Introduction to Supersymmetry

1: What is Supersymmetry?
Among models of electroweak symmetry breaking and physics beyond the Standard Model

**Supersymmetry (SUSY)**

has pride of place. It is an explicit weak-coupling realization of new physics that provides definite answers to all of the relevant questions. For anyone interested in models of new physics, supersymmetry merits serious study.

In this set of lectures, I will explain the formal basis of SUSY and its application to models of elementary particle physics.

A very useful reference on this subject is:


There are also new textbooks on supersymmetry in particle physics, by Drees, Godbole, and Roy, Baer and Tata, and Dreiner, Haber, and Martin.
1. What is supersymmetry?
   symmetry relations of SUSY, construction of Lagrangians

2. The Minimal Supersymmetric Standard Model
   more Lagrangians, definition of the MSSM

3. The Spectrum of Superparticles
   parametrization of SUSY breaking, mass spectrum of the MSSM

4. Supersymmetric Particles at Colliders
   couplings in the Lagrangian, illustrated in physics processes

5. Higgs and Dark Matter in Supersymmetry
   EWSB in supersymmetry, supersymmetric dark matter
We might begin our study by addressing the most important problem with the minimal form of the Standard Model (MSM):

In the MSM, all masses arise from the spontaneous breaking of SU(2) x U(1). This in turn is due to the vacuum expectation value ("vev") of the Higgs scalar field \( \varphi \).

To stabilize a nonzero vev, we postulate a potential

\[
V = \mu^2 |\varphi|^2 + \lambda |\varphi|^4
\]

with \( \mu^2 < 0 \).
Why is $\mu^2 < 0$? "Because it is."

Either sign of $\mu^2$ is possible in principle; there is no preference. $\mu^2$ receives large additive radiative corrections from

\[ \mu^2 = \mu^2_{\text{bare}} + \frac{\lambda}{8\pi^2} \Lambda^2 - \frac{3y_t^2}{8\pi^2} \Lambda^2 + \cdots \]

where $\Lambda$ is the largest momentum at which the MSM is still valid. This already says that the criterion $\mu^2 < 0$ is not a simple condition on the bare value of $\mu^2$.

If $\Lambda \sim m_{\text{Pl}} \sim 10^{19} \text{ GeV}, \ |\mu| \sim 100 \text{ GeV}$, this formalism requires cancellations in the first 36 decimal places. This is called the "gauge hierarchy problem".
There are two ways to solve this problem:

- Lower the cutoff $\Lambda$ to about 1 TeV

For example, postulate that $\varphi$ is not an elementary field but rather is a bound state. Then we need new interactions at 1 TeV ("technicolor") to form this bound state.

Today, this approach is disfavored. Typically, technicolor models give large electroweak corrections and large flavor-changing neutral current amplitudes.
• Insist that $\varphi$ is a fundamental scalar field, but postulate a symmetry that constrains its potential.

In particular, this symmetry should forbid the mass term $\mu^2 |\varphi|^2$.

Violation of the symmetry will re-introduce this term, but hopefully in such a way that we can compute the sign and magnitude.

\[ \delta \varphi = \epsilon \nu \quad \text{shift symmetry} \quad \Rightarrow \quad \text{little Higgs models} \]
\[ \delta \varphi = \epsilon \cdot A \quad \text{gauge symmetry} \quad \Rightarrow \quad \text{extra dimension models} \]
\[ \delta \varphi = \epsilon \cdot \psi \quad \text{chiral symmetry} \quad \Rightarrow \quad \text{supersymmetry models} \]

From here on, we choose the last option and follows its implications to the end.
So, in the rest of this lecture, I would like to work out the symmetry structure that includes the transformation

$$\delta_\epsilon \phi = \epsilon \cdot \psi$$

where $\phi$ is a complex scalar field and $\psi$ is a spin-$\frac{1}{2}$ field. $\epsilon_\alpha$ is a spin-$\frac{1}{2}$ parameter, represented classically by an anticommuting number. In quantum theory, this transformation is generated by

$$[\epsilon \cdot Q, \phi] = \epsilon \cdot \psi$$

where $Q_\alpha$ is a conserved charge: $[Q_\alpha, H] = 0$
This structure looks innocent, but it is not.

Consider the quantity: \( \{Q_\alpha, Q^\dagger_\beta\} \)

This object has the following properties:

- it is a 4-vector: \( \{Q_\alpha, Q^\dagger_\beta\} = 2\gamma^m_{\alpha\beta} R_m \) *
- it is conserved: \([\{Q_\alpha, Q^\dagger_\beta\}, H] = 0\)
- it is nonzero:

\[
\langle \psi | \{Q_\alpha, Q^\dagger_\alpha\} | \psi \rangle = \langle \psi | Q_\alpha Q^\dagger_\alpha | \psi \rangle + \langle \psi | Q^\dagger_\alpha Q_\alpha | \psi \rangle = \|Q^\dagger_\alpha | \psi \rangle \|^2 + \|Q_\alpha | \psi \rangle \|^2
\]

which vanishes only if \( Q_\alpha \) and \( Q^\dagger_\alpha \) annihilate all states.

So, \( R_m \) is a nontrivial conserved 4-vector charge.

* why not a scalar? see below ...
In a relativistic theory, conservation of energy and momentum already imposes severe restrictions on scattering,

E.g., $2 \rightarrow 2$ scattering amplitudes depend only on 1 parameter, the CM polar angle.

Coleman and Mandula proved that, if there is a second conserved vector, the S-matrix must be trivial: $S = 1$. 
So, we have no choice: We must set \( R^m = P^m \)

Then, e.g., \[ \{Q_1, Q_1^\dagger\} = 2(E - P^3) \]

This equation has important implications:

We cannot supersymmetrize just the Higgs field, leaving most of the MSM unchanged. To build a theory with SUSY, all fields in the theory must transform under SUSY.

Even the minimal SUSY extension of the MSM must double the number of particles and fields. This leads to a quite complex theory, but also one with an interesting structure with many implications for the theory of Nature.
To go further, we need to understand 4-d relativistic fermions a little better.

There are two basic spin-1/2 representations of the Lorentz group. Each is 2-dimensional

\[
\psi_L \rightarrow (1 - i\vec{\alpha} \cdot \vec{\sigma}/2 - \vec{\beta} \cdot \vec{\sigma}/2)\psi_L
\]
\[
\psi_R \rightarrow (1 - i\vec{\alpha} \cdot \vec{\sigma}/2 + \vec{\beta} \cdot \vec{\sigma}/2)\psi_R
\]

Let \( c = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \); then

\[
\psi_T^{1L} c \psi_{2L} = -\epsilon_{\alpha\beta} \psi_{1L\alpha} \psi_{2L\beta}
\]

is Lorentz invariant

\( -c\psi_L^* \) transforms like \( \psi_R \)

C or P carries \( \psi_L \leftrightarrow \psi_R \), but these are not symmetries of the Standard Model.
A Dirac fermion can be written as a pair of L-fermions:

\[ \Psi = \begin{pmatrix} \psi_{1L} \\ \psi_{2R} \end{pmatrix} = \begin{pmatrix} \psi_{1L} \\ -c\psi_{2L}^* \end{pmatrix} \]

If we represent

\[ \gamma^m = \begin{pmatrix} 0 & \sigma^m \\ \overline{\sigma}^m & 0 \end{pmatrix} \]

\[ \sigma^m = (1, \vec{\sigma})^m \]

\[ \overline{\sigma}^m = (1, -\vec{\sigma})^m \]

The Dirac Lagrangian becomes

\[ \mathcal{L} = \bar{\Psi}i\gamma \cdot \partial\Psi - M\bar{\Psi}\Psi \]

\[ = \psi_{1L}^\dagger i\overline{\sigma} \cdot \partial\psi_{1L} + \psi_{2L}^\dagger i\overline{\sigma} \cdot \partial\psi_{2L} \]

\[ - (m\psi_{1L}^T c\psi_{2L} - m^* \psi_{1L}^\dagger c\psi_{2L}^*) \]

with \( m = M \). In general, \( m \) can be a complex number.

Note that

\[ \psi_{1L}^T c\psi_{2L} = \psi_{2L}^T c\psi_{1L} \]

\[ (\psi_{1L}^T c\psi_{2L})^* = \psi_{2L}^\dagger (-c)\psi_{1L}^* = -\psi_{1L}^\dagger c\psi_{2L}^* \]
Actually, the most general Lagrangian for massive 4-d fermions has the form

\[ \mathcal{L} = \psi_k^\dagger i\sigma \cdot \partial \psi_k - (m_{jk} \psi_j^T c\psi_k - m_{jk}^* \psi_j^\dagger c\psi_k^*) \]

(Here and henceforth, I drop the subscript L.)

\[ m_{jk} = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix} \] gives a Dirac fermion

with a conserved fermion number

\[ Q\psi_1 = +\psi_1 \quad Q\psi_2 = -\psi_2 \]

\[ m_{jk} = m\delta_{jk} \] gives a Majorana fermion

In general, \( m \) is a complex symmetric matrix, a mixture of Majorana and Dirac mass terms.
The SUSY charges are 4-d fermions. We can write the $Q^\alpha$ as L-fermions, the $Q^\dagger_\alpha$ as R-fermions. The minimal SUSY has two of each: $\alpha = 1, 2$

Now it is clear that $\{Q^\alpha, Q^\dagger_\beta\}$ has no scalar component. The minimal SUSY algebra in 4-d is then:

$$\{Q^\alpha, Q^\dagger_\beta\} = 2\sigma^m_{\alpha\beta} P_m$$

As an action on fields, with anticommuting L-fermion parameters $\xi, \eta$, this takes the form:

$$[\delta\xi, \delta\eta] = 2i[\xi^\dagger\sigma^m\eta - \eta^\dagger\sigma^m\xi] \partial_m$$
Now we can look for representations of this algebra.

The simplest representation includes a complex scalar field $\phi$ and an L-fermion field $\psi$. This is the **chiral supermultiplet**. Its particle content includes

$$
\begin{align*}
2 \text{ bosons } & \phi, \phi^* & + & 2 \text{ fermions } & \psi_L, (\psi^*)_R
\end{align*}
$$

For convenience in writing the algebra, I add a second complex field $F$ that will have no associated particles.

$$
\begin{align*}
\delta_\xi \phi &= \sqrt{2} \xi^T c \psi \\
\delta_\xi \psi &= \sqrt{2} i \sigma^n c \xi^* \partial_n \phi + \sqrt{2} \xi F \\
\delta_\xi F &= -\sqrt{2} i \xi^\dagger \sigma^m \partial_m \psi
\end{align*}
$$

We need to check that this transformation

1. satisfies the fundamental commutation relations
2. leaves a suitable Lagrangian invariant
Check #1 on $\phi$:

$$[\delta_\xi, \delta_\eta] \phi = - \delta_\xi (\sqrt{2} \eta^T c \psi) - (\xi \leftrightarrow \eta)$$

$$= \sqrt{2} \eta^T \sqrt{2} i \sigma^n c \xi^* \partial_n \phi - (\xi \leftrightarrow \eta)$$

$$= 2i \eta^T c \sigma^n c \xi^* \partial_n \phi - (\xi \leftrightarrow \eta)$$

$$= 2i [\xi^\dagger \sigma^n \eta - \eta^\dagger \sigma^n \xi] \partial_n \phi$$

The check on $F$ is equally easy. The check on $\psi$ requires Fierz transformations, e.g.,

$$\eta^*_\alpha \xi_\beta = \frac{1}{2} (\eta^\dagger \sigma_m \xi) \sigma^m_{\beta\alpha}$$

but otherwise it is straightforward.
Check #2 on \( \mathcal{L} = \partial^m \phi^* \partial_m \phi + \psi^\dagger i\sigma \cdot \partial \psi + F^* F \)
using integration by parts freely under \( \int d^4 x \)

\[
\delta \xi \mathcal{L} = \partial^m \phi^* \partial_m (\sqrt{2} \xi^T c \psi) + (-\sqrt{2} \partial^m \psi^\dagger c \xi^*) \partial_m \phi \\
+ \psi^\dagger i\sigma^m \partial_m \left[ \sqrt{2} i \sigma^n c \xi^* \partial_m \phi + \sqrt{2} \xi F \right] \\
+ \left[ \sqrt{2} i \partial_n \phi^* \xi^T c \sigma^n + \sqrt{2} \xi^\dagger F^* \right] i \sigma^m \psi \\
+ F^* \left[ -\sqrt{2} i \xi^\dagger \sigma^m \partial_m \psi \right] + \left[ \sqrt{2} i \partial_m \psi^\dagger \sigma^m \xi \right] F \\
= -\phi^* \sqrt{2} \xi^T c \partial^2 \psi + \sqrt{2} \partial_n \phi^* \xi^T c \sigma^n \sigma^m \partial_n \partial_m \psi \\
+ \sqrt{2} \psi^\dagger c \xi^* \partial^2 \phi - \sqrt{2} \psi^\dagger \sigma^m \sigma^n c \xi^* \partial_m \partial_n \phi \\
+ \sqrt{2} i \psi^\dagger \sigma^m \partial_m F \xi + \sqrt{2} i \partial_m \psi^\dagger \sigma^m \xi F \\
+ \sqrt{2} i \xi^\dagger F^* \sigma^m \partial_m \psi - \sqrt{2} i F^* \xi^\dagger \sigma^m \partial_m \psi \\
= 0
\]
So far, our theory is trivial. But we can give it interactions in a straightforward way.

Let $W(\phi)$ be an analytic function of $\phi$, the "superpotential".

Let

$$\mathcal{L}_W = F \frac{\partial W}{\partial \phi} - \frac{1}{2} \psi^T c \psi \frac{\partial^2 W}{\partial \phi^2}$$

$$\delta_\xi \mathcal{L}_W = F \frac{\partial^2 W}{\partial \phi^2} (\sqrt{2} \xi^T c \psi) - \sqrt{2} F \xi^T c \psi \frac{\partial^2 W}{\partial \phi^2}$$

$$- \sqrt{2} i \xi^+ \sigma^m \partial_m \psi \frac{\partial W}{\partial \phi} - \psi^T c \sqrt{2} i \sigma^n c \xi^* \partial_n \phi \frac{\partial^2 W}{\partial \phi^2}$$

$$- \psi^T c \psi \frac{\partial^3 W}{\partial \phi^3} \sqrt{2} \xi^T c \psi$$

the 2nd line rearranges to:

$$- \sqrt{2} i \xi^+ \sigma^n (\partial_n \psi \frac{\partial W}{\partial \phi} + \psi \partial_n \phi \frac{\partial^2 W}{\partial \phi^2})$$

the 3rd line is proportional to $\psi_\alpha \psi_\beta \psi_\gamma$, which vanishes.
I have now shown that the following Lagrangian is supersymmetric:

\[ \mathcal{L} = \partial^m \phi_k^* \partial_m \phi_k + \psi_k^\dagger i \sigma^m \partial_m \psi_k + F_k^* F_k + (\mathcal{L}_W + h.c.) \]

where \( \mathcal{L}_W \) is built from an analytic function \( W(\phi_k) \)

\[ \mathcal{L}_W = F_k \frac{\partial W}{\partial \phi_k} - \frac{1}{2} \psi_j^T c \psi_k \frac{\partial W}{\partial \phi_j \partial \phi_k} \]

The \( F_k \) are Lagrangian multipliers with constraint equations

\[ F_k^* = - \frac{\partial W}{\partial \phi_k} \]

Eliminating the \( F_k \) using these equations, we find the potential

\[ V_F = \sum_k \left| \frac{\partial W}{\partial \phi_k} \right|^2 \]
It is important that $V \geq 0$ and $V = 0$ only if all $F_k = 0$.

Recall that
\[
\langle 0 | \{ Q_\alpha, Q_\alpha^\dagger \} | 0 \rangle = \langle 0 | (H - P^3) | 0 \rangle
\]
This is $\geq 0$ and is $= 0$ only if $Q_\alpha | 0 \rangle = Q_\alpha^\dagger | 0 \rangle = 0$

Now consider
\[
\langle 0 | [\xi^T c Q, \psi_k] | 0 \rangle = \langle 0 | \sqrt{2} i \sigma^n \xi^* \partial_n \phi_k + \xi F_k | 0 \rangle
\]
\[
= \xi \langle 0 | F_k | 0 \rangle
\]
If the vacuum is supersymmetric, this must vanish. Unbroken SUSY then implies
\[
\langle H \rangle = 0 \quad \langle F \rangle = 0
\]

Because $H \geq 0$, SUSY can be spontaneously broken only if it is impossible to find a state where $\langle H \rangle = 0$. 
These are exact results, and so it must follow that the vacuum energy vanishes in perturbation theory,

\[ \frac{\partial}{\partial m} \phi^* \frac{\partial}{\partial m} \phi + \psi^\dagger i \sigma^m \frac{\partial}{\partial m} \psi - |m \phi + \lambda \phi^2|^2 \]

Eliminating F, we find

\[ W = \frac{m}{2} \phi^2 + \frac{\lambda}{3} \phi^3 \]

I would like to show you another type of cancellation that is also seen in SUSY perturbation theory. Consider

\[ L = \partial^m \phi^* \partial_m \phi + \psi^\dagger i \sigma^m \partial_m \psi - |m \phi + \lambda \phi^2|^2 \]

\[ -\frac{1}{2} (m + 2 \lambda \phi) \psi^T c \psi + \frac{1}{2} (m + 2 \lambda \phi)^* \psi^\dagger c \psi^* \]

\( \phi \) and \( \psi \) have equal masses = |m|. 
From our previous experience, we might expect an additive radiative correction to the mass. Check this at $m = 0$:

$$\psi \rightarrow \psi = 0$$

$$\phi \rightarrow \phi = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{p^2} + \frac{1}{2} (-2i\lambda)(+2i\lambda) \int \frac{d^4 p}{(2\pi)^4} \text{tr}[\frac{i \sigma \cdot p}{p^2} c \frac{i \sigma^T \cdot (-p)^T}{p^2} c] = 0$$
In fact, it may be shown quite generally that the superpotential $W$ receives no radiative corrections. In 1 loop:

\[ \psi \quad \rightarrow \quad \psi = 0 \]

\[ \phi \quad \rightarrow \quad \phi = F_{\phi}^{\phi} + \psi \quad \rightarrow \quad \psi = 0 \]

The field strength renormalization can be nonzero

\[ \psi \rightarrow \psi = i\delta Z \sigma \cdot p \]

\[ \phi \rightarrow \phi = -i\delta Z p^2 \]

so the fields in $W$ can be rescaled by radiative corrections.
Supersymmetry seems to be associated with some generalized notion of space-time. I would now like to make that precise by introducing superspace.

Consider a space with 4 ordinary commuting and 4 anticommuting coordinates: \((x^m, \theta^\alpha, \bar{\theta}^\alpha)\) where \(\bar{\theta}^\alpha\) is the adjoint of \(\theta^\alpha\).

A superfield is a function of the superspace coordinates: \(\Phi(x, \theta, \bar{\theta})\)

It is tempting to define supersymmetry transformations as translations \(\theta \rightarrow \theta + \xi\), but this gives \(\{\delta_\xi, \delta_\eta\} = 0\), which implies a trivial S-matrix.

What we really want is \(\delta_\xi \Phi = Q_\xi \Phi\)

where \(Q_\xi = (-\frac{\partial}{\partial \theta} - i\bar{\theta}\sigma^m \partial_m)\xi + \xi^\dagger (\frac{\partial}{\partial \theta} + i\sigma^m \theta \partial_m)\)

satisfies \([Q_\xi, Q_\eta] = -2i(\xi^\dagger \bar{\sigma}^m \eta - \eta^\dagger \bar{\sigma}^m \xi) \partial_m\)
Two combinations of superspace derivatives commute with $Q_\xi$

$$D_\alpha = \frac{\partial}{\partial \theta_\alpha} - i(\bar{\theta}\sigma^m)_\alpha \partial_m \quad \overline{D}_\alpha = -\frac{\partial}{\partial \bar{\theta}_\alpha} + i(\sigma^m \theta)_\alpha \partial_m$$

These satisfy

$$[D_\alpha, Q_\xi] = [\overline{D}_\alpha, Q_\xi] = 0$$

so we can apply these constraints consistently with supersymmetry:

$$D_\alpha \Phi = 0 \quad \text{or} \quad \overline{D}_\alpha \Phi = 0$$

The solution of $\overline{D}_\alpha \Phi = 0$

is

$$\Phi(x, \theta, \bar{\theta}) = \Phi(x + i\bar{\theta}\sigma^m \theta, \theta)$$

Taylor expand the RHS in $\theta$, taking account of the fact that $\theta$ is anticommuting:

$$\Phi(x, \theta) = \phi(x) + \sqrt{2}\theta^T c\psi(x) + \theta^T c\theta F(x)$$

This gives precisely the chiral supermultiplet.
Anticommuting coordinates are integrated over with \( \int d^2 \theta \)
satisfying \( \int d^2 \theta \ 1 = \int d^2 \theta \ \theta_\alpha = 0 \ \int d^2 \theta \ (\theta^T c \theta) = 1 \)

Now we can compute

\[
\int d^2 \theta \ \Phi(x, \theta) = F(x)
\]

\[
\int d^2 \theta \ W(\Phi) = F \frac{\partial W}{\partial \phi} - \frac{1}{2} \psi^T c \psi \frac{\partial^2 W}{\partial \phi^2}
\]

\[
\int d^2 \theta d^2 \bar{\theta} \ \Phi^\dagger \Phi = \partial^m \phi^* \partial_m \phi + \psi^\dagger i \bar{\sigma} \cdot \partial \psi + F^* F
\]

These formulae produce the invariant Lagrangian of chiral superfields from a superspace point of view.
Superspace implies the following general form for invariant
Lagrangians built from chiral superfields:

$$\int d^4 \theta \ K(\Phi, \Phi) + \int d^2 \theta \ W(\Phi) + \int d^2 \bar{\theta} \ W(\Phi)$$

where \( W(\Phi) \) is an analytic function of complex superfields and
\( K(\Phi, \Phi) \) is a real function of superfields.

If \( K(\Phi, \Phi) \) is nontrivial, the model is a nonlinear sigma model
on a complex manifold with metric

\[
g_{m\bar{n}} = \frac{\partial^2}{\partial \Phi^m \partial \Phi^{\bar{n}}} K(\Phi, \Phi)
\]

such a manifold is called a Kahler manifold; \( K \) is the Kahler
potential.

You see a sense in which supersymmetric QFT is to ordinary QFT
as complex analysis is to real analysis.
It is possible to write perturbation theory in terms of Feynman rules in superspace. I do not have time to describe that formalism in these lectures. However, I would like to state one important consequence of this formalism:

In superspace, all radiative corrections to the effective Hamiltonian are of the form

\[ \int d^4\theta \ X(\Phi, \overline{\Phi}) \]

so it is manifest that the superpotential is not renormalized.
In a supersymmetric generalization of the MSM, the non-renormalization of the superpotential would eliminate the dangerous additive radiative corrections to the Higgs mass.

This makes SUSY a good starting point for the construction of models of physics beyond the SM. We will turn to that model construction in the next lecture.