ELECTRON BUNCHING BY UNIFORM SECTIONS OF DISK-LOADED WAVEGUIDE
PART A: GENERAL STUDY

By
Georges Dôme

Technical Report
Contract AT(04-3)-21
Project Agreement No. 1
M.L. Report No. 780-A
M Report No. 242-A
December 1960

W. W. Hansen Laboratories of Physics
Stanford University
Stanford, California
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*On leave from the Center of Nuclear Sciences of the Royal Military School, Brussels, Belgium.
Preface

Part A of the present report constitutes a general study of the bunching characteristics which can be obtained with a uniform waveguide section used as an electron buncher.

Part B will deal with some particular questions, namely the incidence of a fluctuating injection velocity on the electron bunches, and also a qualitative approach to determine the importance of the beam radial defocusing when passing through the waveguide section.
SUMMARY

Rather than dealing with nonperiodic disk-loaded structures, an exhaustive study is made of the properties of a uniform waveguide section, i.e., a section where the electric-field strength and the phase velocity of the traveling wave are constant along the axis. It appears that if the phase velocity of the wave in a waveguide section is equal to the injection velocity of the unmodulated electrons, this waveguide section can function effectively as a prebuncher and would not require more than a hundred of watts of rf power for this purpose.

This type of buncher is compared with a velocity-modulating cavity followed by a drift space. It is shown that it can achieve better bunching than is obtained by velocity modulation, and is also less sensitive to small fluctuations of the rf power level.
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I. THE DISK-LOADED WAVEGUIDE WITH FIELD STRENGTH AND
PHASE VELOCITY CONSTANT ALONG THE AXIS

Before dealing with waveguides where the field strength and the
phase velocity of the wave vary along the axis, it is worth while to
set down explicit expressions for the phase angle $\Delta$ of the electron
with respect to the wave as a function of the distance $z$ along the
longitudinal axis when the field strength and the phase velocity of
the wave are constant, because in this case there exists an exact
Hamiltonian function to describe the electron's longitudinal motion.$^1$

Using the notation of Chodorow et al.,$^2$ the fundamental equa-
tions of longitudinal electron motion in dimensionless form are

$$\frac{dy}{dz} = -\alpha \sin \Delta \quad \text{and} \quad \frac{d\Delta}{dz} = 2\pi \left( \frac{1}{\beta_w} - \frac{1}{\beta_e} \right)$$  \hspace{1cm} (1.1)

where

$$\gamma = \left( 1 - \beta_e^2 \right)^{-\frac{1}{2}}, \quad \xi = \frac{z}{\lambda}, \quad \alpha = \frac{E_0 e^\lambda}{m_0 c^2},$$

$\lambda$ is the free-space wavelength of the traveling wave, and $E_0$ is the
amplitude of the electric field on the axis.

From Eq. (1.1) we at once obtain

$$\frac{d\Delta}{dz} \cdot dy = -\alpha \sin \Delta \cdot d\Delta,$$

---


$^2$M. Chodorow et al., The Stanford High-Energy Linear Electron
or

\[ \begin{vmatrix} 1 & 1 \\ \beta_w & \beta_e \end{vmatrix} \cdot \frac{dy}{d\gamma} = -\frac{\alpha}{2\pi} \sin \Delta \cdot d\Delta. \]

Further, since \( \frac{d\gamma}{\beta_e} = d(\gamma^2 - 1)^{1/2} \), we may write the last equation as

\[ \frac{dy}{\beta_w} - d(\gamma^2 - 1)^{1/2} = \frac{\alpha}{2\pi} \cdot d\cos \Delta. \quad (1.2) \]

When \( \alpha \) and \( \beta_w \) are independent of \( \xi \), and only in this particular case, Eq. (1.2) may readily be integrated as

\[ \frac{\gamma}{\beta_w} - (\gamma^2 - 1)^{1/2} = \frac{\alpha}{2\pi} \cdot \cos \Delta + H, \quad (1.3) \]

where \( H \) is a constant of motion. It is not difficult to see that, except for a constant factor, \( H \) is exactly the Hamiltonian function of Slater. In the following, we shall write

\[ y = \frac{\gamma}{\beta_w} - \left( \gamma^2 - 1 \right)^{1/2}, \quad (1.4) \]

so that Eq. (1.3) becomes

\[ y = \frac{\alpha}{2\pi} \cdot \cos \Delta + H. \quad (1.5) \]

Equation (1.3) enables us to plot \( (\gamma^2 - 1)^{1/2} \) vs. \( \Delta \) in a phase space, and the graphs so obtained beautifully characterize the electron motion. They are of different types when \( \beta_w = 1 \) and when \( \beta_w < 1 \). In Figs. 1.1 and 1.2 we have reproduced these graphs from Slater.\(^1\)

\(^1\)Slater, op. cit.
Fig. 1.1--Phase space for $\beta_w < 1$.

Fig. 1.2--Phase space for $\beta_w = 1$.
The curves u represent electrons which are unbound to the wave; the curves b, electrons which are bound. When $\beta_w < 1$, the orbits of the bound electrons are closed curves, which the representative points of the electrons traverse in a clockwise direction around a center the ordinate of which represents the momentum of an electron traveling with the velocity of the wave. When $\beta_w = 1$, the center of the bound orbits goes to infinity, and the path of the representative point approaches a vertical asymptote $\Delta = \Delta_\infty$. The preceding graphs are most easily obtained if we make a plot of $y$, defined by Eq. (1.4), as a function of $p = (y^2 - 1)^{1/2}$:

$$ y = \frac{\sqrt{1 + p^2}}{\beta_w} - p \quad (1.6) $$

where $p$ is the ratio of the electron momentum to $m_0 c$ and may take negative as well as positive values. In the $(p, y)$ plane, Eq. (1.6) represents a hyperbola, the asymptotes of which we get at once when we observe that for $p \to \infty$, $y \to p \left[\frac{1}{\beta_w} - 1\right]$; and for $p \to -\infty$, $y \to -p\left[\frac{1}{\beta_w} + 1\right]$. (See Figs. 1.3 and 1.4.)

From Eq. (1.6) we get

$$ y = \frac{1}{\beta_w} \cdot \frac{1 - \beta_e^2 \beta_w^2}{\sqrt{1 - \beta_e^2}} \quad (1.7) $$

and

$$ \frac{dy}{dp} = \frac{p}{\beta_w \cdot \sqrt{1 + p^2}} - 1 = \frac{\beta_e - 1}{\beta_w} $$

Therefore, the minimum value of $y$, which we shall call $b$, corresponds to $\beta_e = \beta_w$ and is given by

$$ - 4 - $$
Fig. 1.3--p versus y for $\beta_w < 1$.

Fig. 1.4--p versus y for $\beta_w = 1$.

- 5 -
Equation (1.5) shows that the interval of variation for \( y \) is \( \alpha/\pi \) wide. From Fig. 1.3 we see that when this interval is \( (b, b + \alpha/\pi) \), the corresponding curve in phase space is just the limit between bound and unbound orbits. It follows that for all the bound orbits we have

\[
y < b + \frac{\alpha}{\pi}
\]

or, with Eqs. (1.7) and (1.8):

\[
\frac{1}{\beta_w} \left[ \frac{1 - \beta_e^2 w}{\sqrt{1 - \beta_e^2}} - \sqrt{1 - \beta_w^2} \right] < \frac{\alpha}{\pi}.
\]

This condition determines the minimum value of \( \alpha \) at which electrons start being bound to the traveling wave, in exact agreement with Eq. (5.7) of reference 2. In a bound orbit the maximum value \( \Delta_m \) of \( \Delta \) corresponds to the minimum value \( b \) of \( y \), as can be seen from Eq. (1.5); whence, for a bound orbit Eq. (1.5) may be written as

\[
y - b = \frac{\alpha}{2\pi} \left( \cos \Delta - \cos \Delta_m \right).
\]

By putting \( \Delta_m = \pi \) we get the limit between bound and unbound orbits.

The minimum value of \( \Delta \) is reached when \( \Delta = 0 \); if we want this minimum value not to be negative we must impose the condition (see Fig. 1.3)

\[
\frac{\alpha}{2\pi} \left( 1 - \cos \Delta_m \right) \leq \frac{1}{\beta} - b \quad \text{or} \quad \cos \Delta_m \geq 1 - \frac{2\pi}{\alpha} \left( \frac{1}{\beta} - b \right).
\]

\(^2\text{Chodorow et al., op. cit.}\)
In particular, this condition is fulfilled by all the bound orbits when

\[
\alpha \leq \frac{1}{\beta} \cdot \frac{1}{1 - \sqrt{1 - \beta^2}} = \beta_w.
\]  

(1.12)

The right-hand side of Eq. (1.12) thus gives the limit that \( \alpha/\pi \) may not exceed in order that no bound electron goes negative in velocity.

Let us call \( y_0 \) the value of \( y \) when the electrons enter the waveguide and rewrite Eq. (1.10) as

\[
2 \cos a = \cos a + \cos (y - b).
\]  

(1.13)

From Fig. 1.1 we then at once get, by putting \( \Delta_m = \pi \) in the preceding equation, the fraction \( F \) of the injected electrons which are bound to the wave:

\[
F = \frac{1}{\pi} \cos^{-1} \left[ -1 + \frac{2\pi}{\alpha} (y_0 - b) \right].
\]  

(1.14)

By Eqs. (1.7) and (1.8), this expression agrees exactly with Eq. (5.8) of reference 2. Now, Eq. (1.11) shows that the last bound orbit where the electron velocity always remains positive corresponds to the value

\[
\cos \Delta_m = 1 - \frac{2\pi}{\alpha} \left( \frac{1}{\beta_w} - b \right).
\]

Putting this value in Eq. (1.13) we get the fraction \( F^+ \) of the injected electrons which, being bound to the wave, never go negative in velocity:

\[\text{Chodorow et al., op.cit.}\]
\[ F^+ = \frac{1}{\pi} \cos^{-1} \left[ 1 - \frac{2\pi}{\alpha} \left( \frac{1}{\beta_w} - y_0 \right) \right] \quad \text{when} \quad \frac{\alpha}{\pi} \geq \frac{1}{\beta_w} - b \quad (1.15) \]

\[ F^+ = F \quad \text{when} \quad \frac{\alpha}{\pi} \leq \frac{1}{\beta_w} - b . \]

All we have written about the case \( \beta_w < 1 \) applies equally well to the case \( \beta_w = 1 \). In particular, Eqs. (1.7), (1.8) and (1.10) become, when \( \beta_w = 1 \),

\[ y = \sqrt{\frac{1 - \beta_w}{1 + \beta_w}} \quad (1.16) \]

\[ b = 0 \]

\[ y = \frac{\alpha}{2\pi} \left( \cos \Delta - \cos \Delta_0 \right) , \quad (1.17) \]

since the asymptotic phase angle \( \Delta_0 \) merely equals \( -\Delta_0 \). The only difference between the two cases arises when we compute the fraction \( F^+ \) of the injected electrons which, being bound to the wave, retain positive velocities throughout their trajectories. In fact, Fig. 1.2 shows that when \( \beta_w = 1 \), only the electrons with a positive \( \Delta_0 \) may go negative in velocity; therefore we have to combine the expressions (1.14) and (1.15) to get \( F^+ \) in the case \( \beta_w = 1 \), and we find that

\[ F^+ = \frac{\cos^{-1} \left[ \frac{2\pi}{\alpha} \left( \frac{1}{\beta_w} - y_0 \right) \right] + \cos^{-1} \left[ \frac{2\pi}{\alpha} \left( 1 - y_0 \right) \right]}{2\pi} \quad \text{when} \quad \frac{\alpha}{\pi} \geq 1 \quad (1.18) \]

\[ F^+ = F = \frac{\cos^{-1} \left[ \frac{2\pi}{\alpha} \left( \frac{1}{\beta_w} - y_0 \right) \right]}{\pi} \quad \text{when} \quad \frac{\alpha}{\pi} \leq 1 . \]
Note: In the case \( \beta_w = 1 \), the fraction of the injected electrons which, being bound to the wave, have a negative velocity at some time in their orbits, is

\[
F' = \frac{\cos^{-1} \left(-1 + \frac{2\pi}{\alpha y_0}\right) - \cos^{-1} \left[1 - \frac{2\pi}{\alpha (1 - y_0)}\right]}{2\pi} \quad \text{when} \quad \frac{\alpha}{\pi} > 1
\]

\[F' = 0 \quad \text{when} \quad \frac{\alpha}{\pi} \leq 1.
\]

As one would expect, we observe that \( F^+ = F - F' \); \( F^+ \) is not equal to \( F(1 - F') \), and this brings up a slight difference between our results and what is stated in reference 2, page 181.

\(^2\text{Chodorow et al., op. cit.}\)
II. THE DISK-LOADED WAVEGUIDE WITH $\alpha$ CONSTANT AND $\beta_w = 1$

From Eq. (1.9) we see that electrons start being bound to the traveling wave when

$$y_0 < \frac{\alpha}{\pi} \quad \text{or} \quad \frac{2\pi}{\alpha} y_0 < 2. \quad (2.1)$$

Now, by Eq. (1.17), the asymptotic phase angle $\Delta_\infty$ of a bound electron which enters the waveguide at an angle $\Delta_0$ is determined by

$$\cos \Delta_\infty = \cos \Delta_0 - \frac{2\pi}{\alpha} y_0. \quad (2.2)$$

Whence the bound orbits correspond to entrance angles $\Delta_0$, the absolute value of which lies between $0$ and $\cos^{-1}[-1 + (2\pi/\alpha) y_0]$. Further, Eq. (1.12) tells us that the bound electrons never go negative in velocity when $\alpha \leq \pi$; but when $\alpha > \pi$, we must impose the condition (1.11), rewritten as

$$\cos \Delta_\infty \geq 1 - \frac{2\pi}{\alpha} \quad (2.3)$$

in order to restrict the bound orbits with $\Delta_0 > 0$ to orbits where the electrons always retain a positive velocity.

This condition does not have to be imposed to the bound orbits with $\Delta_0 < 0$ because in these orbits the electron velocity never becomes negative, as it appears from Fig. 1.2. In terms of the entrance angle $\Delta_0$, the condition (2.3) becomes, by Eq. (2.2),

$$\cos \Delta_0 \geq 1 - \frac{2\pi}{\alpha} (1 - y_0). \quad (2.4)$$
The bound orbits where the electrons always retain a positive velocity are thus confined between $\Delta_0 = 0$ and

(a) $\Delta_0 = -\cos^{-1}\left(-1 + \frac{2\pi}{\alpha} y_0\right)$ for $\Delta_0 < 0$;

the limiting orbit corresponds to $\Delta_\infty = -\pi$;

(b) $\Delta_0 = +\cos^{-1}\left(-1 + \frac{2\pi}{\alpha} y_0\right)$ for $\Delta_0 > 0$ if $\alpha \leq \pi$;

the limiting orbit corresponds to $\Delta_\infty = -\pi$;

(c) $\Delta_0 = +\cos^{-1}\left[1 - \frac{2\pi}{\alpha} (1 - y_0)\right]$ for $\Delta_0 > 0$ if $\alpha > \pi$;

the limiting orbit corresponds to $\Delta_\infty = -\cos^{-1}[1 - (2\pi/\alpha)]$ by Eq. (2.3).

Bunching of the electrons by the regular sections of the accelerator

Since the electron energy at the end of the accelerator is fairly proportional to $-\sin \Delta_\infty$, we demand that the electrons be tied in bunches with a very narrow asymptotic phase-angle width. By Eq. (2.2), assuming that all electrons enter the accelerator with exactly the same velocity, we see that

$$-\sin \Delta_\infty \cdot d\Delta_\infty = -\sin \Delta_0 \cdot d\Delta_0;$$

whence the electrons must be injected at entrance angles very near $\Delta_0 = 0$. From Eq. (2.2) we get the asymptotic phase angle $\Delta_\infty^0$ for the case $\Delta_0 = 0$ by the relation

$$\cos \Delta_\infty^0 = 1 - \frac{2\pi}{\alpha} y_0; \quad (2.5)$$
so we have

\[
\cos \Delta_0^0 - \cos \Delta_\infty = 1 - \cos \Delta_0 .
\]

Let us consider all the electrons which enter the waveguide with phase angles between \(-\Delta_1\) and \(+\Delta_1\). Figure 2.1 shows us that, if we want these electrons to lie in the smallest possible variation interval of \(\sin \Delta_\infty\), we have to take

\[
\frac{1 - \cos \Delta_1}{2} = \cos \Delta_\infty = 1 - \frac{2\pi}{\alpha} y_0 . \tag{2.6}
\]

Thus the quantity \((2\pi/\alpha)y_0\) should be less than but close to 1 in order to have the electrons' energies vary as little as possible.

When the condition (2.6) is fulfilled, the relative spread of the electron energy at the end of a long accelerator is given by

\[
1 - |\sin \Delta_\infty^0|
\]

for all the electrons which enter the accelerator at phase angles between \(-\Delta_1\) and \(+\Delta_1\). For example, if we are interested only in those electrons which have 99 percent or more of the maximum energy, we shall take

\[
|\sin \Delta_\infty^0| = 0.99 \quad \text{or} \quad \Delta_\infty^0 = -81.89^\circ
\]

hence

\[
\cos \Delta_\infty^0 = 0.14107 .
\]

From Eq. (2.6) we then at once get

\[
\cos \Delta_1 = 0.71786, \quad \Delta_1 = 44.12^\circ .
\]

*Here we disregard all the other possible effects (e.g., beam loading, imperfect phasing between sections, etc.) which may cause the electron energy to vary.
Fig. 2.1 The asymptotic phase angles on the circle of unit radius.
The optimum value of \( \alpha \) is obtained from Eq. (2.5):

\[
\frac{2\pi}{\alpha} y_0 = 1 - \cos \Delta_0 = 0.85893,
\]

which condition by Eq. (1.16) we may rewrite as

\[
\alpha = \frac{2\pi}{0.85893} \sqrt{\frac{1 - \beta e_0}{1 + \beta e_0}}.
\]

Equation (2.8) is plotted in Fig. 2.2. In order to prevent electrons injected at phase angles between 0 and \( +\Delta_1 \) from going negative in velocity, we must impose the condition (2.4), i.e.,

\[
\frac{2\pi}{\alpha} - \frac{2\pi}{\alpha} \cdot y_0 \geq 1 - \cos \Delta_1
\]

or, by Eqs. (2.6) and (2.7):

\[
\frac{2\pi}{\alpha} \geq 1 + \cos \Delta_0
\]

In our example this means that we have to restrict the values of \( \alpha \) to

\[
\alpha \leq \frac{2\pi}{1.14107} = 5.506
\]

**Conclusion:** If electrons are injected into the regular sections of the accelerator at phase angles between \(-44.12^\circ\) and \(+44.12^\circ\), with an initial velocity related to the electric field strength \( \alpha \) by Eq. (2.8) or by the graph of Fig. 2.2, their energy spread at the output end of the accelerator will be 0.01, and their phase spread will be \( 8.11^\circ \times 2 = 16.22^\circ \). Moreover, if the value of \( \alpha \) is kept below 5.50, none of these electrons will go negative in velocity in its orbit.
Fig. 2.2 Optimum value of $\alpha$ versus initial electron velocity, when the desired accuracy of the final electron energy is better than 0.01.
In order to get this bunching effect in the regular sections of the accelerator, it is very important that the electrons enter the accelerator with the velocity given by Fig. 2.2. If they enter the accelerator with a velocity nearly equal to that of light, no further bunching can occur since the electrons then move as fast as the traveling wave and their phase angles cannot change.

**Velocity modulation in the electron beam before entering the accelerator**

So far we have assumed that all the electrons have exactly the same velocity when they enter the accelerator; the initial conditions are then represented in phase space by the line \( y_1 \) of Fig. 2.3. In order to be accelerated to a fraction \( \frac{\sin \Delta_0}{\Delta_0} \) or more of the maximum energy, the electrons must have their representative points between the two curves corresponding to \( \Delta_0 \) and \( \Delta_1 = (\Delta_0 - \Delta_0^0) \); therefore, if they enter the accelerator with a lower velocity and thus a higher \( y \), the allowed interval \((-\Delta, +\Delta)\) for their entrance phase angles gets narrower. The widest phase interval \((-\Delta_1, +\Delta_1)\) corresponds to the lowest value \( y_1 \) of \( y \); when all electrons enter the accelerator with the same velocity and thus with the same \( y_0 \), the latter must be chosen equal to \( y_1 \), and this is precisely what has been done above. For other values of \( y, \Delta \) is determined from Eq. (1.17) by the relation

\[
\frac{2\pi}{\alpha} y = \cos \Delta - \cos \Delta_0 = \cos \Delta + \cos \Delta_0^0 ,
\]

whereas for \( y_1 \) we have

\[
\frac{2\pi}{\alpha} y_1 = 1 - \cos \Delta_0 = \cos \Delta_1 + \cos \Delta_0^0 .
\]

If we put \( x = (y - y_1)/(y + y_1) \), we get

\[
\frac{\cos \Delta - \cos \Delta_1}{1 + \cos \Delta} = x \quad \text{and} \quad \cos \Delta = \frac{\cos \Delta_1 + x}{1 - x} . \tag{2.9}
\]
\[ \Delta_0^1 = -\pi - \Delta_0^0 \]

Fig. 2.3 The allowed region in phase space.
The amount of velocity modulation is thus limited by the condition

\[
\frac{\cos \Delta_1 + x}{1 - x} < 1 \quad \text{or} \quad x < \frac{1 - \cos \Delta_1}{2} = \cos \Delta_0. \quad (2.10)
\]

From Eq. (1.4) where \( \beta_w = 1 \) we observe that \( y = \gamma - p \); hence \( x \) may be expressed in terms of the electron momentum and energy, which yields

\[
x = \frac{p_1 - p}{\gamma_1 + \gamma} = \frac{\gamma_1 - \gamma}{p_1 + p}. \quad (2.11)
\]

The phase angle \( \Delta \) as a function of \( \xi \)

Equation (1.1) enables us to determine \( \xi \) as a function of \( \Delta \). Indeed, we may write it as \( (\beta_w = 1) \)

\[
2\pi \cdot \frac{d\xi}{d\Delta} = -\frac{\beta_e}{1 - \beta_e} = -\frac{1}{2} \left[ \frac{1}{y^2} - 1 \right]
\]

by Eq. (1.16). From this we get

\[
4\pi \xi = \int_{\Delta_0}^{\Delta} \left[ 1 - \frac{1}{y^2} \right] \cdot d\Delta
\]

or

\[
\xi = \xi'(\Delta) - \xi'(\Delta_0) \quad (2.12)
\]
where we have put

\[
4\pi \cdot \xi'(\Delta) = \int_0^\Delta \left( 1 - \frac{1}{y^2} \right) \cdot d\Delta = \Delta - \left( \frac{2\pi}{\alpha} \right)^2 \cdot \int_0^\Delta \frac{d\Delta}{(\cos\Delta - \cos\Delta_\infty)^2} 
\]

(2.13)

by Eq. (1.17). For a given value of the asymptotic phase angle \(\Delta_\infty\), \(\xi'(\Delta)\) is thus an odd function of \(\Delta\).

Now, when \(\Delta_0\) changes sign, \(\Delta_\infty\) as determined by Eq. (2.2) remains unchanged; the function \(\xi'(\Delta)\) then remains the same, and \(\xi'(\Delta_0)\) on the right-hand side of Eq. (2.12) merely reverses sign with \(\Delta_0\). Therefore the two curves of \(\Delta\) versus \(\xi\) corresponding to the entrance angles \(+\Delta_0\) and \(-\Delta_0\) are identical except for a translation along the \(\xi\)-axis, and both may be obtained by shifting the plot of the function \(\xi'(\Delta)\) by an amount \(\xi'(\Delta_0)\) either in the positive or in the negative \(\xi\) direction.

Performing the integration in Eq. (2.13), we get

\[
4\pi \cdot \xi' = \Delta - \left( \frac{2\pi}{\alpha} \right)^2 \cdot \frac{1}{\sin^2 \Delta_\infty} \left[ \frac{\sin \Delta}{\cos \Delta - \cos \Delta_\infty} \right. \\
\left. - \cotg \Delta_\infty \cdot \log \frac{1 - \tg \frac{\Delta}{2} \cdot \cotg \frac{\Delta_\infty}{2}}{1 + \tg \frac{\Delta}{2} \cdot \cotg \frac{\Delta_\infty}{2}} \right]. 
\]

(2.14)

Computation of the function \(\xi'\) is made easy when we introduce a new variable \(u\), defined by the relation

\[
\begin{vmatrix}
\Delta \\
tg - \\
\frac{2}{\Delta_\infty} \\
tg \frac{\Delta_\infty}{2} \\
\end{vmatrix} = \begin{vmatrix}
u \\
\frac{u}{2} \\
\frac{u}{2} \\
\end{vmatrix} 
\]

(2.15)
Here \( u \) is real since \(|\Delta|\) is always less than \(|\Delta_\infty|\), as it appears from Eq. (1.17). With the new variable \( u \) we have

\[
\frac{\sin \Delta}{\cos \Delta - \cos \Delta_\infty} = \frac{\text{sh} u}{\sin \Delta_\infty}
\]

\[
l = \frac{\Delta}{\log_2 \left( -\frac{\Delta}{\Delta_\infty} \right)} = -u,
\]

and the function (2.14) reduces to

\[
\xi = \frac{\Delta/2}{\Delta_\infty^2} \frac{\pi}{2} \left( \frac{1}{\sin^3 \Delta_\infty} \right) [\text{sh} u + u \cos \Delta_\infty]. \tag{2.16}
\]

We have computed \( \xi \) as a function of \( \Delta \) for the case \( \alpha = 4 \), \( \beta_{e0} = \frac{1}{2} \). By Eq. (1.16) we then have \( y_0 = 1/\sqrt{3} \), and from Eq. (2.2) we get

\[
\cos \Delta_\infty = \cos \Delta_0 - \frac{\pi}{2} \frac{1}{\sqrt{3}} = \cos \Delta_0 - 0.90690.
\]

The bound orbits are confined between \( \Delta_0 = 0 \) and

\[
\Delta_0 = \pm \cos^{-1} \left( -1 + \frac{2\pi}{\alpha} y_0 \right) = \pm \cos^{-1} (-0.09310) = \pm 95.34^\circ;
\]

but when \( \Delta_0 > 0 \), we know by the condition (2.4) that in order to keep a positive velocity throughout their trajectories the electrons must enter the accelerator at phase angles between 0 and

\[
\cos^{-1} \left[ 1 - \frac{2\pi}{\alpha} (1 - y_0^2) \right] = \cos^{-1} (0.33610) = 70.36^\circ.
\]
Figure 2.4 shows a plot of $\Delta$ versus $\xi$ for entrance phase angles $\Delta_0$ of $0^\circ$, $\pm 30^\circ$, $\pm 60^\circ$, and $\pm 90^\circ$. For $\Delta_0 = +90^\circ$ there is a portion of the curve where electrons go backward of the positive $\xi$ direction, since there they get a negative velocity.

We also observe that $\xi(0) = -\xi'(\Delta_0)$ is very small: it is less than 0.1 for any value of $\Delta_0$.

Comparison with the results of Gallagher

Using an electronic computer, Gallagher has numerically integrated the electron-motion equations (1.1) for the case $\alpha = 4 e^{-0.0315 \xi}$, $\beta_w = 1$, $\beta_e = \frac{1}{2}$. This case differs very little from our case $\alpha = 4$, $\beta_w = 1$, $\beta_e = \frac{1}{2}$, and as a matter of fact the graph obtained by Gallagher coincides very closely with our Fig. 2.4. Moreover, we are able to explain why the computer stopped the computation of the curve $\Delta_0 = +90^\circ$: the electrons in this orbit are pushed into negative velocities by the electric field.

Remark about the case $\Delta_0 = -\pi$

For the limiting orbit $\Delta_\infty = -\pi$ the expression (2.14) of $\xi'$ becomes indeterminate. A way of obtaining the true value of the function $\xi'$ is by expanding the log in series in Eq. (2.14); but it is much easier to go back to Eq. (2.13) and to put $\Delta_\infty = -\pi$ from this point on, from which we get

$$4\pi \xi' = \int_0^\Delta \left[ 1 - \frac{1}{y} \right] \cdot d\Delta = \Delta - \frac{(2\pi)^2}{\alpha} \cdot \int_0^\Delta \frac{d\Delta}{(\cos \Delta + 1)^2}$$

$$= \Delta - \left( \frac{2\pi}{\alpha} \right)^2 \cdot \left[ \frac{1}{2} - \frac{\Delta}{2} + \frac{1}{6} \cdot \frac{\Delta^3}{2} \right].$$

---

$^5$ W. J. Gallagher, private communication.
FIG. 2.4 $\Delta$ versus $\theta$ for $\beta_0 = \frac{1}{2}$, $\alpha = 4$. 

$\Delta_0 = -84.6^\circ$, $\Delta_0 = -92.9^\circ$, $\Delta_0 = -114.0^\circ$, $\Delta_0 = -155.8^\circ$.
Approximate formulas for $\xi >> 1$

When $\xi$ is much greater than 1, $\Delta$ is nearly equal to $\Delta_\infty$, and Eq. (2.15) is no longer adequate for computation of the value of $u$, because $\sinh(u/2)$ is then too close to 1. In this case it is more appropriate to use Eq. (2.17), which may be shown to be exactly equivalent to Eq. (2.15):

$$\cosh u = \frac{\sin^2 \Delta_\infty}{\cos \Delta - \cos \Delta_\infty} - \cos \Delta_\infty. \quad (2.17)$$

For $u >> 1$ this relation may be approximated by

$$e^u \approx \frac{2 \sin^2 \Delta_\infty}{\cos \Delta - \cos \Delta_\infty} - 2 \cos \Delta_\infty. \quad (2.18)$$

$u$ as a function of $\xi$ when $\xi >> 1$

Neglecting quantities which tend to 0 when $\xi$ tends to $\infty$, we may rewrite Eq. (2.16) as

$$\xi' = \frac{\Delta_\infty}{\frac{\xi}{4\pi} - \frac{1}{\alpha^2 \sin^3 \Delta_\infty}} \left[ \frac{e^u}{2} + u \cos \Delta_\infty \right]$$

or

$$\frac{2\alpha^2}{\pi} \sin^3 \Delta_\infty \left( \xi' - \frac{\Delta_\infty}{4\pi} \right) \approx e^u + 2u \cos \Delta_\infty. \quad (2.19)$$

Equation (2.19) is of the type

$$B = e^u + Au$$

which, provided $Au$ is small compared to $e^u$, yields

$$u \approx \left[ 1 - A/B \right] \log B. \quad \text{Since in Eq. (2.12) } \xi' \Delta_0 \text{ is very small,}$$
we may replace \( \xi' \) by \( \xi \) in Eq. (2.19), so that finally this equation is inverted as

\[
u \approx \left[ 1 + \frac{\pi \cos \Delta}{\Delta_{\infty} - \frac{\Delta}{4\pi}} \cdot \frac{1}{\Delta_{\infty} - \frac{\Delta}{4\pi}} \right] \cdot \log \left\{ -\frac{\alpha^2}{\pi} \cdot \sin^3 \Delta_{\infty} \cdot \left( \xi - \Delta_{\infty} \right) \right\} \text{ (2.20)}
\]

The energy \( \gamma \) as a function of \( \xi \)

The electron energy \( \gamma \) is determined at once as a function of \( \Delta \) by Eqs. (1.4) and (1.17):

\[
\gamma + \sqrt{\gamma^2 - 1} = \frac{1}{\alpha} \cdot \frac{1}{\cos \Delta - \cos \Delta_{\infty}}.
\text{ (2.21)}
\]

Now, from Eqs. (2.16) and (2.17) we deduce the exact relation

\[
\xi' = \frac{\Delta}{4\pi} - \frac{\pi}{\Delta_{\infty} - \frac{\Delta}{4\pi}} \cdot \frac{1}{\sin^2 \Delta_{\infty}} \left\{ \frac{\sin \Delta_{\infty}}{\cos \Delta - \cos \Delta_{\infty}} + (u - 1) \cos \Delta_{\infty} - \alpha \right\}.
\]

If we neglect quantities which tend to 0 when \( \xi \to \infty \), and if we again replace \( \xi' \) by \( \xi \), we obtain as an approximate formula

\[
-\alpha \sin \Delta_{\infty} \cdot \left( \xi - \frac{\Delta_{\infty}}{4\pi} \right) \approx \frac{\pi}{\alpha} \cdot \frac{1}{\cos \Delta - \cos \Delta_{\infty}} \cdot \left( \frac{\cos \Delta_{\infty}}{\sin^2 \Delta_{\infty}} \cdot (u - 1) \right)
\]

while Eq. (2.21) reduces to

\[
\gamma \approx \frac{\pi}{\alpha} \cdot \frac{1}{\cos \Delta - \cos \Delta_{\infty}}.
\]

With this last relation, Eq. (2.18) gives

\[
u \approx \log \left\{ \gamma \cdot \frac{\alpha}{\pi} \cdot \sin^2 \Delta_{\infty} \right\} \text{ (2.22)}
\]

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so that finally we have

$$-\alpha \sin \Delta_{\infty} \cdot \left( \xi - \frac{\Delta_{\infty}}{4\pi} \right) \approx \gamma + \frac{\pi \cos \Delta_{\infty}}{\alpha \sin^2 \Delta_{\infty}} \cdot \left[ \log \left( \gamma \cdot \frac{2\alpha}{\pi} \sin^2 \Delta_{\infty} \right) - 1 \right].$$

(2.23)

We observe that when $\xi$ is large, the energy $\gamma$ grows roughly in proportion with $\xi$, and the proportionality factor is $-\alpha \sin \Delta_{\infty}$, as might at once be anticipated from Eq. (1.1).
III. THE DISK-LOADED WAVEGUIDE WITH $\alpha$ CONSTANT AND $\beta_w < 1$

When $\beta_w < 1$ the bound orbits in phase space are closed, and Slater\(^1\) has emphasized the analogy between the electron oscillations about the phase angle $\Delta = 0$ and the pendulum oscillations in a gravitational field. In particular, if the amplitude $\Delta_m$ of the electron oscillations is small, the period $T_0$ of these oscillations is independent of $\Delta_m$ and is related to the period $T$ of the traveling wave by the equation

$$\left(\frac{T}{T_0}\right)^2 = \frac{\alpha}{2\pi} \frac{1}{\beta_w} \left(1 - \beta_w^2\right)^{3/2}.$$  \hspace{1cm} (3.1)

Equation (3.1) is merely Eq. (19) of Slater [Ref. 1, p. 507] translated in our own notations.

Let us now look at Fig. 1.1. If the electrons are injected in the waveguide with the velocity of the wave (i.e., $\beta_e = \beta_w$), all of them will be bound to the wave, and whether they start their travel with a phase angle $+\Delta_m$ or $-\Delta_m$, they will arrive at the phase angle $\Delta = 0$ in roughly the same time $T_0/4$, provided $\Delta_m$ is not too large. Therefore, if we consider a section of waveguide with a length $z = c\beta_w \cdot T_0/4$ or $\xi = z/\lambda = \beta_w \cdot (1/4) \cdot T_0/T$, all the electrons which enter this section with a velocity $\beta_e = \beta_w$ will be bunched around $\Delta = 0$ at the output end of the section.

More precisely, if we call $t$ the exact time it takes an electron which enters the waveguide at the phase angle $\Delta_m$ to arrive at $\Delta = 0$, we see from Fig. 3.1 that the point $z_0$ where this occurs is given by

$$z_0 = c\beta_w \cdot t - \beta_w \frac{\Delta_m}{2\pi}.$$  \hspace{1cm} (26)

---

\(^1\)Slater, op. cit.
Fig. 3.1 Motion of an electron on the wave in the waveguide section
or

\[ 2\pi \xi_0 = \beta_w \cdot \frac{t}{T} = \beta_w \cdot \Delta_m. \]  \(3.2\)

From now on it will be convenient to assume that \(\Delta_m\) may take negative as well as positive values, so that \(\Delta_m\) will represent both in sign and in absolute value the phase angle at which the electrons enter the waveguide with a velocity \(\beta_{e0} = \beta_w\).

The bunching of the electrons around \(\Delta = 0\) at the output end of the section will be all the better as \(\xi_0\) in Eq. (3.2) is less dependent on \(\Delta_m\); whence the first term of \(\xi_0\) must be the most important one, and since \(t\) tends to \(T_0/4\) when \(\Delta_m \to 0\), this means that the ratio \(T_0/T\) must be large or, by Eq. (3.1), that the electric-field strength \(\alpha\) has to be chosen small.

\(\xi\) as a function of \(\Delta\)

In order to investigate the preceding conclusions in more detail, we need the rigorous expression of \(\xi\) as a function of \(\Delta\). Let us start again with Eq. (1.1), which may be written as

\[ 2\pi \cdot \frac{d\xi}{d\Delta} = \frac{\beta_w}{1 - \beta_w/\beta_e} = \frac{\beta_w}{1 - \beta_w \cdot \gamma/\sqrt{\gamma^2 - 1}}. \]  \(3.3\)

We now have to express \(\gamma\) as a function of \(\Delta\) or, what is equivalent by Eq. (1.5), as a function of \(y\). From Eq. (1.4), we get

\[ \sqrt{\gamma^2 - 1} = \frac{\gamma}{\beta_w} \]  \(3.4\)

which readily yields

\[ \frac{\gamma}{\beta_w} = \frac{1}{1 - \beta_w^2} \left[ y \pm \sqrt{\beta_w^2 \cdot y^2 - (1 - \beta_w^2)} \right]. \]
The two solutions for $\gamma/\beta_w$ correspond to the two values of $p$ for a given $y$ in Fig. 1.3, or to the two values of $\sqrt{\gamma^2 - 1}$ for a given $\Delta$ in Fig. 1.1; this latter figure also shows that we must choose the minus sign when $\Delta_m$ is positive and the plus sign when $\Delta_m$ is negative, so that we have

$$\frac{\gamma}{\beta_w} \left(1 - \beta_w^2\right) = y - \text{sgn}(\Delta_m) \cdot \sqrt{\beta_w^2 \cdot y^2 - \left(1 - \beta_w^2\right)}.$$  \hspace{1cm} (3.5)

Going back to Eq. (3.4) we now get

$$1 - \frac{\beta_w}{\beta_e} = 1 - \beta_w \cdot \frac{\gamma}{\sqrt{\gamma^2 - 1}} = \left(1 - \beta_w^2\right) \cdot \frac{1 - \beta_w}{\gamma} \cdot \frac{y}{1 - \beta_w^2}$$

or, by Eq. (3.5):

$$1 - \frac{\beta_w}{\beta_e} = \left(1 - \beta_w^2\right) \cdot \frac{\sqrt{\beta_w^2 \cdot y^2 - \left(1 - \beta_w^2\right)}}{-\text{sgn}(\Delta_m) \cdot y \beta_w^2 + \sqrt{\beta_w^2 \cdot y^2 - \left(1 - \beta_w^2\right)}}.$$  

With this expression Eq. (3.3) becomes

$$2\pi \cdot \frac{d\xi}{d\Delta} = \frac{\beta_w}{1 - \beta_w^2} \left[1 - \text{sgn}(\Delta_m) \cdot \frac{\beta_w^2 \cdot y}{\sqrt{\beta_w^2 \cdot y^2 - \left(1 - \beta_w^2\right)}}\right]$$

and finally, by introducing the minimum value (1.8) of $y$, we reduce this equation to

$$2\pi \cdot b^2 \cdot \frac{d\xi}{d\Delta} = \frac{1}{\beta_w} - \text{sgn}(\Delta_m) \cdot \frac{y}{\sqrt{y^2 - b^2}}.$$  \hspace{1cm} (3.6)
In order to express the right-hand side of Eq. (3.6) as a function of \( \Delta \), we shall use Eq. (1.10), which gives us the difference \( (y - b) \); so we are led to rewrite Eq. (3.6) as

\[
2\pi b^2 \cdot d\xi = \frac{d\Delta}{\beta_w} = \text{sgn} (\Delta_m) \cdot \frac{y - b + 1}{2b + \frac{1}{2}} \cdot \frac{\sqrt{y - b}}{\sqrt{\frac{y + b}{2b}}} \cdot d\Delta \quad (3.7)
\]

while from Eq. (1.10) we have

\[
\frac{y - b}{2b} = \frac{\alpha}{2\pi b} \cdot \left( \sin^2 \frac{\Delta_m}{2} - \sin^2 \frac{\Delta}{2} \right).
\]

The success when integrating Eq. (3.7) depends critically on a good choice of the integration variable. The variable we found the most convenient for this purpose is the angle \( \phi \) defined by

\[
\sin \phi = \frac{\sin \frac{\Delta}{2}}{\sin \frac{\Delta_m}{2}} \quad (3.8)
\]

\( \phi \) may vary from \( +\frac{\pi}{2} \) to \( -\frac{\pi}{2} \). In terms of \( \phi \) we have

\[
\begin{cases}
  \frac{y - b}{2b} = \frac{\alpha}{2\pi b} \sin^2 \frac{\Delta_m}{2} \cdot \cos^2 \phi \\
  \frac{y + b}{2b} = 1 + \frac{\alpha}{2\pi b} \sin^2 \frac{\Delta_m}{2} \cdot \cos^2 \phi
\end{cases} \quad (3.9)
\]

so that

\[
\sqrt{\frac{y - b}{2b}} = \text{sgn} (\Delta_m) \cdot \sqrt{\frac{\alpha}{2\pi b}} \cdot \sin \frac{\Delta_m}{2} \cdot \cos \phi.
\]
From Eq. (3.8) we also get
\[
\cos \frac{\Delta}{2} - d\left(\frac{\Delta}{2}\right) = \sin \frac{\Delta_m}{2} \cdot \cos \phi \cdot d\phi
\]

whence
\[
\frac{d(\Delta/2)}{\sqrt{y-b}/2b} = \text{sgn} (\Delta_m) \cdot \sqrt{\frac{2\pi b}{\alpha}} \cdot \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\Delta_m}{2} \cdot \sin^2 \phi}}.
\]

Introducing now the expressions (3.9) and (3.10) into Eq. (3.7) we obtain
\[
2\pi b^2 \cdot d\xi = \frac{d\Delta}{\beta_w} - \sqrt{\frac{2\pi b}{\alpha}} \cdot \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\Delta_m}{2} \cdot \sin^2 \phi}} \cdot \sqrt{\frac{1}{\sqrt{1 + \frac{\alpha}{2\pi b} \cdot \sin^2 \frac{\Delta_m}{2} \cdot \cos^2 \phi}}}
\]

and
\[
2\pi b^2 \cdot \xi = \frac{\Delta - \Delta_m}{\beta_w} + \sqrt{\frac{2\pi b}{\alpha}} \cdot \sqrt{\int \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\Delta_m}{2} \cdot \sin^2 \phi}}}
\]

\[
\cdot \left(1 + 2 \frac{\alpha}{2\pi b} \cdot \sin^2 \frac{\Delta_m}{2} \cdot \cos^2 \phi \right)
\]

\[
\cdot \left(1 + \frac{\alpha}{2\pi b} \cdot \sin^2 \frac{\Delta_m}{2} \cdot \cos^2 \phi \right)^{-1/2}
\]

\[
\text{sgn} (\Delta_m) \cdot \sqrt{\frac{2\pi b}{\alpha}} \cdot \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\Delta_m}{2} \cdot \sin^2 \phi}}.
\]

\(\text{(3.11)}\)
\[ \xi_0 \text{ is the value of } \xi \text{ for } \Delta = 0: \]

\[
2\pi b^2 \cdot \xi_0 = \frac{\Delta m}{\beta_w} + \sqrt{\frac{2\pi b}{\alpha}} \cdot \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\Delta m}{2} \cdot \sin^2 \phi}} \]

\[
1 + 2 \frac{\alpha}{2\pi b} \sin^2 \frac{\Delta m}{2} \cdot \cos^2 \phi \left(1 + \frac{\alpha}{2\pi b} \sin^2 \frac{\Delta m}{2} \cdot \cos^2 \phi \right)^{-1} \]

\[ (3.12) \]

so we have

\[
2\pi b^2 \cdot (\xi - \xi_0) = \frac{\Delta}{\beta_w} - \sqrt{\frac{2\pi b}{\alpha}} \cdot \int_0^{\phi} \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\Delta m}{2} \cdot \sin^2 \phi}} \]

\[
1 + 2 \frac{\alpha}{2\pi b} \sin^2 \frac{\Delta m}{2} \cdot \cos^2 \phi \left(1 + \frac{\alpha}{2\pi b} \sin^2 \frac{\Delta m}{2} \cdot \cos^2 \phi \right)^{-1} \]

\[ (3.13) \]

Since \( \xi - \xi_0 \) is an odd function of \( \Delta \), the curve \( \Delta(\xi) \) is symmetrical about the point \( \xi = \xi_0 \) on the \( \xi \)-axis.

The integral on \( \phi \) which appears here is of the elliptic type; it is a transcendental function for which, as far as we know, there does not exist any numerical table. Nevertheless, we can obtain a close approximation for this integral if we remember that, in order to keep all the electrons in all orbits positive in velocity, we have to take, by Eqs. (1.12) and (1.8):

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Moreover, as we have already stated, good bunching of the electrons around $\Delta = 0$ occurs when $E_0$ depends very little on $\Delta_m$; by thinking of the right-hand side of Eq. (3.12) as a power series in $\Delta_m$, we see that this will be achieved if $\Delta_m/\beta_w$ stays small compared to $\sqrt{2\pi b/\alpha \cdot \pi/2}$, i.e., if we have

$$\frac{\pi}{\beta_w} < \sqrt{\frac{2\pi b}{\alpha}} \cdot \frac{\pi}{2}$$

or

$$\sqrt{\frac{\alpha}{2\pi b}} < \frac{\beta_w}{2}.$$

So we are led to replace the condition (3.14) by the more stringent one

$$\frac{\alpha}{2\pi b} < \frac{\beta_w^2}{4}.$$ 

Under this assumption the integral in Eq. (3.13) may be expanded as a power series of $\alpha/2\pi b$, following the relation

$$\int_0^\phi \frac{d\phi}{\sqrt{1 - \sin^2 \Delta_m/2 \cdot \sin^2 \phi}} = 1 + \frac{\sin^2 \Delta_m}{\alpha} \cdot \cos^2 \phi
\int_0^\phi \frac{d\phi}{\sqrt{1 + \frac{\alpha}{2\pi b} \cdot \sin^2 \Delta_m/2 \cdot \cos^2 \phi}} = \frac{1}{2} + \frac{3}{2} \frac{\alpha}{2\pi b} \cdot \int_0^\phi \frac{\sin^2 \Delta_m/2 \cdot \cos^2 \phi}{\sqrt{1 - \sin^2 \Delta_m/2 \cdot \sin^2 \phi}} \cdot d\phi + \ldots \quad (3.16)$$

$$-\frac{5}{8} \left(\frac{\alpha}{2\pi b}\right)^2 \cdot \int_0^\phi \frac{\sin^4 \Delta_m/2 \cdot \cos^4 \phi}{\sqrt{1 - \sin^2 \Delta_m/2 \cdot \sin^2 \phi}} \cdot d\phi + \ldots$$

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Each integral appearing on the right-hand side of Eq. (3.16) may be calculated in terms of the elliptic integrals of the first kind $F(A_m/2, \phi)$ and of the second kind $E(A_m/2, \phi)$; but their expressions rapidly become very intricate and moreover are not worth to be evaluating. For this reason we shall limit the expansion to the first two terms of Eq. (3.16), which we can express as

$$
\int_0^\phi \frac{d\phi}{\sqrt{1 - \sin^2 \frac{A_m}{2} \cdot \sin^2 \phi}} = F\left(\frac{A_m}{2}, \phi\right)
$$

and

$$
\int_0^\phi \frac{\sin^2 \frac{A_m}{2} \cdot \cos^2 \phi \cdot d\phi}{\sqrt{1 - \sin^2 \frac{A_m}{2} \cdot \sin^2 \phi}}
\begin{align*}
&= \left(\sin^2 \frac{A_m}{2} - 1\right) \cdot \int_0^\phi \frac{d\phi}{\sqrt{1 - \sin^2 \frac{A_m}{2} \cdot \sin^2 \phi}} \\
&+ \int_0^\phi \sqrt{1 - \sin^2 \frac{A_m}{2} \cdot \sin^2 \phi} \cdot d\phi
\end{align*}
\begin{align*}
&= -\cos^2 \frac{A_m}{2} \cdot F\left(\frac{A_m}{2}, \phi\right) + E\left(\frac{A_m}{2}, \phi\right)
\end{align*}
$$

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Whence Eq. (3.16) finally reduces to

\[
\int_0^\phi \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\Delta_m}{2} \cdot \sin^2 \phi}} \cdot \frac{1 + 2 \frac{\alpha}{2\pi b} \sin^2 \frac{\Delta_m}{2} \cdot \cos^2 \phi}{\sqrt{1 + \frac{\alpha}{2\pi b} \sin^2 \frac{\Delta_m}{2} \cdot \cos^2 \phi}}
\]

\[= F\left(\frac{\Delta_m}{2}, \phi\right) + \frac{3}{2} \frac{\alpha}{2\pi b} \left[ E\left(\frac{\Delta_m}{2}, \phi\right) - \cos^2 \frac{\Delta_m}{2} \cdot F\left(\frac{\Delta_m}{2}, \phi\right) \right] - \ldots \]

(3.17)

When \(\phi = \pi/2\) the elliptic integrals become the complete elliptic integrals of the first kind \(K(\Delta_m/2)\) and of the second kind \(E(\Delta_m/2)\); Eqs. (3.12) and (3.13) may thus be written as

\[2\pi b \cdot \xi_0 = -\frac{\Delta_m}{\beta_w} + \sqrt{\frac{2\pi b}{\alpha}} \cdot K\left(\frac{\Delta_m}{2}\right) + \frac{3}{2} \frac{\alpha}{2\pi b} \left[ E\left(\frac{\Delta_m}{2}\right) - \cos^2 \frac{\Delta_m}{2} \cdot K\left(\frac{\Delta_m}{2}\right) \right] - \ldots \]  

(3.18)

and

\[2\pi b^2 \cdot (\xi - \xi_0) = \frac{\Delta}{\beta_w} - \sqrt{\frac{2\pi b}{\alpha}} \cdot F\left(\frac{\Delta_m}{2}, \phi\right) - \frac{3}{2} \frac{\alpha}{2\pi b} \left[ E\left(\frac{\Delta_m}{2}, \phi\right) - \cos^2 \frac{\Delta_m}{2} \cdot F\left(\frac{\Delta_m}{2}, \phi\right) \right] + \ldots \]  

(3.19)

Practically, the value of \(\sqrt{\alpha/2\pi b}\) will be sufficiently small so that even the third term will be negligible in the preceding equations.

When an electron which entered the waveguide at a phase angle \(\Delta_m\) has arrived to \(\Delta = -\Delta_m\) (i.e., when \(\phi = \pi/2\)), it experiences
exactly the same situation as an electron which enters the waveguide at a phase angle \(-\Delta_m\); and so the curve \(\Delta(\xi)\) representing its path beyond the point \(\Delta = -\Delta_m\) is readily obtained by a displacement along the \(\xi\)-axis of the curve corresponding to an entrance phase angle \(-\Delta_m\).

Remark about the time \(t\) it takes an electron which enters the waveguide to arrive at \(\Delta = 0\)

This time \(t\) may be easily deduced from Eqs. (3.2) and (3.12) as

\[
2\pi = -\Delta_m \cdot \frac{\beta_w^2}{1 - \beta_w^2} + \frac{1}{b^2\beta_w} \cdot \sqrt{\frac{2\pi b}{\alpha}}
\]

\[
\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\Delta_m}{2} \cdot \sin^2 \phi}} = \frac{1 + 2 \frac{\alpha}{2\pi b} \sin^2 \frac{\Delta_m}{2} \cdot \cos^2 \phi}{\sqrt{1 + \frac{\alpha}{2\pi b} \sin^2 \frac{\Delta_m}{2} \cdot \cos^2 \phi}}
\]

(3.20)

It is important to observe that the value of \(t\) does depend on the sign of \(\Delta_m\); and since, as can be readily seen, the time an electron needs to go from \(+\Delta_m\) to \(-\Delta_m\) merely equals \(2t\), this means that the time it takes an electron to go from \(+\Delta_m\) to \(-\Delta_m\) is not the same as the time it takes to go from \(-\Delta_m\) to \(+\Delta_m\). Here we encounter a fundamental difference between the electron oscillations about the phase angle \(\Delta = 0\) and the pendulum oscillations. But if we compute the total time \(T_1\) it takes an electron to achieve a complete loop in phase space, by adding the times for \(+\Delta_m\) and for \(-\Delta_m\) the two \(\Delta_m\) terms of Eq. (3.20) cancel each other, so that we are left simply with the equation
\[
2\pi \frac{T_1}{T} = \frac{1}{b^2 \beta_w} \sqrt{\frac{2\pi b}{\alpha}}
\]

\[
\int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \sin^2 \frac{\Delta m}{2} \cdot \sin^2 \phi}} \cdot \frac{1 + 2 \frac{\alpha}{2\pi b} \sin^2 \frac{\Delta m}{2} \cdot \cos^2 \phi}{\sqrt{1 + \frac{\alpha}{2\pi b} \sin^2 \frac{\Delta m}{2} \cdot \cos^2 \phi}}, \quad (3.21)
\]

Equation (3.21) shows that the period \(T_1\) of the electron oscillations is nearly independent of the amplitude \(\Delta m\) when \(\Delta m\) is small, and that \(T_1\) then approaches the value \(T_0\) given by

\[
\frac{T_0}{T} = \frac{1}{b^2 \beta_w} \sqrt{\frac{2\pi b}{\alpha}}.
\]

With Eq. (1.8), this relation straightforwardly appears equivalent to Eq. (3.1) and so agrees exactly with Slater's results [Ref. 1, p. 507].
Numerical results

We have computed $\xi$ as a function of $\Delta$ for the case $\beta_w = \frac{1}{2}$ (i.e., $b = \sqrt{3}$) and $\sqrt{\alpha/2\pi b} = 0.1$. From Eq. (3.15) we knew that we had to choose $\sqrt{\alpha/2\pi b} < \beta_w/2 = 0.25$; for the sake of simplicity we chose $\sqrt{\alpha/2\pi b} = 0.1$, which corresponds to a value

$$\alpha = 0.01 \cdot 2\pi b = 0.109.$$ 

With such a low value of the quantity $\sqrt{\alpha/2\pi b}$, the accuracy of Eqs. (3.18) and (3.19) is sufficient for making a plot of the function $\Delta$ versus $\xi$ even if the $\sqrt{\alpha/2\pi b}$ terms therein are neglected; we nevertheless have taken these terms into account for our numerical computations. Further, since the curve $\Delta(\xi)$ is symmetrical about the point $\xi = \xi_0$ on the $\xi$-axis, we only need to compute points where $\Delta$ has the same sign as $\Delta_m$, i.e., by Eq. (3.8), points where $\phi$ is positive.

Figure 3.2 shows a plot of $\Delta$ versus $\xi$ for entrance phase angles $\Delta_m$ of $0^\circ$, $\pm 30^\circ$, $\pm 60^\circ$, $\pm 90^\circ$, $\pm 120^\circ$, $\pm 150^\circ$.

Once we have plotted curves $\Delta(\xi)$ for many values of $\Delta_m$, we can draw the envelope of the system. This envelope has a cusp in the neighborhood of the point $\xi_0$ (for $\Delta_m = 0$) and settles the boundary of a region each point of which is the intersection of three curves coming from three different entrance phase angles $\Delta_m$: this means that the electron density is at its highest inside the envelope, or that the electrons are bunched into this region. From Fig. 3.2 we can thus deduce Table I.

If we want the electrons to be bunched more symmetrically about $\Delta = 0$, we need only reduce the dissymmetry between curves for positive and negative $\Delta_m$ by choosing a smaller value for $\alpha$ in Eqs. (3.18) and (3.19). In fact, in the limiting case when $(2/\beta_w) \cdot \sqrt{\alpha/2\pi b}$ tends to 0, these equations become simply

$$\sqrt{\alpha/2\pi b} \cdot 2\pi b^2 \xi_0 = K \left( \frac{\Delta_m}{2} \right)$$

$$\sqrt{\alpha/2\pi b} \cdot 2\pi b^2 (\xi - \xi_0) = - F \left( \frac{\Delta_m}{2}, \phi \right)$$

\[ (3.22) \]
The curves $\Delta(\xi)$ are then perfectly symmetrical about the $\xi$-axis. They are plotted in Fig. 3.3, from which we now deduce Table II.

The bunching obtained in this limiting case is thus slightly better than in Table I. It is entirely determined by the value of the parameter $\sqrt{\alpha/2\pi b} \cdot 2\pi b^2 \xi$ which we shall later see plays exactly the same part that the bunching parameter $r$ of Smârs~⁴ plays for the velocity-modulation type of buncher.

---

FIG. 3.3 $\Delta$ versus $\xi$ for $\beta_{e0} = \beta_w$ and $\frac{2}{\beta_w} \cdot \sqrt{\frac{\alpha}{2\pi b}} \ll 1$. 
### TABLE I. Bunching by a waveguide section with $\beta_w = \frac{1}{2}$, $\alpha = 0.109$.

<table>
<thead>
<tr>
<th>Length $\xi$ of the section</th>
<th>Interval of $\Delta$ at input end of the section</th>
<th>Total Limits</th>
<th>Total Width</th>
<th>Fraction of injected electrons bunched within this interval of $\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.85</td>
<td>$-10^\circ$ + $0^\circ$</td>
<td>$-60^\circ$</td>
<td>0.43</td>
<td></td>
</tr>
<tr>
<td>0.90</td>
<td>$-18^\circ$ + $1^\circ$</td>
<td>$-59^\circ$</td>
<td>0.53</td>
<td></td>
</tr>
<tr>
<td>0.95</td>
<td>$-27^\circ$ + $2^\circ$</td>
<td>$-57^\circ$</td>
<td>0.61</td>
<td></td>
</tr>
<tr>
<td>1.00</td>
<td>$-36^\circ$ + $3^\circ$</td>
<td>$-55^\circ$</td>
<td>0.68</td>
<td></td>
</tr>
<tr>
<td>1.05</td>
<td>$-45^\circ$ + $4^\circ$</td>
<td>$-54^\circ$</td>
<td>0.73</td>
<td></td>
</tr>
<tr>
<td>1.10</td>
<td>$-54^\circ$ + $5^\circ$</td>
<td>$-53^\circ$</td>
<td>0.77</td>
<td></td>
</tr>
<tr>
<td>1.15</td>
<td>$-63^\circ$ + $6^\circ$</td>
<td>$-52^\circ$</td>
<td>0.81</td>
<td></td>
</tr>
<tr>
<td>1.20</td>
<td>$-72^\circ$ + $7^\circ$</td>
<td>$-51^\circ$</td>
<td>0.84</td>
<td></td>
</tr>
<tr>
<td>1.25</td>
<td>$-81^\circ$ + $8^\circ$</td>
<td>$-50^\circ$</td>
<td>0.86</td>
<td></td>
</tr>
<tr>
<td>1.30</td>
<td>$-90^\circ$ + $9^\circ$</td>
<td>$-49^\circ$</td>
<td>0.88</td>
<td></td>
</tr>
</tbody>
</table>

The table shows the fraction of injected electrons bunched within intervals of $\Delta$ for different lengths $\xi$ of the waveguide section, with $\beta_w = \frac{1}{2}$ and $\alpha = 0.109$. The intervals are defined at both input and output ends of the section, with total limits and widths given for each case.
TABLE II. Bunching by a waveguide section with
\[
\frac{2}{\beta_w} \cdot \frac{\sqrt{\alpha}}{2\pi b^2} \text{ very small.}
\]

<table>
<thead>
<tr>
<th>\sqrt{\frac{\alpha}{2\pi b^2}} \cdot 2\pi b^2 \tilde{b}</th>
<th>Interval of ( \Delta ) at output end of the section</th>
<th>Interval of ( \Delta ) at input end of the section</th>
<th>Fraction of injected electrons bunched within this interval of ( \Delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Limits</td>
<td>Total Width</td>
<td>Limits</td>
</tr>
<tr>
<td>1.70</td>
<td>-3°</td>
<td>+3°</td>
<td>6°</td>
</tr>
<tr>
<td>1.80</td>
<td>-7°</td>
<td>+7°</td>
<td>14°</td>
</tr>
<tr>
<td>1.90</td>
<td>-12°</td>
<td>+12°</td>
<td>24°</td>
</tr>
<tr>
<td>2.00</td>
<td>-18°</td>
<td>+18°</td>
<td>36°</td>
</tr>
<tr>
<td>2.10</td>
<td>-24°</td>
<td>+24°</td>
<td>48°</td>
</tr>
<tr>
<td>2.20</td>
<td>-30°</td>
<td>+30°</td>
<td>61°</td>
</tr>
<tr>
<td>2.30</td>
<td>-37°</td>
<td>+37°</td>
<td>74°</td>
</tr>
<tr>
<td>2.40</td>
<td>-43°</td>
<td>+43°</td>
<td>86°</td>
</tr>
<tr>
<td>2.50</td>
<td>-49°</td>
<td>+49°</td>
<td>98°</td>
</tr>
<tr>
<td>2.60</td>
<td>-55°</td>
<td>+55°</td>
<td>110°</td>
</tr>
</tbody>
</table>
The electron energy $\gamma$ along the waveguide

Using Eq. (1.8), we get at once from Eq. (3.5):

$$\gamma \cdot \sqrt{1 - \beta_w^2} = \frac{y}{b} - \text{sgn} (\Delta_m) \cdot 2\beta_w \sqrt{\frac{y - b}{2b} \cdot \frac{y + b}{2b}}.$$  

By Eq. (3.9), this relation may be expressed as

$$\gamma = \frac{1}{\sqrt{1 - \beta_w^2}} \left[ 1 + 2 \frac{\alpha}{2\pi b} \sin^2 \frac{\Delta_m}{2} \cdot \cos^2 \phi \right.$$  

$$- 2\beta_w \frac{\alpha}{2\pi b} \sin \frac{\Delta_m}{2} \cdot \cos \phi \cdot \sqrt{1 + \frac{\alpha}{2\pi b} \sin^2 \frac{\Delta_m}{2} \cdot \cos^2 \phi} \right].$$  

(3.23)

If we think of the right-hand side of Eq. (3.23) as expanded in a power series of $\sqrt{\alpha/2\pi b}$, by the condition (3.15) we only need to retain the first-power term in order to get a fairly close approximation of $\gamma$:

$$\gamma \approx \frac{1}{\sqrt{1 - \beta_w^2}} \left[ 1 - 2\beta_w \sqrt{\frac{\alpha}{2\pi b} \sin \frac{\Delta_m}{2} \cdot \cos \phi} \right].$$  

(3.24)

Equation (3.24) shows, in agreement with Fig. 1.1, that the maximum deviation of the electron velocity from the phase velocity of the traveling wave occurs at $\phi = 0$ (i.e., $\Delta = 0$). But whatever the value of $\Delta_m$, the relative variation of the electron energy along the waveguide is always less than $2\beta_w \sqrt{\alpha/2\pi b}$.

The relative energy spread of the electrons at the output end of a waveguide section with a length roughly equal to $\xi_0$ for $\Delta_m = 0$ is thus $2\beta_w \sqrt{\alpha/2\pi b}$, and from Eq. (3.12) appears to be inversely proportional to the length of the buncher when the latter is varied by varying the electric-field strength $\alpha$. This conclusion is
particularly obvious when \( \frac{2}{\beta_w} \sqrt{\alpha/2\pi b} \) is very small, for the bunching achieved by the waveguide section is then entirely determined by our bunching parameter \( \sqrt{\alpha/2\pi b} \cdot 2\pi b \beta_z \), which with Eq. (1.8) may be written as

\[
\sqrt{\frac{\alpha}{2\pi b}} \cdot 2\pi b \beta_z \beta_z = \left( \frac{2\beta_w}{\beta_w} \right) \sqrt{\frac{\alpha}{2\pi b}} \cdot \frac{b^2}{\beta_w} \pi \psi
\]

\[
= \left( \frac{\delta \gamma}{\gamma} \right)_{\text{max}} \cdot \frac{b^2}{\beta_w} \pi \psi = \left( \frac{\delta \gamma}{\gamma} \right)_{\text{max}} \cdot b^3 \pi \psi . \quad (3.25)
\]

We must keep in mind that the relative energy spread discussed here is the energy spread relative to the total electron energy (rest mass energy + kinetic energy).

From now on we shall always assume that \( \frac{2}{\beta_w} \sqrt{\alpha/2\pi b} \) is small compared with unity. In this case, from Eq. (3.24), rewritten as

\[
\gamma \frac{1}{\sqrt{1 - \beta_w^2}} = \frac{\beta_w}{\sqrt{1 - \beta_w^2}} \cdot 2 \sqrt{\frac{\alpha}{2\pi b}} \sin \frac{\Delta m}{2} \cdot \cos \phi , \quad (3.26)
\]

it is very easy to obtain, by the relation \( \delta p = \delta \gamma / \beta_e \), the electron momentum spread:

\[
\frac{1}{p} = - \frac{1}{\beta_w} \cdot 2 \sqrt{\frac{\alpha}{2\pi b}} \sin \frac{\Delta m}{2} \cdot \cos \phi , \quad (3.27)
\]

and using the relations \( \beta_e^2 = 1 - 1/y^2 \), \( \delta \beta_e / \beta_e^2 = \delta \gamma / \beta_e^3 y^3 = \delta \gamma / p^3 \), we at once deduce the electron velocity spread:

\[
\frac{1}{\beta_w} - \frac{1}{\beta_e} = \frac{1 - \beta_w^2}{\beta_w} \cdot 2 \sqrt{\frac{\alpha}{2\pi b}} \sin \frac{\Delta m}{2} \cdot \cos \phi . \quad (3.28)
\]
Except for constant factors, the right-hand sides of Eqs. (3.26), (3.27) and (3.28) are identical with \( \sin(\Delta_m/2) \cdot \cos \phi \). This function has been plotted in Fig. 3.4 for various electron orbits (as it was in Fig. 1.1), and from the data of Fig. 3.3 it was then possible to draw curves of constant \( \xi \) in phase space. These constant-\( \xi \) curves show beautifully the bunching action of the waveguide: they start with the horizontal axis, and their rotation about the origin in a clockwise direction changes the initial phase modulation of the electrons into a velocity modulation.

Remark

In order to show the worth of the approximation (3.27), we shall now deduce the rigorous expression for \( p \) from Eqs. (3.4) and (3.5):

\[
p = \frac{\beta_w^2}{1 - \beta_w^2} \left[ y - \frac{\text{sgn}(\Delta_m)}{\beta_w^2} \sqrt{\frac{\beta_w^2 \cdot y^2 - (1 - \beta_w^2)}{2}} \right].
\]

With Eq. (1.8) we get

\[
p = \frac{1}{b} \left[ y - \frac{\text{sgn}(\Delta_m)}{\beta_w} \cdot \frac{2}{y - b} \sqrt{\frac{y - b}{y + b}} \right]
\]

or, by Eq. (3.9):

\[
p = \frac{1}{b} \left[ 1 + 2 \frac{\alpha}{2\pi b} \sin^2 \frac{\Delta_m}{2} \cdot \cos^2 \phi \right.
\]

\[
\left. - \frac{2}{\beta_w^2} \sqrt{\frac{\alpha}{2\pi b} \sin \frac{\Delta_m}{2} \cdot \cos \phi \cdot \sqrt{1 + \frac{\alpha}{2\pi b} \sin^2 \frac{\Delta_m}{2} \cdot \cos^2 \phi}} \right].
\]

Comparing Eq. (3.27) with this rigorous expression of \( p \), we see that it gives \( p \) with an excellent approximation when \( (2/\beta_w^2) \sqrt{\alpha/2\pi b} \ll 1 \); moreover, this latter assumption appears to be equivalent to \( (\delta p)_{\max}/p \ll 1 \).
FIG. 3.4--The constant-$\xi$ curves in phase space, when $\beta_{e0} = \beta_w$ and $\frac{2}{\beta_w} \sqrt{\frac{\alpha}{2\pi b}} \ll 1$. 
The effect on the electron bunching of a short drift space following the waveguide section

In order to provide a transition between the waveguide buncher, where \( \beta_w = \beta_{e0} \) and where the electric field has to be weak, and the regular sections of the accelerator where \( \beta_w = 1 \) and where the electric field is strong, it may be necessary to have a short drift space. The question then arises what would be the effect of this drift space on the electron bunches as they go out of the waveguide buncher.

In the drift space the time of arrival of an electron is a linear function of \( \xi \), such that (see Fig. 3.5)

\[
\frac{d(\omega t)}{d\xi} = \omega \lambda, \quad \frac{dt}{dz} = \frac{2\pi}{\beta_e}
\]

whence

\[
\frac{d\Delta}{d\xi} = -\frac{2\pi}{\beta_e} = 2\pi \left( \frac{1}{\beta_w} - \frac{1}{\beta_e} \right) \frac{2\pi}{\beta_w}.
\]  

Comparing this relation with Eq. (1.1), we see that the slope of the curves \( \Delta(\xi) \) has a discontinuity \(-2\pi/\beta_w\) at the output end of the buncher; but this discontinuity is the same for all electrons and therefore does not affect their bunching: it merely results in an additional phase shift \(-2\pi/\beta_w\xi\) for all electrons at a given distance \( \xi \) in the drift space, following the relation

\[
\Delta = \Delta_0 + 2\pi \left( \frac{1}{\beta_w} - \frac{1}{\beta_e} \right) \xi - \frac{2\pi}{\beta_w} \xi.
\]

Disregarding this phase shift, we may thus consider that the curves \( \Delta(\xi) \) continue as straight lines in the drift space, with the slope they have as they enter it. From Eq. (3.28) we see that we have at once a plot of the distribution of these slopes with the constant-\( \xi \) curves in Fig. 3.4; moreover, since roughly speaking electrons are
Fig. 3.5 Drift space following the waveguide section.
being bunched when the slope $\frac{d\Delta}{d\xi}$ at a given $\xi$ is a decreasing function of $\Delta$, and are being debunched when $\frac{d\Delta}{d\xi}$ is an increasing function of $\Delta$. Fig. 3.4 shows that at the output end of the waveguide buncher electrons with small phase angles $\Delta$ experience a de-bunching effect while electrons at large phase angles are still being bunched. However, these bunching or debunching effects are the same whether the waveguide section ends or not, and they thus remain the same if the drift space is not too long.

More precisely, since the curves $\Delta(\xi)$ in Fig. 3.3 are nearly straight lines for values of the bunching parameter between 1.8 and 2.6 (this is because these curves have a point of inflection at $\Delta = 0$), they may be replaced by their tangents in at least a 0.2-wide interval of the bunching parameter; in other words, in a waveguide buncher the length of which shall correspond to a bunching parameter near 2, one may change the output end into a drift space even as long as 0.1 times the total length of the buncher without practically affecting the bunching characteristics of the system.

**Limitation of the field strength $\alpha$ in the waveguide section**

So far we have seen that in order to get a good bunching of the electrons by a waveguide section with $\beta_w < 1$, the electric-field strength therein must be limited by the condition (3.15); but another limitation arises from the fact that the electron beam is no longer monoenergetic when it leaves the waveguide buncher, and that its velocity modulation must be kept sufficiently small. In a precise manner, the representative points of the electrons in phase space when they enter the accelerator must fall within the region comprised between the two curves of Fig. 2.3 if we want these electrons to get a fraction $|\sin \Delta^0_\infty|$ or more of the maximum energy at the end of the accelerator.

This means that, in any case, the velocity modulation of the electron beam after bunching is limited by the condition (2.10):

$$x < \cos \Delta^0_\infty$$

(3.31)
where \( x \) is the quantity given by Eq. (2.11) as

\[
x = \frac{p_{1} - p}{\gamma_1 + \gamma} = \frac{\gamma_1 - \gamma}{p_1 + p}.
\] (3.32)

Taking now \( p_1 \) and \( p \) as the extreme values of the electron momentum at the output end of the buncher, we see from Fig. 3.4 and from Eq. (3.27) that

\[
\frac{\gamma(p_{1} - p)}{\gamma_1 + \gamma} = 2 \sqrt{\frac{\alpha}{2\pi b}}.
\]

So we get the very simple result

\[
x = 2 \sqrt{\frac{\alpha}{2\pi b}}.
\] (3.33)

by which the condition (3.31) reduces to

\[
\sqrt{\frac{\alpha}{2\pi b}} < \frac{\cos \Delta_0}{2}.
\] (3.34)

As the relative energy spread at the output end of an electron accelerator is always preferred to be small, \( \Delta_0 \) shall be chosen close to \(-90^0\): Eq. (3.34) appears thus as a much more stringent limitation of the electric-field strength in the waveguide buncher than Eq. (3.15).

Figure 3.6 shows the complete phase space as the electrons enter the accelerator; the curve which represents the electrons has been reproduced from Fig. 3.4 for a value 2.2 of the bunching parameter, and the two curves representing (as in Fig. 2.3) the electron motion in the accelerator for the two asymptotic phase angles \( \Delta_0 \) and \( \Delta_1 = -\pi + \Delta_0 \) settle the boundary of the region within which electrons are accelerated to a fraction \( |\sin \Delta_0| \) or more of the maximum energy.
FIG. 3.6--Phase space for both the buncher and the accelerator.
In order to get as many electrons as possible between these two curves, we must take \( p_1 \) as the minimum point of the upper curve; and the condition (3.34) merely means that \( p \) must exceed the minimum point of the lower curve.

Once \( \alpha \) has been chosen in accordance with Eq. (3.34), \( p \) is determined, and the optimum value of the bunching parameter then corresponds to the constant-\( \xi \) curve in phase space which touches the \( \Delta^1 \) curve just after its minimum at \(-\Delta'\). Now, on Fig. 3.4 we can see that the point of contact \( A \) between the two curves will always have an ordinate close to \(-0.56\); this means that the corresponding ordinate \( p' \) in Fig. 3.6 will be such that

\[
\frac{p_1 - p'}{p_1 - p} = \frac{1.56}{2} = 0.78,
\]

and by Eq. (3.32) the corresponding value of \( x \) will be

\[
x' = \frac{p_1 - p'}{\gamma_1 + \gamma} \approx x \cdot \frac{p_1 - p'}{p_1 - p} = 0.78 \cdot x. \tag{3.35}
\]

From Eq. (2.9) it is then easy to get the abscissa of \( A \); and since this is very nearly equal to \(-\Delta'\), we may write to a good approximation

\[
\cos \Delta' \approx \cos \frac{\Delta + x'}{1 - x}. \tag{3.36}
\]

As \((-\Delta', +\Delta')\) is the phase interval within which each value of \( \Delta \) on the constant-\( \xi \) curve is attained from three different entrance angles \( \Delta_m \), it is nothing but the output phase interval that we have taken to characterize the bunching properties of the waveguide section in Table II: interpolation in this table thus gives at once the value to be chosen for the bunching parameter and the fraction of electrons bunched within this phase interval \((-\Delta', +\Delta')\).
Strictly speaking, the electrons bunched within this phase interval 
\((-\Delta', +\Delta')\) are not quite exactly those which fall between the two curves 
corresponding to the asymptotic phase angles \(\Delta_\infty^0\) and \(\Delta_\infty^1\) in the acceler-
ator; but, as can be seen from Fig. 3.6, the difference is very small and 
may be neglected. In fact, the constant-\(e\) curve has even more electrons 
in the region bounded by the two curves \(\Delta_\infty^0\) and \(\Delta_\infty^1\) than it has between 
the two vertical lines \(-\Delta'\) and \(+\Delta'\). In other words, the fraction of 
electrons bunched within the phase interval \((-\Delta', +\Delta')\) may be regarded 
as (and is even a bit less than) the fraction of injected electrons which 
will be accelerated to an energy at least equal to \(|\sin \Delta_\infty^0|\) times the 
maximum energy at the end of the accelerator.

Let us consider the example of Section II again:

\[
|\sin \Delta_\infty^0| = 0.99 \quad \cos \Delta_\infty^0 = 0.14107 .
\]

From Eq. (2.6) we then have

\[
\cos \Delta_\perp = 1 - 2 \cos \Delta_\infty^0 = 0.71786 \quad \Delta_\perp = 44.12^\circ ,
\]

while Eq. (2.9) gives

\[
x = \frac{\cos \Delta - \cos \Delta_\perp}{1 + \cos \Delta} .
\]

\(x'\) and \(\Delta'\) are then readily computed from Eqs. (3.35) and (3.36).

Finally, by interpolating between the figures of Table II we get 
Table III; the last column therein we simply obtain when we divide 
the bunching parameter \(\sqrt{\alpha/2\pi b} \cdot 2\pi b^2 \xi\) by \(x\).
TABLE III. Fraction of injected electrons which will get 99 percent or more of the maximum energy in the accelerator, and length of the waveguide buncher, as functions of the electric-field strength $\alpha$ in the buncher, when there is no phase shift between buncher and accelerator.

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>$x = 2 \frac{\alpha}{2\pi b}$</th>
<th>$\Delta'$</th>
<th>$\sqrt[4]{\frac{\alpha}{2\pi b}} \cdot 2\pi b^2 \xi$</th>
<th>Fraction of electrons bunched within the phase interval (-$\Delta'$, +$\Delta'$)</th>
<th>$\pi b^2 \xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0^\circ$</td>
<td>0.14107</td>
<td>21.5°</td>
<td>2.06</td>
<td>0.71</td>
<td>14.6</td>
</tr>
<tr>
<td>23°</td>
<td>0.10551</td>
<td>29.3°</td>
<td>2.18</td>
<td>0.76</td>
<td>20.7</td>
</tr>
<tr>
<td>30°</td>
<td>0.07940</td>
<td>33.8°</td>
<td>2.25</td>
<td>0.79</td>
<td>28.3</td>
</tr>
<tr>
<td>35°</td>
<td>0.05568</td>
<td>37.3°</td>
<td>2.31</td>
<td>0.81</td>
<td>41.5</td>
</tr>
<tr>
<td>40°</td>
<td>0.02728</td>
<td>41.0°</td>
<td>2.36</td>
<td>0.83</td>
<td>86.5</td>
</tr>
<tr>
<td>44.12°</td>
<td>0.00000</td>
<td>44.12°</td>
<td>2.42</td>
<td>0.84</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>
For a given value of the field strength in the buncher (i.e., for a
given $x$ or a given momentum width $p_1 - p$) it is possible to have a
still larger fraction of the electrons between the two curves $\Delta^0_\infty$ and $\Delta^1_\infty$
if an appropriate phase shift is introduced between the buncher and the
accelerator, so that the optimum constant-$\xi$ curve would then have two
points of contact with the $\Delta^1_\infty$ curve: one (B) just after its maximum
at $+\Delta''$, and one (A) just after its minimum at $-\Delta'$. For example, we see
in Fig. 3.6 (where $\Delta = 23^0$) that the constant-$\xi$ curve could be shifted
by about $5^0$ to the right and then changed into a constant-$\xi$ curve with a
half-width of $35^0$ instead of $29.3^0$; the optimum value of the bunching
parameter would then increase slightly, and the fraction of electrons which
will get 99 percent or more of the maximum energy would rise from 0.75 to
0.79.

In order to compute these optimum conditions more precisely, we first
observe in Fig. 3.4 that the point of contact B between the constant-$\xi$
curves and the $\Delta^1_\infty$ curve in phase space will always have an ordinate close
to $+0.50$; in other words, the corresponding ordinate $p''$ in Fig. 3.6
will be such that
\[
\frac{p_1 - p''}{p_1 - p} = \frac{0.50}{2} = 0.25
\]
and by Eq. (3.32) the corresponding value of $x$ will be
\[
x'' = \frac{p_1 - p''}{\gamma_1 + \gamma} \approx x \cdot \frac{p_1 - p''}{p_1 - p} = 0.25x.
\]  

(3.37)

From Eq. (2.9) we then get the abscissa of B; and since this is very
nearly equal to $+\Delta''$, we have to a good approximation
\[
\cos \Delta'' \approx \frac{\cos \Delta_1 + x''}{1 - x''}.
\]  

(3.38)

The phase interval ($-\Delta'$, $+\Delta''$) available for the optimum constant-$\xi$ curve
in phase space has a total width of $\Delta' + \Delta''$, instead of $2\Delta'$; from this
new value, interpolation in Table II will give the best value to be chosen
TABLE IV. Fraction of injected electrons which will get 99 percent or more of the maximum energy in the accelerator, and length of the waveguide buncher, as functions of the electric-field strength $\alpha$ in the buncher, when there is an optimum phase shift between buncher and accelerator.

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>$x=2\sqrt{\frac{\alpha}{2\pi b}}$</th>
<th>$\frac{\alpha}{2\pi b} \cdot 10^4$</th>
<th>$\Delta'$</th>
<th>$\Delta^*$</th>
<th>$\Delta' + \Delta^*$</th>
<th>$\sqrt{\frac{\alpha}{2\pi b}} \cdot 2\pi b^2 \xi$</th>
<th>Fraction of electrons bunched within the phase interval $(-\Delta', +\Delta')$</th>
<th>$\pi b^2 \xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0.14107</td>
<td>49.76</td>
<td>21.5°</td>
<td>38.7°</td>
<td>60.2°</td>
<td>2.20</td>
<td>0.77</td>
<td>15.6</td>
</tr>
<tr>
<td>23°</td>
<td>0.10551</td>
<td>27.84</td>
<td>29.3°</td>
<td>40.2°</td>
<td>69.5°</td>
<td>2.27</td>
<td>0.80</td>
<td>21.5</td>
</tr>
<tr>
<td>30°</td>
<td>0.07940</td>
<td>15.76</td>
<td>33.8°</td>
<td>41.2°</td>
<td>75.0°</td>
<td>2.31</td>
<td>0.81</td>
<td>29.1</td>
</tr>
<tr>
<td>35°</td>
<td>0.05568</td>
<td>7.75</td>
<td>37.3°</td>
<td>42.1°</td>
<td>79.4°</td>
<td>2.35</td>
<td>0.82</td>
<td>42.2</td>
</tr>
<tr>
<td>40°</td>
<td>0.02728</td>
<td>1.86</td>
<td>41.0°</td>
<td>43.1°</td>
<td>84.1°</td>
<td>2.39</td>
<td>0.83</td>
<td>87.4</td>
</tr>
<tr>
<td>44.12°</td>
<td>0.00000</td>
<td>0.00</td>
<td>44.12°</td>
<td>44.12°</td>
<td>88.24°</td>
<td>2.42</td>
<td>0.84</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>
for the bunching parameter and the fraction of electrons bunched within this phase interval.

As we already pointed out for the interval \((-\Delta', +\Delta')\), the electrons bunched within the phase interval \((-\Delta', +\Delta')\) are not quite exactly those which fall between the two curves \(\Delta^0_{\infty}\) and \(\Delta^1_{\infty}\) of Fig. 3.6; but the difference is very small and may still be neglected. In fact, the optimum constant-\(\xi\) curve has a bit less electrons in the region bounded by the two curves \(\Delta^0_{\infty}\) and \(\Delta^1_{\infty}\) than it has between the two vertical lines \(-\Delta'\) and \(+\Delta'\), so that the fraction of electrons bunched within the phase interval \((-\Delta', +\Delta')\) is slightly higher than (but very close to) the fraction of injected electrons which will be accelerated to an energy at least equal to \(\sin \Delta^0_{\infty}\) times the maximum energy at the end of the accelerator.

Turning back to our example of Section II, we have \(\beta''\) and \(\Delta''\) computed from Eqs. (3.37) and (3.38); by interpolating in Table II we then get Table IV, which supersedes Table III.

In particular, when \(\beta_w = \frac{1}{2}\) in the waveguide buncher we have \(b^2 = 3\) by Eq. (1.8), and Table IV gives the following figures.

<table>
<thead>
<tr>
<th>(x)</th>
<th>(a\cdot10^3)</th>
<th>Fraction of electrons which will get 99 percent or more of the maximum energy</th>
<th>Length (\xi) of the buncher</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.14107</td>
<td>5.4.15</td>
<td>0.77</td>
<td>1.66</td>
</tr>
<tr>
<td>0.10551</td>
<td>30.30</td>
<td>0.80</td>
<td>2.28</td>
</tr>
<tr>
<td>0.07940</td>
<td>17.15</td>
<td>0.81</td>
<td>3.09</td>
</tr>
<tr>
<td>0.05568</td>
<td>8.43</td>
<td>0.82</td>
<td>4.48</td>
</tr>
<tr>
<td>0.02728</td>
<td>2.02</td>
<td>0.83</td>
<td>9.28</td>
</tr>
<tr>
<td>0.00000</td>
<td>0.00</td>
<td>0.84</td>
<td>(\infty)</td>
</tr>
</tbody>
</table>

The first row corresponds to the highest value of \(a\) which is permissible from Eq. (3.34). When \(a\) is decreased and the buncher lengthened, more electrons are in the condition to be accelerated.

---

*Phases are now referred to the accelerator and no longer to the buncher.*
to 99 percent at least of the maximum energy at the end of the accelerator; but the difference is not appreciable and does not call for such a lengthening of the buncher, since in any case the needed electric field remains weak and is very easy to produce.

Comparison with the velocity-modulation type of buncher

In this type of buncher, the original electron beam moving with a velocity $\beta_{e0}$ passes through the gap of an rf excited cavity where it gets an energy modulation proportional to the sinusoidal voltage across the gap, and then goes into bunches in a drift space which follows the resonant cavity. The energy modulation of the beam is thus a sinusoidal function of time:

$$\delta \gamma = (\delta \gamma)_{max} \cdot \sin \omega t.$$ 

Now, if we think in terms of phase angles rather than in terms of arrival times for the electrons, using the notations of Fig. 3.5 we have

$$\delta \gamma = (\delta \gamma)_{max} \cdot \sin (-\Delta_0)$$

and we may write as in Eq. (3.29):

$$\frac{d\Delta}{d\xi} = \frac{2\pi}{\beta_e} = 2\pi \left( \frac{1}{\beta_{e0}} - \frac{1}{\beta_e} \right) = \frac{2\pi}{\beta_{e0}}.$$ 

$\beta_{e0}$ here plays exactly the same part as $\beta_w$ in Eq. (3.29), so we have as in Eq. (3.30):

$$\Delta = \Delta_0 + 2\pi \left( \frac{1}{\beta_{e0}} - \frac{1}{\beta_e} \right) \xi - \frac{2\pi}{\beta_{e0}} \xi.$$ 

The last term in this relation merely shifts the phase of all electrons by the same amount $-\left(2\pi/\beta_{e0}\right)\xi$ at a given distance $\xi$ in the drift space; it therefore has no effect on the bunching of the electrons and
may be disregarded for the sake of simplicity. We are thus left with only the equation

$$\Delta = \Delta_0 + 2\pi \left( \frac{1}{\beta_e} - \frac{1}{\beta_0} \right) \xi \ . \quad (3.40)$$

If the peak voltage in the gap is sufficiently weak so the relative variation of electron momentum remains small, we may use the differential relation \( \delta p = \delta \gamma / \beta_e \) to get the electron momentum modulation from Eq. (3.39):

$$\delta p = \frac{(\delta \gamma)_{\text{max}}}{\beta_0} \cdot \sin (-\Delta_0) \ \quad (3.41)$$

and using the relation \( \delta \beta_e / \beta_e^2 = \delta \gamma / p^3 \) we at once deduce the electron velocity modulation:

$$\frac{1}{\beta_0} - \frac{1}{\beta_e} = \frac{(\delta \gamma)_{\text{max}}}{p_0^3} \cdot \sin (-\Delta_0) \ ,$$

where \( p_0 \) represents the momentum of the unmodulated electrons in units of \( m_0 c \). For a waveguide buncher with \( \beta_w = \beta_0 \) this momentum would be identical to \( 1/b \), as can be seen from Eq. (1.8); for the sake of uniformity we shall keep the same notation in both cases and use \( 1/b \) to represent the momentum of the unmodulated electrons here also.

Whence we shall write

$$\frac{1}{\beta_0} - \frac{1}{\beta_e} = (\delta \gamma)_{\text{max}} \cdot b^3 \cdot \sin (-\Delta_0)$$

and Eq. (3.40) now becomes

$$\Delta = \Delta_0 - (\delta \gamma)_{\text{max}} \cdot b^3 \cdot 2\pi \xi \cdot \sin \Delta_0 \ .$$
The functions \( \Delta(\xi) \) are thus entirely determined by the value of the parameter \((\delta\gamma)_{\text{max}} \cdot b^3 \cdot 2\pi\xi\), which Simmons [Ref. 4, p. 4] in his Eq. (2) called the bunching parameter \( r \):

\[
\Delta = \Delta_0 - r \cdot \sin \Delta_0
\]  

(3.42)

and

\[
r = (\delta\gamma)_{\text{max}} \cdot b^3 \cdot 2\pi\xi.
\]  

(3.43)

With Eq. (3.25) we now see at once the relationship between the bunching parameter we introduced for the waveguide buncher, and the bunching parameter \( r \):

\[
\sqrt{\frac{x}{2\pi b^2 \xi}} \cdot 2\pi b^2 \xi = \frac{r}{2}.
\]  

(3.44)

So, for the same injection velocity of the electrons at the input, for the same energy spread at the output, and for the same length of the buncher, the bunching parameter of the waveguide buncher is exactly equal to half the bunching parameter of the velocity-modulation buncher.

The curves \( \Delta(\xi) \), or more precisely \( \Delta(r) \), are easy to plot, since by Eq. (3.42) they are straight lines which start from \( \Delta_0 \) on the \( \Delta \)-axis and cross the \( r \)-axis at \( r_0 = \Delta_0 / \sin \Delta_0 \); they are drawn in Fig. 3.7 for the same values of entrance phase angles as in Fig. 3.3.

As in Fig. 3.3 also, the system of curves \( \Delta(\xi) \) corresponding to the different values of \( \Delta_0 \) has an envelope which determines the boundary of the region where the electrons are best bunched. The point where a line \( \Delta_0 \) touches this envelope is obtained by differentiating Eq. (3.42) with respect to \( \Delta_0 \); its abscissa is thus given by

\[
r = \frac{1}{\cos \Delta_0}.
\]  

(3.45)
FIG. 3.7 $\Delta$ versus $\xi$ in drift space, after a small velocity modulation $\left[ \frac{(\delta p)_{\text{max}}}{p} \ll 1 \right]$. 
This shows at once that the two straight lines $\Delta_0 = \pm 90^\circ$ are asymptotes of the envelope. The latter is made up of two parts which both start with a horizontal tangent from the point $r = 1$ on the $r$-axis and are symmetrical about this axis.

Considering the electrons inside the envelope in Fig. 3.7, we now deduce Table V in the same manner we deduced Table II from Fig. 3.3. The figures so obtained agree closely with those deduced from graph 3 of Smår [Ref. 4, p. 15].

Comparison of Tables II and V shows precisely what already appears from a comparison of Figs. 3.3 and 3.7: the bunching achieved by a uniform waveguide section with $\beta_w = \beta_{e0}$ is better than that obtained by velocity modulation.

In both cases the useful values of the bunching parameter are about the same; by Eqs. (3.44) and (3.43) this means that for a given energy spread of the outgoing electrons, the waveguide buncher shall be about twice as long as the drift space, or that for the same length as the drift space, it will deliver electrons with twice the energy spread of the latter.

Nevertheless, the only meaningful comparison between the two types of bunchers must be made in phase space, in order to take into account also the bunching achieved by the accelerator itself.

For this reason Fig. 3.8 shows the constant-$\xi$ (or constant-$r$) curves as they appear in phase space after a small velocity modulation. These curves are easy to plot, since Eq. (3.41) already gives the $p(\Delta_0)$ curve for $r = 0$; and from this curve, by using Eq. (3.42), we at once get $p(\Delta)$ curves for other values of $r$.

If we want the electrons to get a fraction $|\sin \Delta_0|$ or more of the maximum energy at the end of the accelerator, their velocity modulation when they enter the accelerator must be limited according to the condition (2.10) or (3.31), which may be readily expressed in terms of momentum or energy modulation when we rewrite Eq. (3.32) as

$$x = \frac{(\delta p)_{\text{max}}}{\gamma_0} = \beta_{e0} \frac{(\delta p)_{\text{max}}}{p_0}$$
TABLE V. Bunching by a small velocity modulation.

<table>
<thead>
<tr>
<th>r</th>
<th>Interval of Δ at the output end of the drift space</th>
<th>Interval of Δ at the input end of the drift space</th>
<th>Fraction of injected electrons bunched within this interval of Δ</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Limits</td>
<td>Total Width</td>
<td>Limits</td>
</tr>
<tr>
<td>1.20</td>
<td>-4.5° + 4.5°</td>
<td>9°</td>
<td>-68.5° +68.5°</td>
</tr>
<tr>
<td>1.40</td>
<td>-11.5° +11.5°</td>
<td>23°</td>
<td>-92° +92°</td>
</tr>
<tr>
<td>1.60</td>
<td>-20° +20°</td>
<td>40°</td>
<td>-107.5° +107.5°</td>
</tr>
<tr>
<td>1.80</td>
<td>-29.5° +29.5°</td>
<td>59°</td>
<td>-119.5° +119.5°</td>
</tr>
<tr>
<td>2.00</td>
<td>-39° +39°</td>
<td>78°</td>
<td>-128.5° +128.5°</td>
</tr>
<tr>
<td>2.20</td>
<td>-49.5° +49.5°</td>
<td>99°</td>
<td>-136.5° +136.5°</td>
</tr>
<tr>
<td>2.40</td>
<td>-59.5° +59.5°</td>
<td>119°</td>
<td>-143° +143°</td>
</tr>
<tr>
<td>2.60</td>
<td>-70° +70°</td>
<td>140°</td>
<td>-148.5° +148.5°</td>
</tr>
</tbody>
</table>
FIG. 3.8--The constant-$\xi$ curves in phase space, after a small velocity modulation $\left[\frac{(6p)_{\text{max}}}{p} \ll 1\right]$. 
and

\[ x = \frac{(\delta \gamma)_{\text{max}}}{p_0} = \frac{1}{\beta e_0} \cdot \frac{(\delta \gamma)_{\text{max}}}{\gamma_0} \]  

(3.46)

or

\[ \frac{(\delta p)_{\text{max}}}{p_0} = \frac{x}{\beta e_0} \quad \text{and} \quad \frac{(\delta \gamma)_{\text{max}}}{\gamma_0} = x \cdot \beta e_0 . \]  

(3.47)

Whence the condition (3.31):

\[ x < \cos \Delta_0^0 \]

automatically insures that our fundamental assumption \((\delta p)_{\text{max}}/p_0 << 1\) is verified.

Once the total momentum modulation \(p_1 - p\) or, what is equivalent, the value of \(x\) has been chosen in accordance with this condition, the optimum value of \(r\) is determined and corresponds to that constant-\(\xi\) curve in phase space which, provided an appropriate phase shift is introduced between the velocity-modulating cavity and the accelerator, touches the \(\Delta_{\infty}^1\) curve at two points of contact (see Fig. 3.6): one (A) just after its minimum at \(-\Delta'\), and one (B) just after its maximum at \(+\Delta''\).

Now, on Fig. 3.8 we can see that the point of contact A between the two curves will in all practical cases have an ordinate about \(-0.86\), whereas the point of contact B will have an ordinate about \(+0.74\); this means that the corresponding ordinates \(p'\) and \(p''\) in Fig. 3.6 will roughly be such that

\[ \frac{p_1 - p'}{p_1 - p} = \frac{1.86}{2} = 0.93 \quad \text{and} \quad \frac{p_1 - p''}{p_1 - p} = \frac{0.26}{2} = 0.13 . \]

Equations (3.35) and (3.37) are thus replaced by

\[ x' = 0.93 \, x \quad \text{and} \quad x'' = 0.13 \, x . \]
Putting these values in Eqs. (3.36) and (3.38) we get

$$\cos \Delta \approx \frac{\cos \Delta_1 + 0.93 x}{1 - 0.93 x} \quad \text{and} \quad \cos \Delta'' \approx \frac{\cos \Delta_1 + 0.13 x}{1 - 0.13 x}.$$

From this we find the total width $\Delta' + \Delta''$ of phase interval which is available for the optimum constant-\(\xi\) curve in phase space between its maximum and its minimum in $\Delta$. Since this interval is precisely the output phase interval which appears in Table V, interpolation through this table will give at once the optimum value of the bunching parameter $r$ and the fraction of electrons bunched within the phase interval $(-\Delta', +\Delta''$); as can be seen from Fig. 3.6, this fraction represents fairly well the fraction of injected electrons which after bunching have arrived between the two curves $\Delta_0$, $\Delta_1$ and which will thus be accelerated to an energy at least equal to $|\sin \Delta_0|$ times the maximum energy at the end of the accelerator.

Finally, we readily get the length $\xi$ of the buncher when we observe that with the relation

$$x = \frac{(6\gamma)_{\text{max}}}{P_0} = (6\gamma)_{\text{max}} \cdot b,$$

deduced from Eq. (3.46), Eq. (3.43) may be rewritten as

$$\frac{r}{x} = 2\pi b^2 \xi.$$

Taking the example of Section II again, $x$ has the same values as in Tables III and IV, since these are determined by Eq. (2.9); from the new values of $\Delta'$ and $\Delta''$ we deduce Table VI by interpolating in Table V.

From a practical point of view it is interesting to have the maximum energy variation $(6\gamma)_{\text{max}}$ expressed as a fraction of the

*Note that phase angles are here referred to the accelerator and not to the buncher.*
TABLE VI. Fraction of injected electrons which will get 99 percent or more of the maximum energy in the accelerator, and length of the drift space, as functions of the peak voltage in the cavity gap, when there is an optimum phase shift between buncher and accelerator.

<table>
<thead>
<tr>
<th>$\Delta$</th>
<th>$x = (\delta \gamma)_{\text{max}} \cdot b$</th>
<th>$\Delta'$</th>
<th>$\Delta''$</th>
<th>$\Delta' + \Delta''$</th>
<th>$r = x \cdot 2\pi b^2 \xi$</th>
<th>Fraction of electrons bunched within the phase interval (-$\Delta'$, +$\Delta''$)</th>
<th>$2\pi b^2 \xi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0°</td>
<td>0.14107</td>
<td>12.2°</td>
<td>41.4°</td>
<td>53.6°</td>
<td>1.74</td>
<td>0.65</td>
<td>12.4</td>
</tr>
<tr>
<td>23°</td>
<td>0.10551</td>
<td>25.2°</td>
<td>42.1°</td>
<td>67.3°</td>
<td>1.89</td>
<td>0.69</td>
<td>17.9</td>
</tr>
<tr>
<td>30°</td>
<td>0.07940</td>
<td>31.3°</td>
<td>42.6°</td>
<td>73.9°</td>
<td>1.95</td>
<td>0.70</td>
<td>24.6</td>
</tr>
<tr>
<td>35°</td>
<td>0.05568</td>
<td>35.7°</td>
<td>43.1°</td>
<td>78.8°</td>
<td>2.00</td>
<td>0.71</td>
<td>35.9</td>
</tr>
<tr>
<td>40°</td>
<td>0.02728</td>
<td>40.3°</td>
<td>43.6°</td>
<td>83.9°</td>
<td>2.05</td>
<td>0.73</td>
<td>75.2</td>
</tr>
<tr>
<td>44.12°</td>
<td>0.00000</td>
<td>44.12°</td>
<td>44.12°</td>
<td>88.24°</td>
<td>2.10</td>
<td>0.74</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>
kinetic energy \((\gamma_0 - 1)\) of the unmodulated electrons; their ratio
(which is called \(\alpha\) by Smårs [Ref. 4, p. 4]) we obtain at once from
Eq. (3.47), and from the relation
\[
\frac{\gamma}{\gamma - 1} = \frac{1 + \sqrt{1 - \beta^2}}{\beta^2},
\]
as
\[
\frac{(6\gamma)_{\max}}{\gamma_0 - 1} = x \cdot \frac{1 + \sqrt{1 - \beta_{e0}^2}}{\beta_{e0}}.
\]
In particular, when \(\beta_{e0} = \frac{1}{2}\), we have \(b^2 = 3\), and Table VI gives the
following figures:

<table>
<thead>
<tr>
<th>(x)</th>
<th>(\frac{(6\gamma)_{\max}}{\gamma - 1})</th>
<th>Fraction of electrons which will get 99 percent or more of the maximum energy</th>
<th>Length (\xi) of the buncher</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.14107</td>
<td>0.5265</td>
<td>0.65</td>
<td>0.66</td>
</tr>
<tr>
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<td>1.30</td>
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<td>0.2078</td>
<td>0.71</td>
<td>1.91</td>
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<td>0.1018</td>
<td>0.73</td>
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<tr>
<td>0.00000</td>
<td>0.0000</td>
<td>0.74</td>
<td>(\infty)</td>
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The first row corresponds to the highest value of \((6\gamma)_{\max}\) which is
permissible with the condition (2.10). Now, since \((6\gamma)_{\max}/(\gamma_0 - 1)\)
represents the ratio between the peak voltage in the cavity gap and
the gun voltage which gives the electrons a velocity \(\beta_{e0}\), it would
be inconvenient to have it much higher than 0.1 (which is the value
chosen by Smårs [Ref. 4, p. 11]), but it would also be useless to have

- 69 -
it lower than 0.1 because the buncher length would then rise without appreciable increase of the fraction of electrons which are in the condition to be accelerated to 99 percent or more of the maximum energy at the end of the accelerator.

**General Conclusion**

The comparison between Tables IV and VI when \( \beta_w = \beta_e = \frac{1}{2} \) confirms what has already been stated from the comparison of Tables II and V: for a given value of \( x \), i.e., for a given energy spread of the outgoing electrons, the waveguide buncher is at least 2.3 times as long as the drift space which should follow a velocity-modulating cavity; but the fraction of electrons which satisfy the conditions required to get 99 percent or more of the maximum energy at the end of the accelerator is about 0.80 for a waveguide buncher instead of 0.70 for a velocity-modulating buncher.

Both types of buncher may be shortened by increasing the electric-field strength if we allow a larger electron energy spread at the output; this is easy to do for a waveguide buncher since the needed electric-field strength \( \alpha \) in any case remains as low as a few hundredths, whereas it is much more inconvenient to do for a velocity-modulation buncher because the peak voltage in the cavity gap would then be very high (16 kv for a drift-space length \( \xi = 2 \)). Even though such a shortening always involves a slight decrease of the fraction of electrons which will get at least 99 percent of the maximum energy at the end of the accelerator, this fraction remains greater for a waveguide buncher than it is for a drift space, whatever its length may be. Therefore, in practice a waveguide buncher should not be longer than a velocity-modulation one; both types will work satisfactorily with a length equal to 3 or 4 wavelengths.

Further, a waveguide buncher does not need more rf power than a velocity-modulating cavity; due to the low values of \( \alpha \), the necessary rf peak power is of the order of tens or hundreds of watts. Moreover, from Eqs. (3.43) and (3.44) we see that the bunching parameter is proportional to the electric-field strength in a velocity-modulating cavity, while in a waveguide buncher it varies as the
square root of the electric-field strength; the latter thus appears to be less sensitive to a small fluctuation of the feeding rf power level than the velocity-modulation buncher.

Finally, we note that the electron density (per unit of phase interval) at the output end of both types of buncher is at its highest near the envelope of the curves $\Delta(\xi)$ in Figs. 3.2, 3.3 and 3.7; in other words, electrons are concentrated mainly near the two phase angles which were chosen as limits of the output phase interval in Tables I, II and V.
IV. THE DISK-LOADED WAVEGUIDE WHEN THE FIELD STRENGTH AND THE PHASE VELOCITY OF THE WAVE VARY ALONG THE AXIS

The complexity of the functions $\xi(\Delta)$ which appear in Eqs. (2.14) and (3.11) as rigorous solutions of the electron-motion equations (1.1) when $\alpha$ and $\beta_w$ are constant suggests that it is probably hopeless to try to get analytic solutions of these equations when $\alpha$ and/or $\beta_w$ vary along the waveguide.

Several authors, after choosing more or less arbitrary trial functions for $\alpha(\xi)$ and $\beta_w(\xi)$, solved Eqs. (1.1) on an electronic computer for different values of the entrance phase angle $\Delta_0$, and so deduced the bunching characteristics of a waveguide section where $\alpha$ and $\beta_w$ exhibit the assumed variations as functions of $\xi$.

Ponce de Leon varied systematically the functions $\alpha(\xi)$ and $\beta_w(\xi)$; with the optimum buncher he thus found, 93 percent of the injected electrons are bunched within an asymptotic phase interval of $25.8^\circ$ around $-60^\circ$ when they arrive at the end of the accelerator, and they then have an energy spread of 20 percent. But the theoretical problem of determining what would be the optimum functions $\alpha(\xi)$ and $\beta_w(\xi)$, without first assuming these functions to be known, seems insoluble.

The trial functions which have been assumed by all authors had imposed the following conditions:

$$\begin{align*}
\alpha & \text{ rises from a low value to the accelerator value;} \\
\beta_w & \text{ rises from } \beta_{e0} \text{ to 1.}
\end{align*}$$

This latter condition brings the electron velocity close to that of light at the output of the buncher, and we have emphasized the fact that this prevents the electrons from a further bunching in the regular sections of the accelerator. On the contrary, a

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2 Chodorow et al., op. cit.

velocity-modulation buncher or a waveguide section with $\beta_w$ constant and equal to $\beta_{c0}$ does not accelerate the electrons and thus fully utilizes the important bunching properties of the accelerator itself.

Taking this effect into account, we have shown that a disk-loaded waveguide section of uniform periodicity used as a buncher would bring 80 percent of the injected electrons in the condition to be accelerated to an energy differing by less than 1 percent from the maximum energy at the end of the accelerator. If we compare this figure of 80 percent with the optimum figure that has ever been predicted for a tapered disk-loaded structure, namely, 95 percent for the buncher designed for the Mark III accelerator, it appears that the difficulties of building such a tapered structure are not balanced by a significant improvement of the system's bunching characteristics.

In summary, the idea of a nonperiodic disk-loaded structure does not seem worth utilizing for electron bunching.

*Note that with this buncher the expected energy spread of the electrons at the end of the accelerator is 2 percent.

1Chodorow et al., op. cit.

2Chodorow et al., op. cit.
ACKNOWLEDGMENT

The writer is greatly indebted to Dr. R. B. Neal for many valuable discussions while carrying out this work.
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