frac{\alpha^2}{2} \left( x^2 - \xi^2 \right) + \cos \theta
\]

\[
\left[ \frac{\alpha^2}{2} \left( x^2 - \xi^2 \right) - \cos \theta \right] \frac{d^2x}{dz^2} + \alpha^2 x \left( \frac{dx}{dz} \right)^2 + \alpha^2 x = 0
\]  \(\text{(44)}^*\)

If the length of the quadrupole is small in comparison to the average radius of curvature, \(2L/R \ll 1\), it is possible to describe the trajectory by a parabola:

\[ x = a + bz + cz^2 \]

The three coefficients of this expansion are given by \((44)\) and the input condition by:

*If we assume \(x\) and \(\theta\) small enough to neglect the second and higher order terms, the equation is \(a^2x/dz^2 - \alpha^2x = 0\); if \(\alpha^2 > 0\), the quadrupole is equivalent to a divergent lens in the horizontal plane; if \(\alpha^2 < 0\), to a convergent lens in this plane.
\[ \xi = a - bL + cL^2 \]
\[ \tan \theta = b - 2cL \]
\[ 2c \left[ \frac{\alpha^2}{2} (a^2 - \xi^2) - \cos \theta \right] + \alpha^2 a b^2 + \alpha^2 a = 0 \]

By combining these equations it follows that
\[ b = \frac{2(\xi - a) + L \tan \theta}{L} \]
\[ c = \frac{(\xi - a) + L \tan \theta}{L^2} \]
\[ -2 \left[ (\xi - a) + L \tan \theta \right] \left[ \frac{\alpha^2}{2} (a^2 - \xi^2) - \cos \theta \right] + \alpha^2 a \left[ 2 (\xi - a) + L \tan \theta \right]^2 + \alpha^2 a L^2 = 0 \]

The central trajectory (subscript \( o \) for its parameters) generally has to be symmetrical about the center of the system. If this is so, the following relations must hold:
\[ b_0 = 0 \]
\[ 2(\xi_0 - a_0) + L \tan \theta_0 = 0 \]
or
\[ a_0 = \xi_0 + \frac{L}{2} \tan \theta_0 \]

and
\[ \xi_0 = \frac{\sin \theta_0 + \frac{\alpha_0^2 L^2}{2} \tan \theta_0 \left( 1 - \frac{\tan^2 \theta_0}{4} \right)}{\alpha_0^2 L \left( 1 - \frac{\tan^2 \theta_0}{2} \right)} \]

Equation (47) gives the relation between \( \xi_0 \theta_0 \) and \( \alpha_0^2 \) that must be satisfied in order to have a symmetrical central trajectory. We can see that \( \xi_0 \theta_0 \) has the opposite sign of \( \alpha_0^2 \) (Fig. 13). If a ray is defined by the following parameters...
Convergent lens in the vertical plane.

\[ \alpha_o^2 > 0 \]
\[ \xi_o \theta_o < 0 \]

Divergent lens in the vertical plane.

\[ \alpha_o^2 < 0 \]
\[ \xi_o \theta_o > 0 \]

Fig. 13
\[ \xi = \xi_0 + d\xi_0 \]
\[ \theta = \theta_0 + d\theta_0 \]
\[ \alpha^2 = \alpha_0^2 + d\alpha^2 \]

Equation (46) will give us the corresponding variations \( da, db, dc \) of \( a, b, \) and \( c \):

\[
\left( \frac{\partial f}{\partial a} \right)_0 \, da + \left( \frac{\partial f}{\partial \alpha^2} \right)_0 \, d\alpha^2 + \left( \frac{\partial f}{\partial \theta} \right)_0 \, d\theta_0 + \left( \frac{\partial f}{\partial \xi} \right)_0 \, d\xi_0 = 0
\]

where \( f(a, \xi, \theta, \alpha^2) = 0 \) represents Eq. (36) and \( \left( \frac{\partial f}{\partial \xi} \right)_0 \) is calculated for

\[
a = a_0 = \xi_0 + \frac{L}{2} \tan \theta_0
\]
\[ \alpha^2 = \alpha_0^2 \]
\[ \theta = \theta_0 \]
\[ \xi = \xi_0 \]

\[
\left( \frac{\partial f}{\partial a} \right)_0 = \alpha_0^2 L^2 \left( 1 - \tan^2 \theta_0 \right) - 2 \cos \theta_0 \approx \alpha_0^2 L^2 - 2
\]

\[
\left( \frac{\partial f}{\partial \alpha^2} \right)_0 = \frac{L^2}{2} \tan \theta_0 \left( 1 - \frac{\tan^2 \theta_0}{4} \right) + \frac{L^2 \xi_0}{2} \left[ 1 - \frac{\tan^2 \theta_0}{2} \right] - \frac{L \xi_0^2 \tan \theta_0}{2}
\]

\[
= - \frac{L \sin \theta_0}{\alpha_0^2} - \frac{L \xi_0^2 \tan \theta_0}{\alpha_0^2} \approx - \frac{L \theta_0}{\alpha_0^2}
\]

\[
\left( \frac{\partial f}{\partial \xi} \right)_0 = 2 \cos \theta_0 - \frac{\alpha_0^2 L^2 \tan \theta_0}{4} \approx 2
\]

\[
\left( \frac{\partial f}{\partial \theta} \right)_0 = -2L \left( 1 + \tan^2 \theta_0 \right) \left[ \frac{\alpha_0^2 L^2}{8} \tan^2 \theta_0 + \frac{\alpha_0^2 L \xi_0 \tan \theta_0}{2} - \cos \theta_0 \right]
\]

\[
- L \frac{\sin^2 \theta_0}{\cos \theta_0} \approx 2L
\]
and

\[ da = -\frac{(\partial f/\partial x^2)}{(\partial f/\partial a)} \frac{dx^2}{d\theta} - \frac{(\partial f/\partial \theta)}{(\partial f/\partial a)} d\theta - \frac{(\partial f/\partial \xi)}{(\partial f/\partial a)} d\xi \]

\[ db = \frac{2da}{L} - \frac{2d\xi}{L} - (1 + \tan^2 \theta) \frac{d\theta}{L} = \frac{2da}{L} - \frac{2d\xi}{L} - d\theta \]

\[ d\theta = \frac{da}{L^2} - \frac{d\xi}{L^2} - \frac{1 + \tan^2 \theta}{L} \frac{d\theta}{L^2} = \frac{da}{L^2} - \frac{d\xi}{L^2} - \frac{d\theta}{L} \]

with

\[ dx^2 = -\alpha_o^2 \frac{\Delta P}{P_0} \]

The general trajectory has the following form:

\[ x(z) = x_o(z) + A(z)d\xi + B(z)d\theta + C(z) \frac{\Delta P}{P_0} \]

where \( x_o(z) \) represents the central trajectory

\[ x_o(z) = \left( \xi_o + \frac{L}{2} \tan \theta_0 \right) - \tan \theta_0 \frac{z^2}{2L} \]

and

\[ A(z) = -\frac{(\partial f/\partial \xi)}{(\partial f/\partial a)} \left[ \frac{1 + 2z}{L} + \frac{z^2}{L^2} \right] - \frac{2z}{L} + \frac{z^2}{L^2} \]

\[ B(z) = -\frac{(\partial f/\partial \theta)}{(\partial f/\partial a)} \left[ \frac{1 + 2z}{L} + \frac{z^2}{L^2} \right] - (1 + \tan^2 \theta_0) \left( z + \frac{z^2}{L} \right) \]

\[ C(z) = \alpha_o^2 \frac{(\partial f/\partial x^2)}{(\partial f/\partial a)} \left[ \frac{1 + 2z}{L} + \frac{z^2}{L^2} \right] \]
In first approximation we shall assume that $\theta_0$ is small enough to neglect $\theta_0^2$ and higher powers of $\theta_0$. The coefficients $A$, $B$ and $C$ of a trajectory are then given at the output of the quadrupole by the relations

$$A(L) \simeq \frac{2 + 3\alpha_0^2L^2}{2 - \alpha_0^2L^2}$$

$$B(L) \simeq 2L \frac{2 + \alpha_0^2L^2}{2 - \alpha_0^2L^2}$$

$$C(L) \simeq 4L \frac{\theta_0}{2 - \alpha_0^2L^2}$$

To first order in $\theta_0$

$$d\theta = \frac{dA}{dz} d\xi_0 + \frac{dB}{dz} d\theta_0 + \frac{dC}{dz} \Delta \rho$$

and

$$\left(\frac{d\theta}{d\xi_0}\right)_{z=L} = -\frac{(\partial f/\partial \xi)_0}{(\partial f/\partial \alpha)_0} \left(\frac{4}{L}\right) - \frac{4}{L} \frac{2 + \alpha_0^2L^2}{2 - \alpha_0^2L^2}$$

$$\left(\frac{d\theta}{d\theta_0}\right)_{z=L} = -\frac{(\partial f/\partial \theta)_0}{(\partial f/\partial \alpha)_0} \left(\frac{4}{L}\right) - 3(1 + \tan^2\theta_0) \frac{2 + 3\alpha_0^2L^2}{2 - \alpha_0^2L^2}$$

$$\left(\frac{\Delta \rho}{\rho_0} \right)_{z=L} = \alpha_0^2 \frac{(\partial f/\partial \alpha^2)_0}{(\partial f/\partial \alpha)_0} \left(\frac{4}{L}\right) \approx \frac{4\theta_0}{2 - \alpha_0^2L^2}$$
3.3 Matrix formulation

It is more convenient to define the input conditions for the central trajectory by \( \theta_0 \eta_0 \), as shown on Fig. 14, and to take \( A_0 r_1' \) and \( A_0 r_2' \) as reference transverse axes to define the structure of the beam at the input and at the output of the quadrupole.

We have the following relations (if \( \theta_0 \) is small enough):

\[
\begin{align*}
\frac{dr_1}{d\xi_0} &= \cos \theta_0 \quad \frac{dr_2}{d\xi_0} \approx \frac{d\xi_{\text{output}}}{d\xi_0} \\
\frac{d\theta_1}{d\theta_0} &= \frac{d\theta_2}{d\theta_0} \approx \frac{d\theta_{\text{output}}}{d\theta_0}
\end{align*}
\]

\[
\begin{pmatrix}
\frac{dr_2}{d\xi_0} \\
\frac{d\theta_2}{d\theta_0} \\
\frac{\Delta p}{p_0}
\end{pmatrix} = M
\begin{pmatrix}
\frac{dr_1}{d\xi_0} \\
\frac{d\theta_1}{d\theta_0} \\
\frac{\Delta p}{p_0}
\end{pmatrix}
\]
where \( M \) is the matrix defined by (49) and (50), and also

\[
\begin{pmatrix}
\frac{dr'_2}{dr'_1} \\
\frac{d\theta'_2}{d\theta'_1} \\
\frac{\Delta p}{p_0}
\end{pmatrix} =
\begin{pmatrix}
1 & -L & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} M
\begin{pmatrix}
1 & -L & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\frac{dr'_2}{dr'_1} \\
\frac{d\theta'_2}{d\theta'_1} \\
\frac{\Delta p}{p_0}
\end{pmatrix}
\]

\[
\frac{dr'_2}{dr'_1} \approx 1 \quad \frac{d\theta'_2}{d\theta'_1} \approx \frac{4\alpha^2 L^2}{L^2 - \alpha^2 L^2}
\]

\[
\frac{dr'_2}{dr'_1} \approx 1 \quad \frac{d\theta'_2}{d\theta'_1} \approx 1
\]

\[
\left( \frac{dr'_2}{dr'_1} \right) \approx 0 \quad \left( \frac{d\theta'_2}{d\theta'_1} \right) \approx 0
\]

From Fig. 14 we can see

\[
\eta_0 = \xi_0 + L \tan \theta_0 = \frac{-2\theta_0 L}{\alpha^2 L^2}
\]

### 3.4 Path length

The general orbit in the horizontal plane is given by

\[
x(z) = x_0(z) + A(z) \, d\xi_0 + B(z) \, d\theta_0 + C(z) \, \frac{\Delta p}{p_0}
\]

The path length between the input and the output of the quadrupole is then

\[
\mathcal{L} = \int_{-L}^{+L} \sqrt{1 + \left( \frac{dx}{dz} \right)^2} \, dz = \int_{-L}^{+L} \sqrt{1 + \left( x'_0 + A' d\xi_0 + B' d\theta_0 + C \frac{\Delta p}{p_0} \right)^2} \, dz
\]

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and to first order

\[ \mathcal{L} = \int_{-L}^{+L} \frac{dz}{\sqrt{1 + x'^2}} \left\{ 1 + \frac{A'x'^2}{1 + x'^2} \frac{d\xi_0}{1 + x_0^2} + \frac{B'x'^2}{1 + x'^2} \frac{d\theta_0}{1 + x_0^2} + \frac{C'x'^2}{1 + x'^2} \frac{\Delta p}{p_0} \right\} \]

\[ \mathcal{L} = \mathcal{L}_0 + d\xi_0 \int_{-L}^{+L} \frac{A'x'^2}{1 + x'^2} \, dz + d\theta_0 \int_{-L}^{+L} \frac{B'x'^2}{1 + x'^2} \, dz + \frac{\Delta p \int_{-L}^{+L} C'x'^2}{p_0} \, dz \]

\[ x_0 = a_0 + C_0 z^2 \]

\[ x'_0 = 2C_0 z \quad \text{and} \quad 2C_0L = \tan \theta_0 \]

The coefficients of the path length expansion are as follows:

\[ (L|d\xi_0) = \frac{1}{C_0 L^2} \left( 1 + \frac{(d\xi/dz)_0}{(df/da)_0} \right) \left\{ \frac{1}{2} \log \left[ \frac{1 + 4C_0^2L^2}{1 + 4C_0^2L^2} \right] - C_0L \sqrt{1 + 4C_0^2L^2} \right\} \]

\[ \approx - 4C_0L \left( 1 + \frac{(d\xi/dz)_0}{(df/da)_0} \right) \approx - 2\theta_0 \frac{a^2L^2}{2 - a^2L^2} \approx (L|d\xi_1) \]

\[ (L|d\theta_0) = \frac{L}{C_0 L^2} \left[ \frac{1}{L (df/da)_0} + (1 + \tan^2 \theta_0) \right] \left\{ \frac{1}{2} \log \left[ \frac{1 + 4C_0^2L^2}{1 + 4C_0^2L^2} \right] - C_0L \sqrt{1 + 4C_0^2L^2} \right\} \]

\[ \approx - 2\theta_0L \frac{a^2L^2}{2 - a^2L^2} \approx (L|d\theta_1) \]

(52)
\[
\left( L \left| \frac{\Delta p}{p_0} \right. \right) = \frac{\alpha_o^2}{C_o^2L^2} \left( \frac{\partial f}{\partial a} \right)_o \left\{ \frac{1}{2} \log \left( \sqrt{1 + 4C_oL^2} + 2C_oL \right) - C_oL \sqrt{1 + 4C_oL^2} \right\} \\
\qquad \quad = 2L \frac{\theta_o^2}{2 - \alpha_o^2L^2}
\]

If we use the matrix written for the symmetry plane (center) of the quadrupole, we obtain

\[
\left( L \left| dr_1' \right. \right) = -2\theta_o \frac{\alpha_o^2L^2}{2 - \alpha_o^2L^2}
\]

\[
\left( L \left| ds_1' \right. \right) = 0
\]

\[
\left( L \left| \frac{\Delta p}{p_0} \right. \right) = 2L \frac{\theta_o^2}{2 - \alpha_o^2L^2}
\]

The matrix in this case is

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
\frac{4\alpha_o^2L^2}{L(2 - \alpha_o^2L^2)} & 1 & \frac{4\theta_o^2}{2 - \alpha_o^2L^2} & 0 \\
0 & 0 & 1 & 0 \\
-2\theta_o \frac{\alpha_o^2L^2}{2 - \alpha_o^2L^2} & 0 & 2L \frac{\theta_o^2}{2 - \alpha_o^2L^2} & 1
\end{pmatrix}
\]
In the vertical plane there is no change for the matrix:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
-\frac{\alpha^2 L^2}{L} & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

3.5 Parameters of the isochronous system (Fig. 15).

*From Eq. (42) we have \( \frac{d^2y}{ds^2} = -\alpha^2y \left[ \frac{\alpha^2}{2} (y^2 - x^2) + B \right] \) and to first order \( \frac{d^2y}{ds^2} = -\alpha^2 By \). In this approximation \( B = 1 \), and \( \frac{d^2y}{ds^2} + \alpha^2 y = 0 \).
The transfer matrix for the displaced quadrupole lens can be written as

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
- \frac{4f}{4f + L} & 1 & \frac{8f \theta_0}{4f + L} & 0 \\
0 & 0 & 1 & 0 \\
\frac{2L \theta_0}{4f + L} & 0 & \frac{4Lf \theta_0^2}{4f + L} & 1
\end{pmatrix}
\]

where \( f \) is the focal length of the quadrupole lens

\[
\frac{1}{f} = -2a^2 L
\]

We shall take \( n = \frac{1}{2} \) magnets. In this case the vertical and the horizontal foci of the first magnet must occur at the same location, namely in the plane of symmetry. This occurs for

\[
l = \frac{p \sqrt{2}}{\tan \alpha'} \quad \text{with} \quad \alpha' = \frac{\alpha}{\sqrt{2}}
\]
For this condition, the total transfer matrix for the entire system is

\[
\begin{pmatrix}
-1 & \frac{2\rho \sqrt{2}}{\sin \alpha'} \left[ \cos \alpha' - \frac{2\sqrt{2} \rho}{(4f + L) \sin \alpha'} \right] & \frac{4\rho}{4f + L} - \frac{2\sqrt{2} (\rho - t\theta_0)}{(4f + L) \sin \alpha'} & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & \frac{4\rho}{4f + L} - \frac{2\sqrt{2} (\rho + L\theta_0/4)}{(4f + L) \sin \alpha'} & \frac{4\rho}{4f + L} + \theta_0 + \alpha' + \frac{L\theta_0^2/\rho}{4f + L} & 1
\end{pmatrix}
\]
The conditions for isochronism are, to first order to \( \theta_0 \),

\[
1 - \frac{2\sqrt{2} (\rho - f\theta_0)}{(4f + L) \sin \alpha'} = 0
\]

\[
\alpha + \theta_0 - \frac{4\rho}{4f + L} = 0
\]

\[
\alpha + \theta_0 = \frac{D}{2}
\]

where \( D \) is the total deflection of the system.

This system has one degree of freedom remaining. We shall take \( \alpha \) as the variable parameter:

\[
\frac{\rho}{\rho} = \frac{(4/D\sqrt{2}) \sin (\alpha/\sqrt{2}) - 1}{\alpha - (D/2)}
\]

\[
\theta_0 = \frac{D}{2} - \alpha
\]

\[
\frac{L}{4\rho} = \frac{2}{D} - \frac{(4/D\sqrt{2}) \sin (\alpha/\sqrt{2}) - 1}{\alpha - (D/2)}
\]

The average radius of curvature of the central trajectory in the quadrupole is

\[
R = \left| \frac{L}{\theta_0} \right|
\]

and to a good approximation

\[
\frac{\rho}{\rho} \approx \frac{2}{D}
\]

\[
\frac{2L}{\rho} \approx \frac{2\alpha^3}{3(2\alpha - D)}
\]

\[
\frac{R}{\rho} \approx \frac{2\alpha^3}{3(2\alpha - D)^2}
\]
3.6 Discussion of the results (Fig. 16).

The results are given by the following table:

<table>
<thead>
<tr>
<th>$(\theta_0)$</th>
<th>$\alpha$</th>
<th>$2L/\rho$</th>
<th>$R/\rho$</th>
<th>$\eta_0/\rho$</th>
<th>$L/\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2°30'$</td>
<td>$47°30'$</td>
<td>4.36</td>
<td>50.1</td>
<td>0.22</td>
<td>2.4</td>
</tr>
<tr>
<td>$5°$</td>
<td>$50°$</td>
<td>2.55</td>
<td>14.6</td>
<td>0.44</td>
<td>2.3</td>
</tr>
<tr>
<td>$10°$</td>
<td>$55°$</td>
<td>1.7</td>
<td>4.9</td>
<td>0.88</td>
<td>2.1</td>
</tr>
<tr>
<td>$15°$</td>
<td>$60°$</td>
<td>1.5</td>
<td>2.9</td>
<td>1.32</td>
<td>1.9</td>
</tr>
</tbody>
</table>

A physical solution exists if $2L/\rho < 2L/\rho$. This limitation does not occur for values of $\theta_0$ greater than $2°30'$, as shown by the above table. If a small $\theta_0$ is taken, the radius of curvature in the quadrupole is large and the aperture can be small; however, in this case the compensation of the path length is not very strong and the quadrupole has to be very long. A good compromise seems to be $|\theta_0| = 10^\circ$. The length of the quadrupole is then reasonable, but the aperture is very large: $\eta_0 = 0.88/\rho$. It would be reasonable, if this solution is chosen, to design specially the quadrupole in order to reduce the total dimensions of the magnet. It is also necessary to keep $\rho$ as small as possible. For 40 Mev and 5000 gauss, $\rho = 10.5$ inches. If we want to decrease this radius by a factor of 2, the field in the bending magnets is then too large for easy pulsed operation. However, it is always possible to add other magnets along the main accelerator to avoid a deflection of the main beam.
The best nonisochronous deflection system for injection that we have been able to devise seems to limit the angle of injectors to $45^\circ$ in order to avoid a large debunching. The simplest system is one with two bending and one quadrupole magnets. The bending magnets can be of either the $n = \frac{1}{2}$ type or the uniform field type with rotated pole faces.

Injection at $90^\circ$ implies the use of an isochronous system. The most attractive one seems to be the off-center quadrupole system, but it requires the design of a special quadrupole magnet. The system with three n-type magnets and two quadrupoles is also feasible and, in this case, all the components of the system are used in a classical way.

The low value of the magnetic field (500 gauss) permits the possibility of pulsed operation. The effect of the remanent field on the main beams can probably be compensated for by a normal steering coil. If DC operation is chosen, it is necessary to introduce along the accelerator another magnetic device for compensating for the effect of the last magnet of the deflecting system on the main beams.
APPENDIX A: SUGGESTIONS FOR EXPERIMENTAL TESTING OF THE DEBUNCHING OF A DEFLECTION SYSTEM

Before using a deflection system in the machine, it would be useful to test the system for debunching. The calculations described in this report are only a first-order approximation that does not take into account fringe-field effects, misalignment of the components, second-order effects, etc. A simple way of testing the system is described below.

After one section we use the magnetic inflector with slits that define a narrow energy spread (0.5% for example). Then we compare the phases of the signals induced in 2 cavities by the beam, one before and the other after the system of deflection. It is possible to have this phase shift as a function of the magnetic field of the bending magnets in a range corresponding to the energy spread of the beam to be transferred into the machine (± 5% for example). By this method, the two signals compared do not have the same amplitude for all the range of energy spread explored. If the ratio of the signals is more than 10 db, the sensitivity for phase comparison is very poor. It is possible to modify the design to avoid this difficulty by adding a small section after the buncher in order to work with the same deflected current across 0.5% slits.
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