The Physical Renormalization of Quantum Field Theories

Michael Binger

PhD Defense

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Advisor : Stan Brodsky

Outline

I. Preliminaries (13)

1. Motivations for Physical Renormalization
2. Lessons from QED
3. Applying these lessons to non-abelian theories

II. Gauge Coupling Unification in Physical Schemes (3)

III. The Gauge-Invariant Three Gluon Vertex (31)

1. General structure and symmetries
2. Supersymmetric relations
3. The form factors in physical processes
4. Effective charge and effective scale
5. Mass effects

$$\alpha_{eff}(p_1^2, p_2^2, p_3^2)$$
$$Q_{eff}^2(p_1^2, p_2^2, p_3^2)$$

IV. Pinch-Technique Effective Charge at 2-loops (2)

V. Future Directions and Summary (4)

Big Picture Motivation

- Large Hadron Collider (LHC) at CERN

  14 TeV proton-proton collisions

  Due to come online in late 2007

NEW PHYSICS ?!?!

- International Linear Collider (ILC)

  Precision electron-positron collider

Precision tests of strong (QCD) and electroweak interactions

→ Precision tests of renormalization

Renormalization = method of getting finite predictions from the seemingly infinite results of quantum field theories

Motivation for Physical Renormalization Schemes

Problems with the usual approach :

- *Scale and Scheme Ambiguity*
- *Analyticity and Mass Thresholds*
- *Convergence of the Series*

Scale and Scheme Ambiguity

In any perturbative series $$R(Q) = \sum_{n=0}^{N} R_n(Q,\mu)\alpha_s^n(\mu)$$

You can change the scale of the last term :

$$\alpha_s(\tilde{\mu}) = \alpha_s(\mu) - \frac{(\alpha_s(\mu))^2}{2\pi}\beta_0 \log(\tilde{\mu}/\mu)$$

Or the scheme of the last term :

$$\tilde{\alpha}_s(\mu) = \alpha_s(\mu) + C(\alpha_s(\mu))^2$$

The result is formally the same to the order calculated

⇨ The prediction is ambiguous

Analyticity and Mass Thresholds

$\overline{MS}$ does not have automatic decoupling of heavy particles

⇨ Must define a set of schemes in each desert region and match

$$\alpha_s^{(f)}(M_Q) = \alpha_s^{(f+1)}(M_Q)$$

- The coupling has discontinuous derivative at the matching point
- At higher orders the coupling itself becomes discontinuous!
- Does not distinguish between spacelike and timelike momenta

"AN ANALYTIC EXTENSION OF THE MS-BAR RENORMALIZATION SCHEME"
S. Brodsky, M. Gill, M. Melles, J. Rathsman. **Phys.Rev.D58:116006,1998**

Convergence of the Series

It is commonly believed that the series diverges!

Renormalons

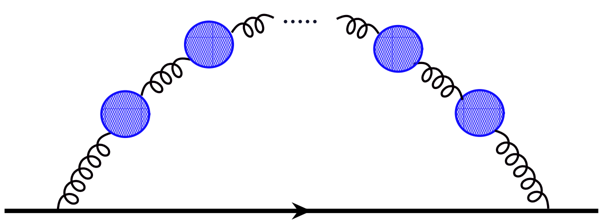
$$R(Q) = \sum_{n=0}^{N} R_n(Q,\mu)\alpha_s^n(\mu) \qquad R_n \propto n!$$

$$\int d^4k\,\alpha_s(k^2) f(k^\mu, p_i^\mu) \rightarrow \infty$$

From the $k^2 \approx 0$ region

‘n’ insertions

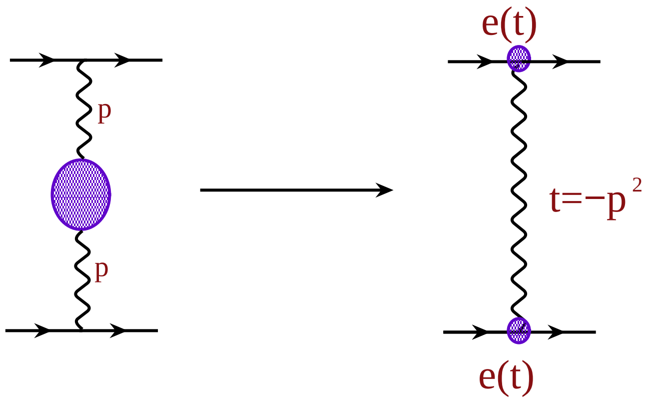
$$\sum_{n=0}^{\infty}$$



Lessons from QED (I)

QED has a naturally defined physical effective charge

$$e^- e^- \to e^- e^-$$



+ the u-channel crossed graph

Effective Charge

$$\overline{MS}$$

(A) $M(t,u) = M_t e^2(t) + M_u e^2(u)$ (B) $M_{\overline{MS}}(t,u) = (M_t + M_u) e^2_{\overline{MS}}(\mu_R^2)$

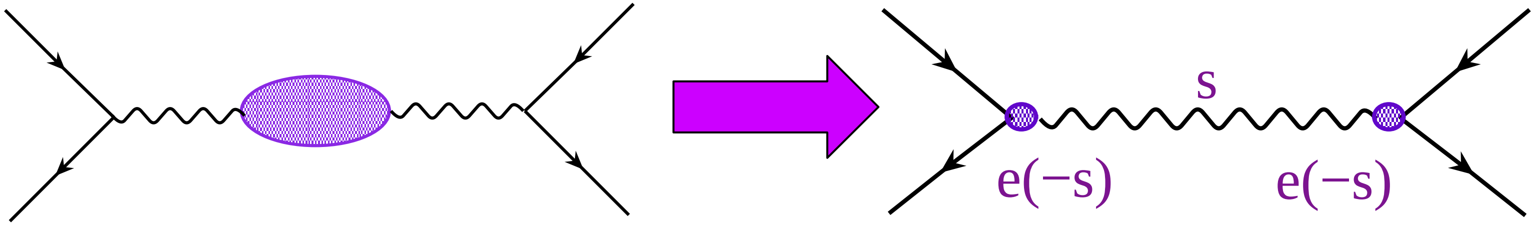
The scale μ_R in (B) is usually guessed. To get it correctly :

- Compute the cross section in both (A) and (B), and compare
- BLM scale fixing using partial info about the higher order terms

Otherwise you have to calculate to a higher order (here the full one-loop graphs)

Lessons from QED (II)

$$e^+e^- \to \mu^+\mu^-$$



$$\text{Im}\, e^2(-s) \propto \text{Im}\, \Pi_{\gamma\gamma}(s)$$

The QED Effective Charge

- Complex
- Analytic through mass thresholds
- Distinguishes between timelike and spacelike momenta

QED Lesson III Cancelled

QED *in principle* has a renormalon divergence at very large energies

But QED merges into the electroweak theory well below this Landau pole

Not much help for the very low energy renormalon problems of QCD

Lessons from QED : Summary

- Effective couplings are complex analytic functions with the correct threshold structure expected from unitarity

- Multiple “renormalization” scales appear

- The scales are unambiguous since they are physical kinematic invariants

- Optimal improvement of perturbation theory

Applying These Lessons To Non-Abelian Theories

- Seems difficult since the running coupling is not usually defined from a gauge-invariant gluon self-energy
- Thus, we need a ***natural diagrammatic method*** of getting gauge-invariant Green's functions

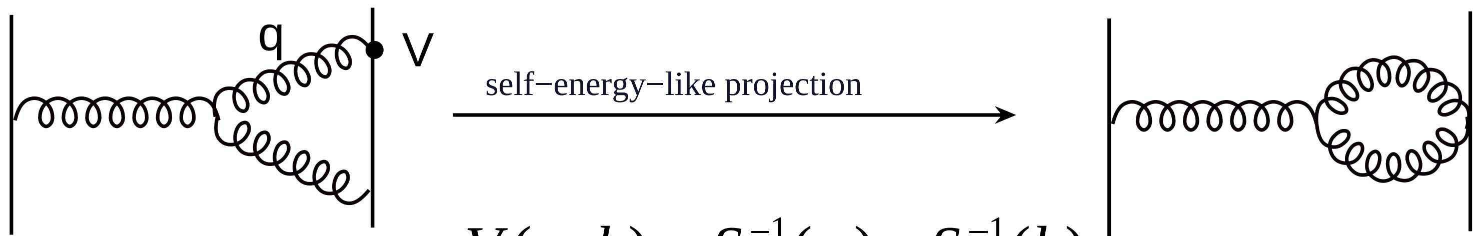
Luckily we have two….

- ***Pinch-Technique (PT)***

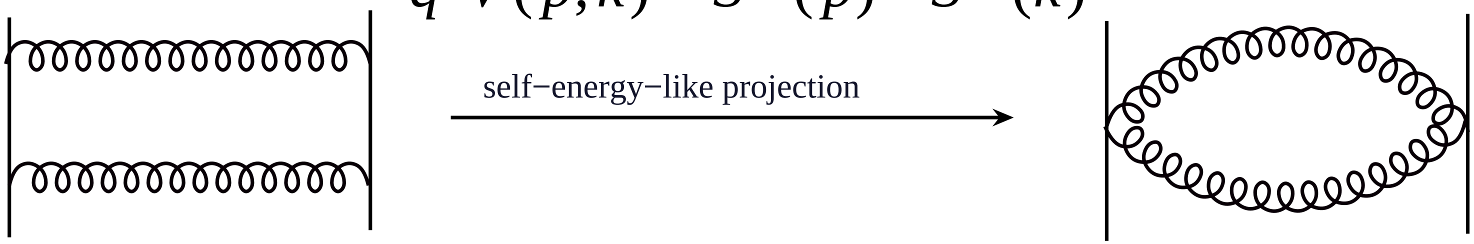
- ***Background Field Method (BFM)***

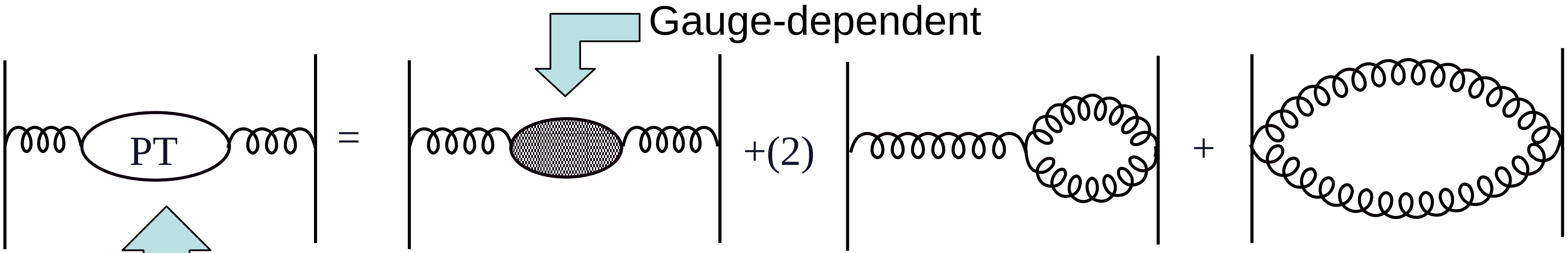
The Pinch Technique

(Cornwall, Papavassiliou)



$$q \cdot V(p,k) = S^{-1}(p) - S^{-1}(k)$$





Gauge-invariant gluon self-energy!

Background Field Method

- Gauge field is split into quantum (Q) and background (B) parts

$$A_\mu = B_\mu + Q_\mu$$

B_μ → External legs

Q_μ → Loops

Choose the gauge fixing function appropriately

$$G^a = (D^\mu(B))^{ac} Q^c_\mu = \partial^\mu Q^a_\mu + g f^{abc} B^b_\mu Q^\mu_c$$

(In the conventional formulation $G^a = \partial^\mu A^a_\mu$)

⇒ ***Green's functions are background gauge-invariant***

But they do (lightly) depend on the quantum gauge-fixing parameter

⇒ Only through UV finite terms (Kallosh)

A Happy Marriage

PT = BFM in quantum Feynman gauge (BFMFG)

Proven by Binosi and Papavassiliou to all orders

also = star-scheme for electroweak theory at one-loop (Kennedy and Lynn)

PT/BFMFG Green's functions have excellent properties :

- Non-abelian analogs of QED with simple Ward ID's
- Lead to analytic effective charges
- Can be derived from unitarity (optical theorem)
- Correct asymptotic UV behavior

$$\Pi_{PT}(p^2) \propto \beta_0 \log(p^2) + \cdots$$

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→ $\alpha_{eff}(p_1^2, p_2^2, p_3^2)$

$Q_{eff}^2(p_1^2, p_2^2, p_3^2)$

IV. Pinch-Technique Effective Charge at 2-loops (2)

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Unification in Physical Schemes

"PHYSICAL RENORMALIZATION SCHEMES AND GRAND UNIFICATION"
M.B. and Stanley J. Brodsky. **Phys.Rev.D69:095007,2004**

$$\alpha_i(Q) = \frac{\alpha_i(Q_0)}{1+\hat{\Pi}_i(Q)-\hat{\Pi}_i(Q_0)}$$

i=1,2,3

$$\hat{\Pi}_i(Q) = \frac{\alpha_i}{4\pi}\sum_p \beta_i^{(p)}\left(L_{s(p)}(Q^2/m_p^2)+\cdots\right)$$

"log-like" function:

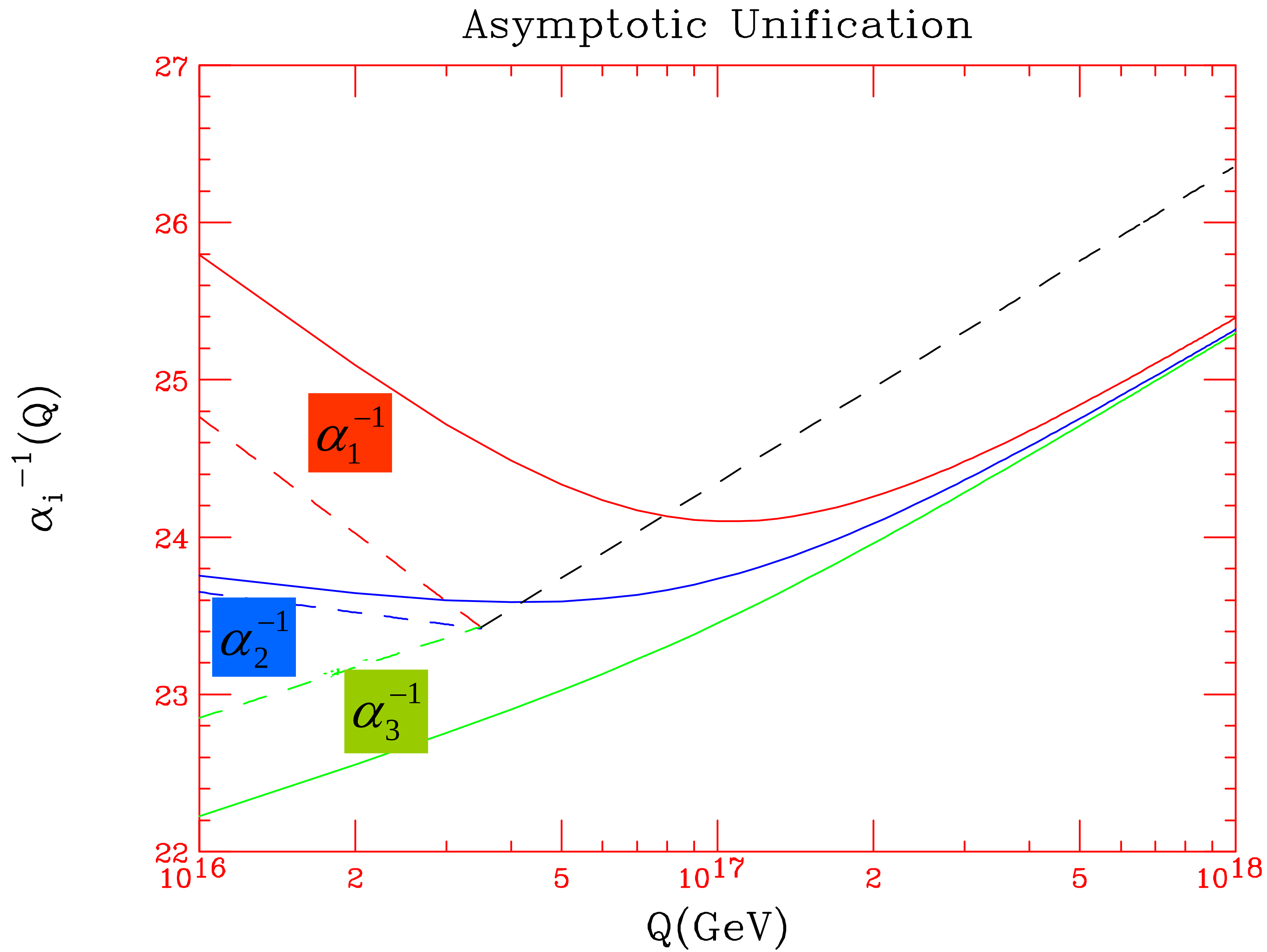
$$L_{s(p)} \approx \log(e^{\eta_p} + Q^2/m_p^2)$$

$$\eta_p = 8/3,\, 5/3,\, 40/21$$

For spin s(p) = 0, ½, and 1

⇨ Elegant and natural formalism for all threshold effects

(Automatically included)



Asymptotic Unification
$\alpha_i^{-1}(Q)$
27
26
25
24
23
22
10^{16}
2
5
10^{17}
2
5
10^{18}
Q(GeV)
α_1^{-1}
α_2^{-1}
α_3^{-1}

Unification in Physical Schemes

- Smooth analytic threshold behavior with automatic decoupling
- More directly reflects the unification of the forces
- Higher "unification" scale than usual

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$\Rightarrow$

$$\alpha_{eff}(p_1^2, p_2^2, p_3^2)$$

$$Q_{eff}^2(p_1^2, p_2^2, p_3^2)$$

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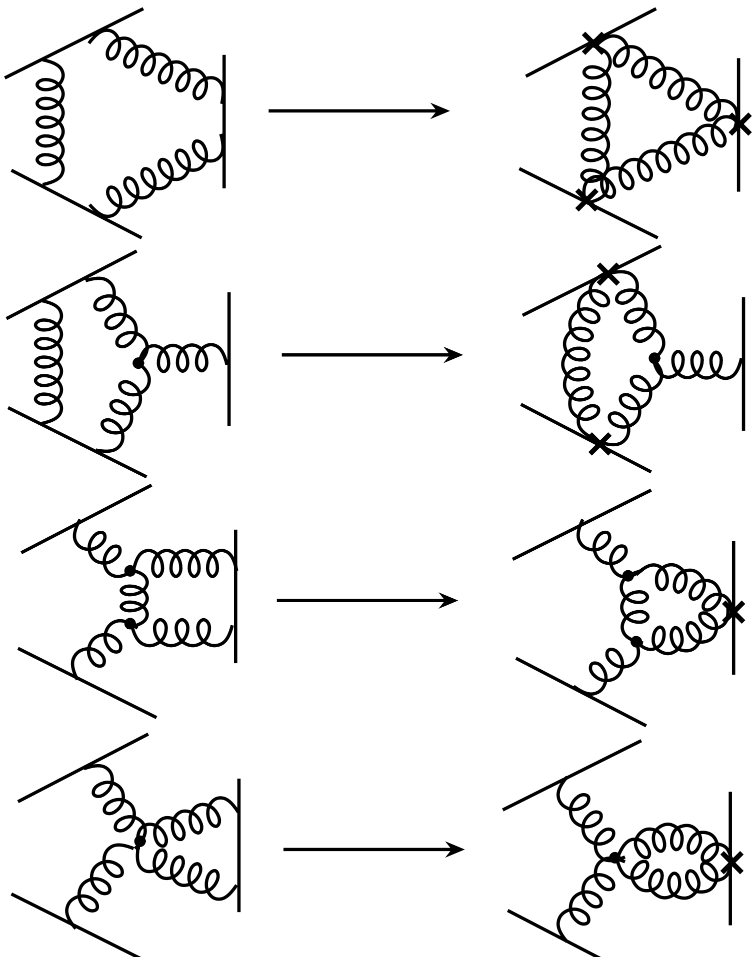
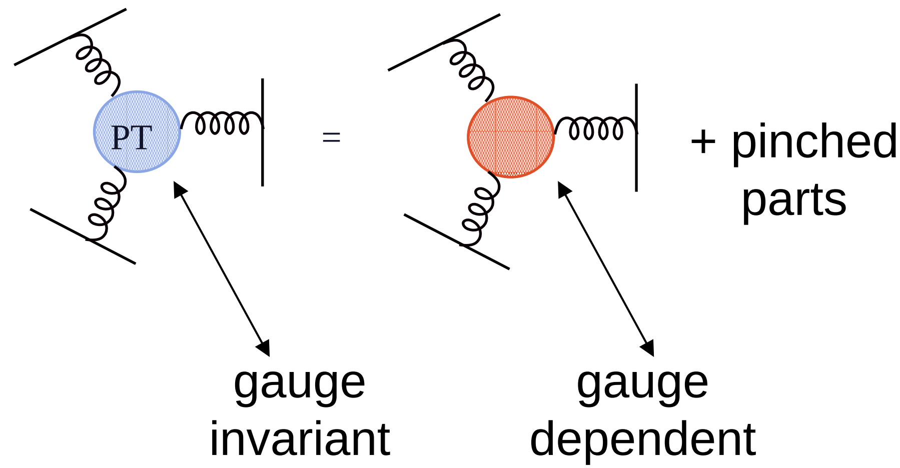
The Gauge Invariant Three Gluon Vertex

Cornwall and Papavassiliou performed the PT construction :



The “pinched” parts are added to the “regular” 3 gluon vertex



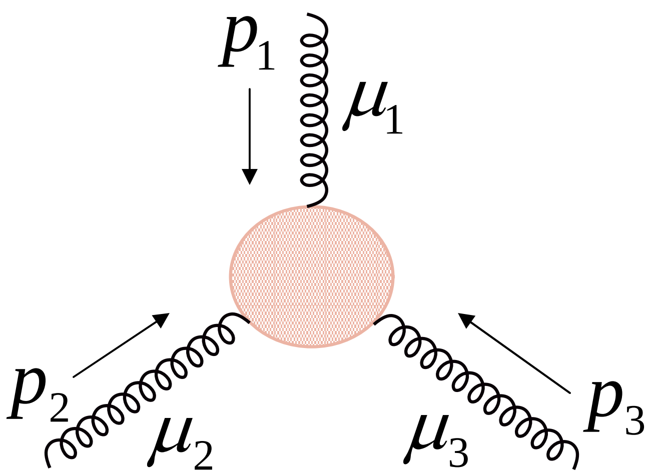
Later shown to = BFMFG

Integrals were not evaluated…

General Structure of the Three-Gluon Vertex

"THE FORM-FACTORS OF THE GAUGE-INVARIANT THREE-GLUON VERTEX"
M.B. and Stanley J. Brodsky. **hep-ph/0602199.** Submitted to PRD

$$\hat{\Gamma}_{\mu_1\mu_2\mu_3} =$$



3 index tensor $\hat{\Gamma}_{\mu_1\mu_2\mu_3}$ built out of $g_{\mu\nu}$ and p_1, p_2, p_3

with $p_1 + p_2 + p_3 = 0$

⇒ 14 basis tensors and form factors

General Structure of the Three-Gluon Vertex

Simple (QED-like) Ward ID

$$p_3^{\mu_3}\hat{\Gamma}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3) = t_{\mu_1\mu_2}(p_2)\left[1+\hat{\Pi}(p_2)\right] - t_{\mu_1\mu_2}(p_1)\left[1+\hat{\Pi}(p_1)\right]$$

where $t_{\mu\nu}(p) = p^2 g_{\mu\nu} - p_\mu p_\nu$

⇒ One form factor always = 0 (not obvious) ⇒ 13 nonzero form factors

Convenient Tensor Bases

Physical $\pm$ Basis

- Written in terms of linear combinations of momenta called "+" and "-" momenta such that $p_+ \cdot V_{ext} = 0$

  by elementary Ward IDs

- Maximum # of FF's vanish when in a physical matrix element

- Good for real scattering problems

LT Basis

- Longitudinal (L) FF's :

$$p_3^{\mu_3} \cdot \hat{\Gamma}^{(L)}_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) \neq 0$$

- Transverse (T) FF's :

$$p_3^{\mu_3} \cdot \hat{\Gamma}^{(T)}_{\mu_1\mu_2\mu_3}(p_1, p_2, p_3) = 0$$

- Good for theoretical work and solving Ward ID

Complementary in their relation to current conservation (Ward ID's)

WARNING : DO NOT ATTEMPT TO READ THE FORMULA BELOW

1 The longitudinal form factors

$$\Pi(p^2) = ig^2\beta_0(d)\int\frac{d^dl}{(2\pi)^d}\frac{1}{l^2(l+p)^2}, \qquad (1)$$

where $\beta_0(d)$ is given by

$$\beta_0(d) = \frac{7d-6}{2(d-1)}C_2(G) - \frac{2(d-2)}{(d-1)}\sum_f T_f N_f - \frac{1}{(d-1)}\sum_s T_s N_s \qquad (2)$$

$$\begin{aligned}
\overline{A}_{12} &= \frac{\Pi(p_1^2)+\Pi(p_2^2)}{2} \\
\overline{B}_{12} &= \frac{\Pi(p_1^2)-\Pi(p_2^2)}{2} \\
\overline{C}_{12} &= \frac{\Pi(p_1^2)-\Pi(p_2^2)}{p_1^2-p_2^2} \\
\overline{S} &= 0,
\end{aligned} \qquad (3)$$

$$\begin{aligned}
\overline{A}_{12}(G) &= \frac{7d-6}{2(d-1)}\frac{1}{2}(J_1+J_2) \\
\overline{A}_{12}(Q) &= \frac{2-d}{(d-1)}\frac{1}{2}(J_1+J_2) \\
\overline{A}_{12}(S) &= -\frac{1}{2(d-1)}\frac{1}{2}(J_1+J_2),
\end{aligned} \qquad (4)$$

2 The transverse form factors

$$a = p_1^2 \quad b = p_2^2 \quad c = p_3^2 \quad \alpha = p_1\cdot p_2 \quad \beta = p_2\cdot p_3 \quad \gamma = p_3\cdot p_1 \qquad (5)$$

$$\begin{aligned}
\mathcal{Q} &= \alpha+\beta+\gamma \\
\mathcal{K} &= \alpha\beta+\beta\gamma+\gamma\alpha \\
\mathcal{P} &= \alpha\beta\gamma.
\end{aligned} \qquad (6)$$

$$\begin{aligned}
\Sigma_{Qa}(\overline{F}_{12}) &= -\frac{(d-10)}{2\mathcal{K}}\left(\alpha J + \frac{2\alpha(J_1-J_2)-\beta(J_2-J_3)-\gamma(J_3-J_1)}{\beta-\gamma}\right) \\
\Sigma_{Qa}(\overline{H}) &= \frac{(d-10)}{2}J.
\end{aligned} \qquad (1)$$

$$\begin{aligned}
\overline{F}_{12}(G) &= \frac{1}{2\mathcal{K}^2}\Bigg(J\Big(10\mathcal{P}+c(\mathcal{K}-7\alpha^2-2\beta\gamma)\Big) \\
&+ \left(1-\frac{(d+1)\beta\gamma}{\mathcal{K}}\right)\left(\mathcal{P}J+\alpha\gamma J_1+\alpha\beta J_2+\beta\gamma J_3-\frac{\mathcal{K}}{d-1}(J_1+J_2+J_3)\right) \\
&+ \frac{7d-6}{2(d-1)(d-2)}\Big[8\mathcal{P}+(d-4)\alpha c^2+(d-2)(4\mathcal{K}\alpha-c\beta\gamma)\Big]\frac{J_1-J_2}{a-b} \\
&+ \frac{1}{d-1}\left[2\beta^2(d-2)-\frac{5d-2}{2}\mathcal{K}-\frac{7d-6}{d-2}\alpha\beta-\frac{2d^2-15d+14}{d-2}\alpha\gamma\right](J_2-J_3) \\
&- \frac{1}{d-1}\left[2\gamma^2(d-2)-\frac{5d-2}{2}\mathcal{K}-\frac{7d-6}{d-2}\alpha\gamma-\frac{2d^2-15d+14}{d-2}\alpha\beta\right](J_3-J_1)\Bigg)
\end{aligned} \qquad (2)$$

and

$$\begin{aligned}
\overline{H}(G) &= -\frac{1}{2\mathcal{K}^2}\Bigg(J\left[8\mathcal{K}^2+(d-2)\mathcal{P}\mathcal{Q}+(d+1)\frac{abc\mathcal{P}}{\mathcal{K}}\right] \\
&+ \frac{d-2}{d-1}\Big[\alpha(\mathcal{K}-2\alpha\gamma)(J_1-J_2)+\beta(\mathcal{K}-2\beta\alpha)(J_2-J_3)+\gamma(\mathcal{K}-2\beta\gamma)(J_3-J_1)\Big] \\
&+ \mathcal{P}\frac{d+1}{d-1}\left[-(J_1+J_2+J_3)+\frac{d-1}{\mathcal{K}}(\alpha\gamma J_1+\alpha\beta J_2+\beta\gamma J_3)\right]\Bigg).
\end{aligned} \qquad (3)$$

WARNING : DO NOT ATTEMPT TO READ THE FORMULA BELOW

1 The Form Factors in the Physical Basis

$$
\begin{aligned}
\Sigma_{qa}(A_{12}) &= \frac{(d-10)}{4\mathcal{K}}\Big(abcJ + a\beta J_1 + b\gamma J_2 + c\alpha J_3\Big) \\
\Sigma_{qa}(B_{12}) &= \frac{(d-10)}{4\mathcal{K}}\Big((\gamma-\beta)abJ + (2\alpha+\beta)aJ_1 - (2\alpha+\gamma)bJ_2 - \alpha(\beta-\gamma)J_3\Big) \\
\Sigma_{qa}(C_{12}) &= -\frac{(d-10)}{4\mathcal{K}}\Big(\alpha cJ + \gamma J_1 + \beta J_2 + cJ_3\Big) \\
\Sigma_{qa}(D_{12}) &= 0 \qquad (1)\\
\Sigma_{qa}(H) &= 0 \\
\Sigma_{qa}(S) &= 0,
\end{aligned}
$$

and the remaining sums (for A_{23}, etc.) are related trivially by permutations $\alpha \to \beta \to \gamma \to \alpha$, $a \to b \to c \to a$, $J_1 \to J_2 \to J_3 \to J_1$.

The gluon form factors in d dimensions are

$$
\begin{aligned}
-4\mathcal{K}^2 A_{12}(G) &= abcJ(7\mathcal{K}+\beta\gamma) + aJ_1\Big(7\mathcal{K}\beta + \beta^2\gamma + \mathcal{K}\gamma\frac{d-2}{d-1}\Big) \qquad (2)\\
&+ bJ_2\Big(7\mathcal{K}\gamma + \beta\gamma^2 + \mathcal{K}\beta\frac{d-2}{d-1}\Big) + cJ_3\Big(7\mathcal{K}\alpha + \mathcal{P} + \mathcal{K}c\frac{d-2}{d-1}\Big) \\
-4\mathcal{K}^2 B_{12}(G) &= abJ(7\mathcal{K}+\beta\gamma)(\gamma-\beta) + aJ_1\Big(7\mathcal{K}\beta - b\gamma(\beta-\gamma) + \mathcal{K}\frac{2\alpha(7d-6)+\gamma}{d-1}\Big) \\
&- bJ_2\Big(7\mathcal{K}\gamma + a\beta(\beta-\gamma) + \mathcal{K}\frac{2\alpha(7d-6)+\beta}{d-1}\Big) + (\gamma-\beta)J_3\Big(7\mathcal{K}\alpha + \mathcal{P} + \mathcal{K}c\frac{d-2}{d-1}\Big)
\end{aligned}
$$

1

$$
\begin{aligned}
16\mathcal{K}^3 C_{12}(G) &= cJ\Big(3\mathcal{K}^2(10\alpha+c) + \mathcal{K}(\alpha^3 - 6c\beta\gamma) + \mathcal{P}\mathcal{K}(d+4) - \mathcal{P}(d+1)(\alpha^2+2\beta\gamma)\Big) \\
&+ J_1\Big(\mathcal{K}^2\Big[\frac{(3-2d)}{d-1}\alpha - \gamma\frac{(d^2-30d+24)}{d-1} + 3\beta\Big] + \mathcal{P}(d+1)\Big[\frac{\mathcal{K}}{d-1} + \gamma(2Q-3\alpha)\Big] \\
&+ \gamma\mathcal{K}\Big[\frac{(d^2-3)}{d-1}\alpha^2 - 6\beta^2 + \frac{(d^2-8d+9)}{d-1}\beta\gamma - 4\gamma^2\frac{(d-2)}{d-1}\Big]\Big) \qquad (1)\\
&+ J_2\Big(\mathcal{K}^2\Big[\frac{(3-2d)}{d-1}\alpha - \beta\frac{(d^2-30d+24)}{d-1} + 3\gamma\Big] + \mathcal{P}(d+1)\Big[\frac{\mathcal{K}}{d-1} + \beta(2Q-3\alpha)\Big] \\
&+ \beta\mathcal{K}\Big[\frac{(d^2-3)}{d-1}\alpha^2 - 6\gamma^2 + \frac{(d^2-8d+9)}{d-1}\beta\gamma - 4\beta^2\frac{(d-2)}{d-1}\Big]\Big) \\
&+ cJ_3\Big(\frac{(30d-31)}{d-1}\mathcal{K}^2 + \mathcal{K}\Big[\alpha^2 - 4c^2\frac{(d-2)}{d-1} - \frac{(d^2-4d+1)}{d-1}\beta\gamma\Big] + (d+1)\mathcal{P}(2Q-3\alpha)\Big)
\end{aligned}
$$

$$
\begin{aligned}
16\mathcal{K}^3 D_{12}(G) &= ab(a-b)J\Big(\mathcal{K}(Q+2\alpha) - (d+1)\mathcal{P}\Big) - aJ_1\Big(\frac{d^2-4}{d-1}\mathcal{K}^2 \qquad (2)\\
&+ \mathcal{K}\Big[\beta^2 - \beta\gamma\frac{(d^2-3)}{d-1} - \alpha\beta\frac{(d^2-4d+1)}{d-1} + 4\alpha^2\frac{(d-2)}{d-1}\Big] - (2\alpha+\beta)\mathcal{P}(d+1)\Big) \\
&+ bJ_2\Big(\frac{(d^2-4)}{d-1}\mathcal{K}^2 + \mathcal{K}\Big[\gamma^2 - \beta\gamma\frac{(d^2-3)}{d-1} - \alpha\gamma\frac{(d^2-4d+1)}{d-1} + 4\alpha^2\frac{(d-2)}{d-1}\Big] \\
&- (2\alpha+\gamma)\mathcal{P}(d+1)\Big) + (a-b)J_3\Big(\frac{(4d-7)}{d-1}\mathcal{K}^2 + \mathcal{K}\Big[3\alpha^2 - \beta\gamma\frac{(d^2-3)}{d-1}\Big] - \alpha\mathcal{P}(d+1)\Big)
\end{aligned}
$$

1

THESE ARE THE RESULTS FOR ALL FORM FACTORS IN d-DIMENSIONS…

WARNING : DO NOT ATTEMPT TO READ THE FORMULA BELOW

$$
\begin{aligned}
16\mathcal{K}^3 H(G) = {} & abcJ\Big(\mathcal{P}(d+1) - \mathcal{K}Q\Big) \\
+ \quad & aJ_1\left(\frac{3-2d}{d-1}\mathcal{K}^2 + \left[\frac{d^2-3}{d-1}\alpha\gamma - \beta^2\right]\mathcal{K} + (d+1)\beta\mathcal{P}\right) \\
+ \quad & bJ_2\left(\frac{3-2d}{d-1}\mathcal{K}^2 + \left[\frac{d^2-3}{d-1}\alpha\beta - \gamma^2\right]\mathcal{K} + (d+1)\gamma\mathcal{P}\right) \qquad (1) \\
+ \quad & cJ_3\left(\frac{3-2d}{d-1}\mathcal{K}^2 + \left[\frac{d^2-3}{d-1}\beta\gamma - \alpha^2\right]\mathcal{K} + (d+1)\alpha\mathcal{P}\right)
\end{aligned}
$$

$$
\begin{aligned}
16\mathcal{K}^3 S(G) = {} & (a-b)(b-c)(c-a)J\Big(3\mathcal{K}Q - (d+1)\mathcal{P}\Big) \qquad (2) \\
+ \quad & (b-c)J_1\left(\frac{3\mathcal{K}^2}{d-1} + \mathcal{K}\left[4a^2\frac{d-2}{d-1} + \alpha\gamma\frac{d^2-4d+1}{d-1} - 3\beta^2\right] - (d+1)\mathcal{P}(2Q-3\beta)\right) \\
+ \quad & (c-a)J_2\left(\frac{3\mathcal{K}^2}{d-1} + \mathcal{K}\left[4b^2\frac{d-2}{d-1} + \alpha\beta\frac{d^2-4d+1}{d-1} - 3\gamma^2\right] - (d+1)\mathcal{P}(2Q-3\gamma)\right) \\
+ \quad & (a-b)J_3\left(\frac{3\mathcal{K}^2}{d-1} + \mathcal{K}\left[4c^2\frac{d-2}{d-1} + \beta\gamma\frac{d^2-4d+1}{d-1} - 3\alpha^2\right] - (d+1)\mathcal{P}(2Q-3\alpha)\right)
\end{aligned}
$$

$$
\begin{aligned}
A_0 &= \frac{1}{3}\Big(A_{12} + A_{23} + A_{31}\Big) \\
A_+ &= \frac{1}{3}\Big(A_{12} + \lambda A_{23} + \bar{\lambda} A_{31}\Big) \equiv A_1 + iA_2 \qquad (3) \\
A_- &= \frac{1}{3}\Big(A_{12} + \bar{\lambda} A_{23} + \lambda A_{31}\Big) \equiv A_1 - iA_2,
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
A_1 &= \frac{1}{3}\Big(A_{12} - \frac{1}{2}(A_{23} + A_{31})\Big) \\
A_2 &= \frac{\sqrt{3}}{6}\Big(A_{23} - A_{31}\Big). \qquad (4)
\end{aligned}
$$

1

$$
\begin{aligned}
A^*_\pm(a,b,c) &= A_\mp(-a,-b,-c) \\
B^*_0(a,b,c) &= B_0(-a,-b,-c) \\
B^*_\pm(a,b,c) &= B_\mp(-a,-b,-c) \\
C^*_0(a,b,c) &= -C_0(-a,-b,-c) \\
C^*_\pm(a,b,c) &= -C_\mp(-a,-b,-c) \qquad (1) \\
D^*_0(a,b,c) &= -D_0(-a,-b,-c) \\
D^*_\pm(a,b,c) &= -D_\mp(-a,-b,-c) \\
H^*(a,b,c) &= -H(-a,-b,-c) \\
S^*(a,b,c) &= -S(-a,-b,-c).
\end{aligned}
$$

$$
F(\lambda a, \lambda b, \lambda c) = F(a,b,c) = F(a/c, b/c, 1) \qquad \lambda > 0. \qquad (2)
$$

$$
\begin{aligned}
\Sigma_{QG}(A_0) &= \frac{(d-10)}{4\mathcal{K}}\Phi_0 \qquad \Sigma_{QG}(A_\pm) = 0 \\
\Sigma_{QG}(B_0) &= -\frac{(d-10)}{8\mathcal{K}}B_0(G) \\
\Sigma_{QG}(B_\pm) &= \frac{(d-10)}{36\mathcal{K}}\left(-3\Big(Q\Phi_2 + \mathcal{K}J(\beta-\gamma)\Big) \pm i\sqrt{3}\Big(Q\Phi_1 - \mathcal{K}J(Q-3\alpha)\Big)\right) \\
\Sigma_{QG}(C_0) &= \frac{(d-10)}{6}J \\
\Sigma_{QG}(C_\pm) &= \frac{(d-10)}{24\mathcal{K}}\Big(\Phi_1 \pm i\sqrt{3}\Phi_2\Big) \qquad (3) \\
\Sigma_{QG}(D_0) &= \Sigma_{QG}(D_\pm) = 0 \\
\Sigma_{QG}(H) &= \Sigma_{QG}(S) = 0
\end{aligned}
$$

$$
\begin{aligned}
\Phi_0 &= abcJ + a\beta J_1 + b\gamma J_2 + c\alpha J_3 \\
\Phi_1 &= (\mathcal{K} - 3\beta\gamma)J - 3\gamma J_1 - 3\beta J_2 + 3(\beta+\gamma)J_3 \qquad (4) \\
\Phi_2 &= \alpha(\beta-\gamma)J + (2\alpha+\gamma)J_1 - (2\alpha+\beta)J_2 + (\beta-\gamma)J_3.
\end{aligned}
$$

1

INCLUDING MASSIVE FERMIONS, SCALARS, AND GAUGE BOSONS…

WARNING : DO NOT ATTEMPT TO READ THE FORMULA BELOW

1 Mass Effects

$$\Pi_1(MQ) = \frac{d-2}{1-d} J_{1M} + 2M^2 \left(\frac{2J_{1M} - (d-2)T_M}{a(1-d)} \right)$$

$$\Pi_1(MS) = \frac{1}{2(1-d)} J_{1M} - M^2 \left(\frac{2J_{1M} - (d-2)T_M}{a(1-d)} \right) \qquad (1)$$

$$F(MS) = F(S)\Big|_M + \delta_{MS}(F)$$

$$F(MQ) = F(Q)\Big|_M + \delta_{MQ}(F) \qquad (2)$$

$$\delta_{MQ} = -2\delta_{MS}$$

$$\delta_{MS}(\overline{F}_{12}) = -\frac{2M^2}{\mathcal{K}^2} \left[\left(\frac{\mathcal{K} - 3\beta\gamma}{d-2} \right) J_M + \frac{\mathcal{P} - 2\alpha\mathcal{K} - \gamma^2(3\alpha + 2\gamma - \beta)}{a(\beta-\gamma)(d-1)} J_{1M} \right. \qquad (3)$$

$$\left. - \frac{\mathcal{P} - 2\alpha\mathcal{K} - \beta^2(3\alpha + 2\beta - \gamma)}{b(\beta-\gamma)(d-1)} J_{2M} - \frac{2c}{d-1} J_{3M} + \frac{(2-d)\mathcal{K}\alpha}{2(d-1)ab} T_M \right]$$

and

$$\delta_{MS}(\overline{H}) = \frac{2M^2}{\mathcal{K}^2} \left[\frac{3\mathcal{P}}{d-2} J_M - \frac{\mathcal{K} - 2\alpha\gamma}{d-1} J_{1M} - \frac{\mathcal{K} - 2\alpha\beta}{d-1} J_{2M} - \frac{\mathcal{K} - 2\beta\gamma}{d-1} J_{3M} + \frac{\mathcal{K}(d-2)}{2(d-1)} T_M \right]. \qquad (4)$$

$$\delta_{MS}(A_{12}) = \frac{M^2}{\mathcal{K}} \left[\frac{c\alpha}{d-2} J_M + \frac{\gamma J_{1M} + \beta J_{2M} + c J_{3M}}{d-1} \right] \qquad (5)$$

$$\delta_{MS}(B_{12}) = -\frac{M^2}{\mathcal{K}} \left[\frac{(a-b)\alpha}{d-2} J_M + \frac{(2\alpha+\gamma)J_{1M} - (2\alpha+\beta)J_{2M} + (\beta-\gamma)J_{3M}}{d-1} \right] \qquad (6)$$

$$\delta_{MS}(C_{12}) = \frac{M^2}{4\mathcal{K}^2} \left[\frac{8\mathcal{P} + \alpha^2 c + 3\beta^2(\alpha-\gamma) + 3\gamma^2(\alpha-\beta)}{d-2} J_M \right.$$

$$+ \frac{8\mathcal{P} + \alpha^2(\beta-\gamma) + \gamma^2(5\alpha + 7\beta + 4\gamma)}{a(d-1)} J_{1M} \qquad (1)$$

$$+ \frac{8\mathcal{P} + \alpha^2(\gamma-\beta) + \beta^2(5\alpha + 7\gamma + 4\beta)}{b(d-1)} J_{2M}$$

$$\left. + \frac{4c^2 + 2\beta\gamma - \mathcal{K}}{d-1} J_{3M} + \frac{(d-2)\mathcal{K}(\alpha^2 + 2\beta\gamma + 3\mathcal{K})}{(d-1)2ab} T_M \right]$$

$$\delta_{MS}(D_{12}) = \frac{M^2}{4\mathcal{K}^2} \left[\frac{(\mathcal{K} + 3\alpha^2)(a-b)}{d-2} J_M + \frac{\mathcal{K} + 4\alpha^2 + 2\alpha\gamma}{d-1} J_{1M} \right. \qquad (2)$$

$$\left. - \frac{\mathcal{K} + 4\alpha^2 + 2\alpha\beta}{d-1} J_{2M} + \frac{(2\beta\gamma - 3\mathcal{K})(a-b)}{c(d-1)} J_{3M} + \frac{(d-2)\mathcal{K}(a-b)}{2(d-1)c} T_M \right]$$

$$\delta_{MS}(H) = \frac{M^2}{4\mathcal{K}^2} \left[\frac{2\mathcal{P} + abc}{2-d} J_M + \frac{\mathcal{K} - 2\alpha\gamma}{d-1} J_{1M} + \frac{\mathcal{K} - 2\alpha\beta}{d-1} J_{2M} + \frac{\mathcal{K} - 2\beta\gamma}{d-1} J_{3M} \right.$$

$$\left. \left. - \frac{(d-2)\mathcal{K}}{2(d-1)} T_M \right) \right] \qquad (3)$$

$$\delta_{MS}(S) = -\frac{M^2}{4\mathcal{K}^2} \left[\frac{3(a-b)(b-c)(c-a)}{2-d} J_M + \frac{(4a^2 + 2\alpha\gamma - 3\mathcal{K})(b-c)}{a(d-1)} J_{1M} \right.$$

$$+ \frac{(4b^2 + 2\alpha\beta - 3\mathcal{K})(c-a)}{b(d-1)} J_{2M} + \frac{(4c^2 + 2\beta\gamma - 3\mathcal{K})(a-b)}{c(d-1)} J_{3M}$$

$$\left. \left. - \frac{(d-2)(a-b)(b-c)(c-a)\mathcal{K}}{2(d-1)abc} T_M \right) \right]. \qquad (4)$$

USING BOTH DIMENSIONAL REGULARIZATION AND DIMENSIONAL REDUCTION

Form Factors : Supersymmetric Relations

- Any form factor can be decomposed :

$$F = C_A F_G + 2\sum_f T_f F_Q + 2\sum_s T_s F_S$$

G = gluons
Q = quarks
S = scalars

C_A, T_f, T_s are color factors

- Individually, F_G, F_Q, F_S are complicated…

Form Factors : Supersymmetric Relations (Massless)

….but certain linear sums are simple :

$$\Sigma_{QG}(F) \equiv \frac{d-2}{2} F_Q + F_G \longrightarrow 0$$

for 7 of the 13 FF's (in physical basis)

→ Simple N=1 SUSY contribution in d=4

$$F_G + 4F_Q + (10-d)F_S = 0$$

For all FF's !!

→ N=4 SUSY in d=4 gives 0

These are off-shell generalizations of relations found in SUSY scattering amplitudes by Z. Bern, L.J. Dixon, D.C. Dunbar, and D.A. Kosower (NPB 425,435)

Form Factors : Supersymmetric Relations (Massive)

Equal masses for massive gauge bosons (MG), quarks (MQ), and scalars (MS)

$$F_{MG} + 4F_{MQ} + (9-d)F_{MS} = 0$$

1 d.o.f. “eaten” by MG

Massive gauge boson (MG) inside of loop might be the X and Y gauge bosons of SU(5), for example

External gluons remain unbroken and massless

$$\Sigma_{MQG}(F) \equiv \frac{d-1}{2} F_{MQ} + F_{MG}$$ is simple

Summary of Supersymmetric Relations

Massless	**Massive**
$F_G + 4F_Q + (10-d)F_S = 0$	$F_{MG} + 4F_{MQ} + (9-d)F_{MS} = 0$
$\Sigma_{QG}(F) \equiv \frac{d-2}{2} F_Q + F_G$	$\Sigma_{MQG}(F) \equiv \frac{d-1}{2} F_{MQ} + F_{MG}$
= simple	= simple

Form Factors : Consequences of Supersymmetric Relations

For any SUSY each of the 13 FF's are $\propto \beta_0$ even though only one FF is directly related to coupling renormalization

$$\beta_0(d) = \frac{7d-6}{2(d-1)} C_A - \frac{2(d-2)}{d-1} \sum_f T_f - \frac{1}{d-1} \sum_f T_s$$

$$\xrightarrow{d=4} \frac{11}{3} C_A - \frac{4}{3} T_f - \frac{1}{3} T_s$$

⇨ Contributions of gluons, quarks, and scalars have same functional form

Form Factors Without Supersymmetry (in d=4)

Seven FF's have

$$\Sigma_{QG}(F) = 0 \quad \Longrightarrow \quad F = \left(N_c - N_f + \frac{1}{2} N_s \right) F_G$$

FF of tree level tensor

$$A_0 \propto \left(\frac{11}{3} N_c - \frac{2(3d-8)}{3(d-2)} \sum_f T_f - \frac{2}{3(d-2)} \sum_s T_s \right)$$

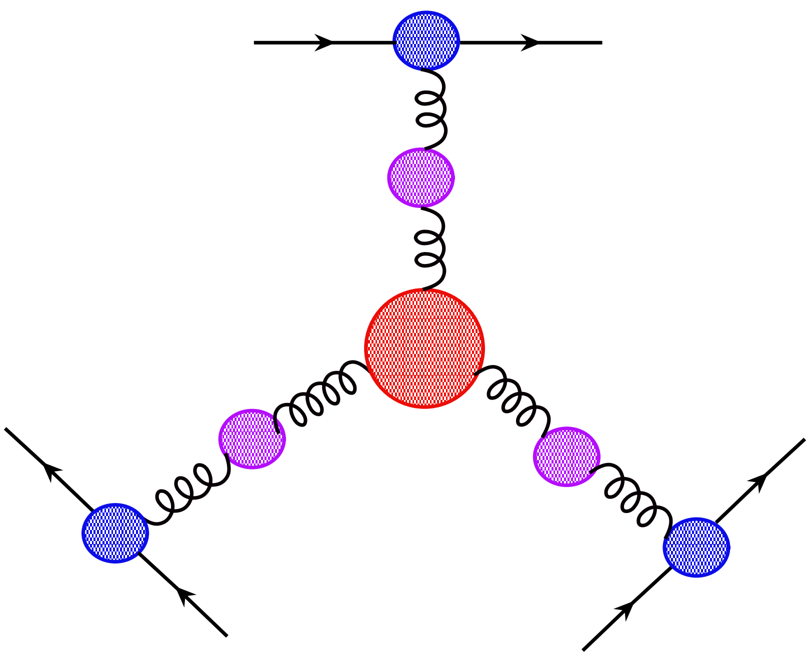
$$\xrightarrow{d\,=\,4} \quad \left(\frac{11}{3} N_c - \frac{4}{3} T_f - \frac{1}{3} T_s \right) = \beta_0$$

Another FF has $B_0 \propto \left(4N_c - N_f \right)$ $\qquad B_0(S) = 0$

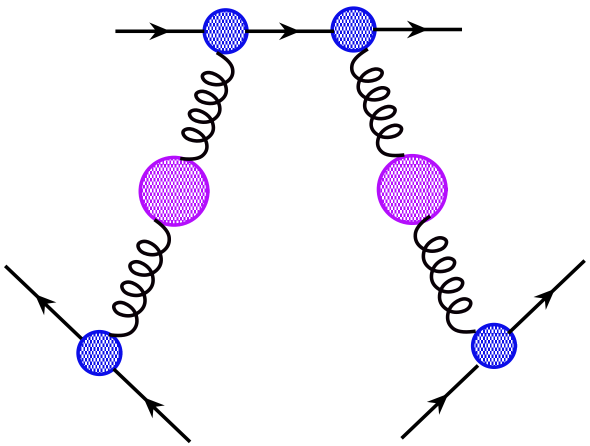
3 Gluon Vertex In Scattering Amplitudes

Pinch-Technique approach :

fully dress with gauge-invariant Green's functions



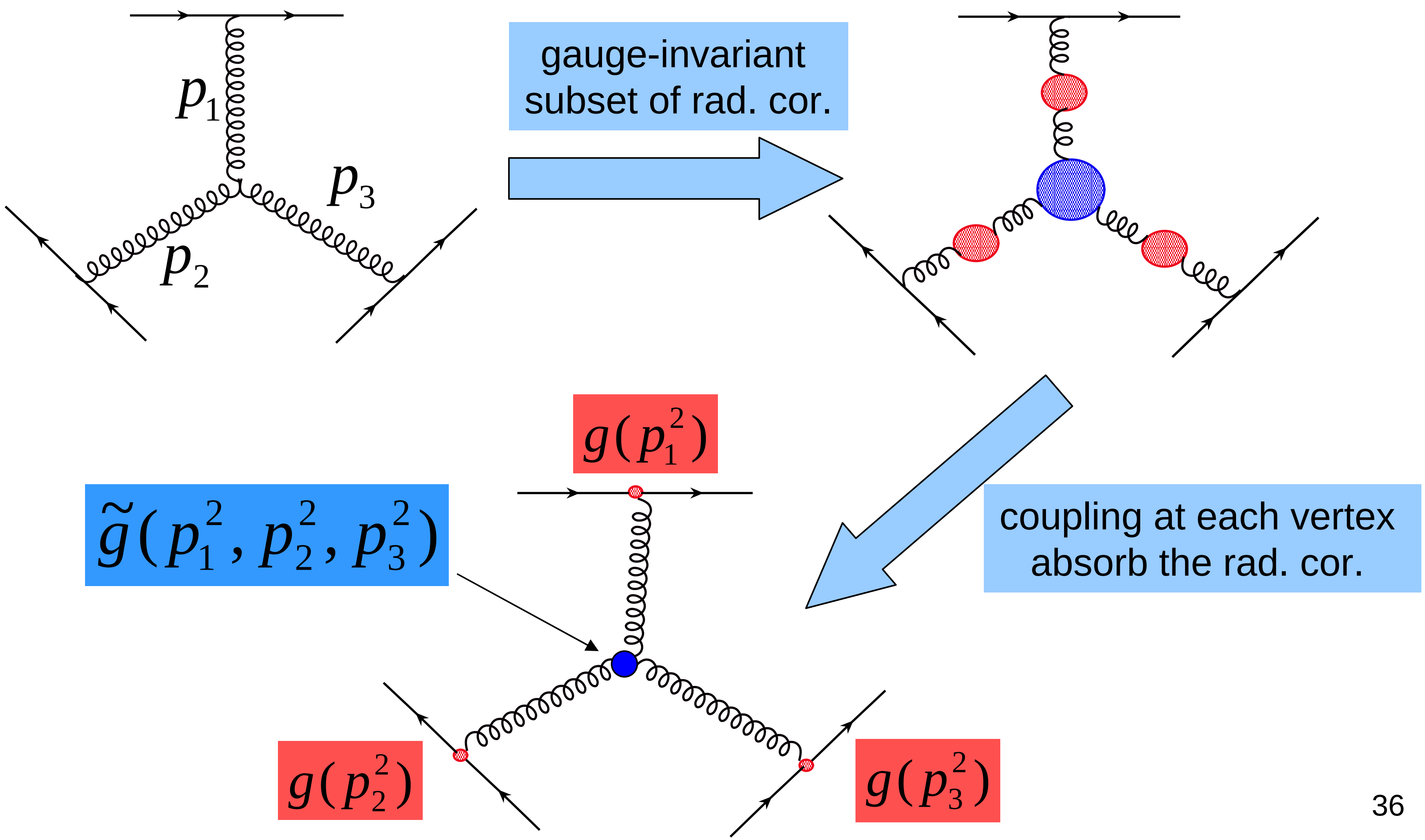
(A)



(B)

+ perms

Multi-scale Renormalization of the Three-Gluon Vertex



3 Gluon Vertex In Scattering Amplitudes

$$Amplitude = color \times vertices \times g(a)g(b)g(c)$$

$$\times g_{bare}\left[(1+A_0)\hat{t}_0 + A_+\hat{t}_+ + A_-\hat{t}_- + H\hat{h}\right]$$

$g_{bare}(1+A_0)$: $\tilde{g}(a,b,c)$

$A_+\hat{t}_+ + A_-\hat{t}_- + H\hat{h}$: Other tensors and form factors

$\hat{t}_0$: Tree level tensor structure :

$$\hat{t}_0 = (p_1 - p_2)^{\mu_3} g^{\mu_1\mu_2} + (p_2 - p_3)^{\mu_1} g^{\mu_2\mu_3} + (p_3 - p_1)^{\mu_2} g^{\mu_3\mu_1}$$

Form factors A_0, A_+, A_-, H depend on these $\begin{cases} a = p_1^2 \\ b = p_2^2 \\ c = p_3^2 \end{cases}$

3 Scale Effective Charge

$$\tilde{\alpha}(a,b,c) \equiv \frac{\tilde{g}^2(a,b,c)}{4\pi}$$

(First suggested by H.J. Lu)

$$\frac{1}{\tilde{\alpha}(a,b,c)} = \frac{1}{\alpha_{bare}} + \frac{1}{4\pi}\beta_0\left(L(a,b,c) - \frac{1}{\varepsilon} + \cdots \right)$$

$$\frac{1}{\tilde{\alpha}(a,b,c)} = \frac{1}{\tilde{\alpha}(a_0,b_0,c_0)} + \frac{1}{4\pi}\beta_0\left[L(a,b,c) - L(a_0,b_0,c_0)\right]$$

L(a,b,c) = 3-scale “log-like” function

L(a,a,a) = log(a)

3 Scale Log-Like Function

$$L(a,b,c) = \frac{1}{K}\left(\alpha\gamma \log a + \alpha\beta \log b + \beta\gamma \log c - abc\bar{J}(a,b,c)\right) + \Omega$$

$$K = \alpha\beta + \beta\gamma + \gamma\alpha$$

Master triangle integral can be written in terms of Clausen functions

$$Cl_2(\theta) = \mathrm{Im}\, Li_2\left(e^{i\theta}\right)$$

$$\Omega \approx 3.125$$

$$\alpha = p_1 \cdot p_2 = \frac{1}{2}\left(c - a - b\right)$$

$$\beta = p_2 \cdot p_3 = \frac{1}{2}\left(a - b - c\right)$$

$$\gamma = p_3 \cdot p_1 = \frac{1}{2}\left(b - c - a\right)$$

$$a = p_1^2$$

$$b = p_2^2$$

$$c = p_3^2$$

3 Scale Effective Scale

$$L(a,b,c) \equiv \log\left(Q^2_{eff}(a,b,c)\right) + i\,\mathrm{Im}\,L(a,b,c)$$

Governs strength of the three-gluon vertex

$$\frac{1}{\tilde{\alpha}(a,b,c)} = \frac{1}{\tilde{\alpha}(a_0,b_0,c_0)} + \frac{1}{4\pi}\beta_0\left[L(a,b,c) - L(a_0,b_0,c_0)\right]$$

$$\hat{\Gamma}_{\mu_1\mu_2\mu_3} \propto \sqrt{\tilde{\alpha}(a,b,c)}$$

Generalization of the BLM scale to the 3-gluon vetex

Properties of the Effective Scale

$$Q^2_{eff}(a,b,c) = Q^2_{eff}(-a,-b,-c)$$

$$Q^2_{eff}(\lambda a,\lambda b,\lambda c) = |\lambda|\, Q^2_{eff}(a,b,c)$$

$$Q^2_{eff}(a,a,a) = |a|$$

$$Q^2_{eff}(a,-a,-a) \approx 5.54\,|a|$$

$$Q^2_{eff}(a,a,c) \approx 3.08\,|c| \quad \text{for} \quad |a| >> |c|$$

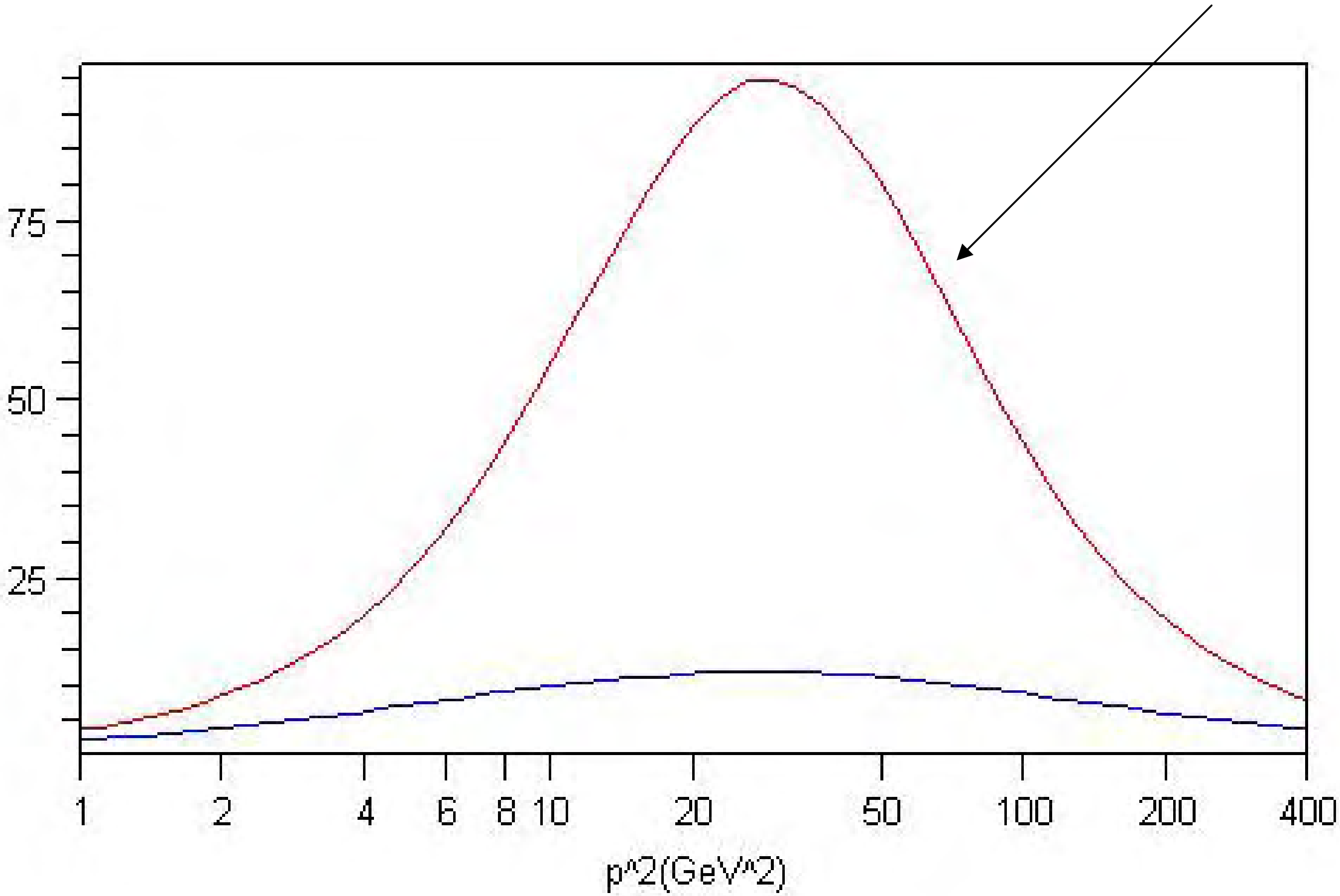
$$Q^2_{eff}(a,-a,c) \approx 22.8\,|c| \quad \text{for} \quad |a| >> |c|$$

$$Q^2_{eff}(a,b,c) \approx 22.8\,\frac{|bc|}{|a|} \quad \text{for} \quad |a| >> |b|, |c|$$

The Effective Scale

$Q_{eff}^2\left(10\,\mathrm{GeV}^2, 10\,\mathrm{GeV}^2, p^2\right)$

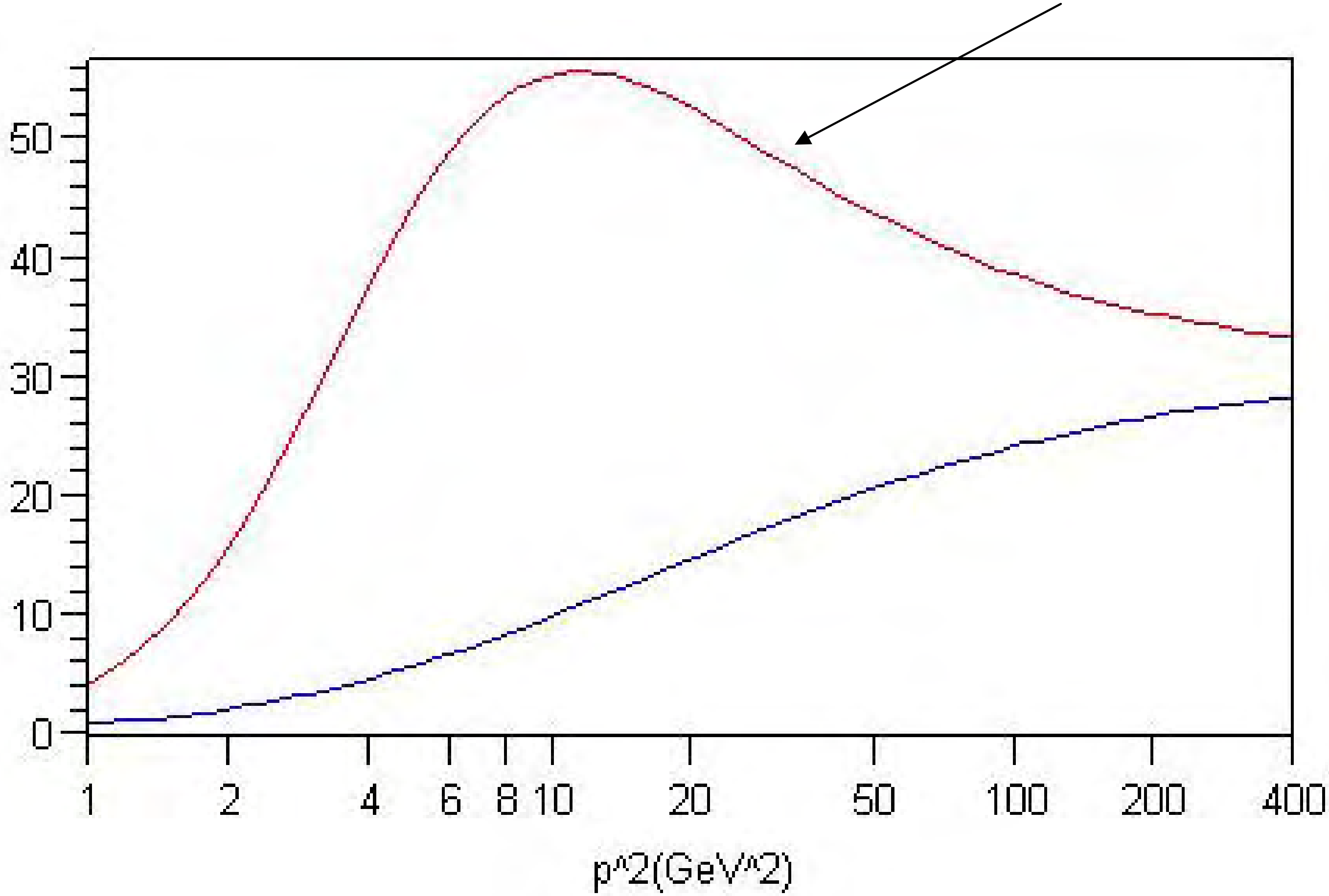
$Q_{eff}^2\left(-10\,\mathrm{GeV}^2, -10\,\mathrm{GeV}^2, p^2\right)$



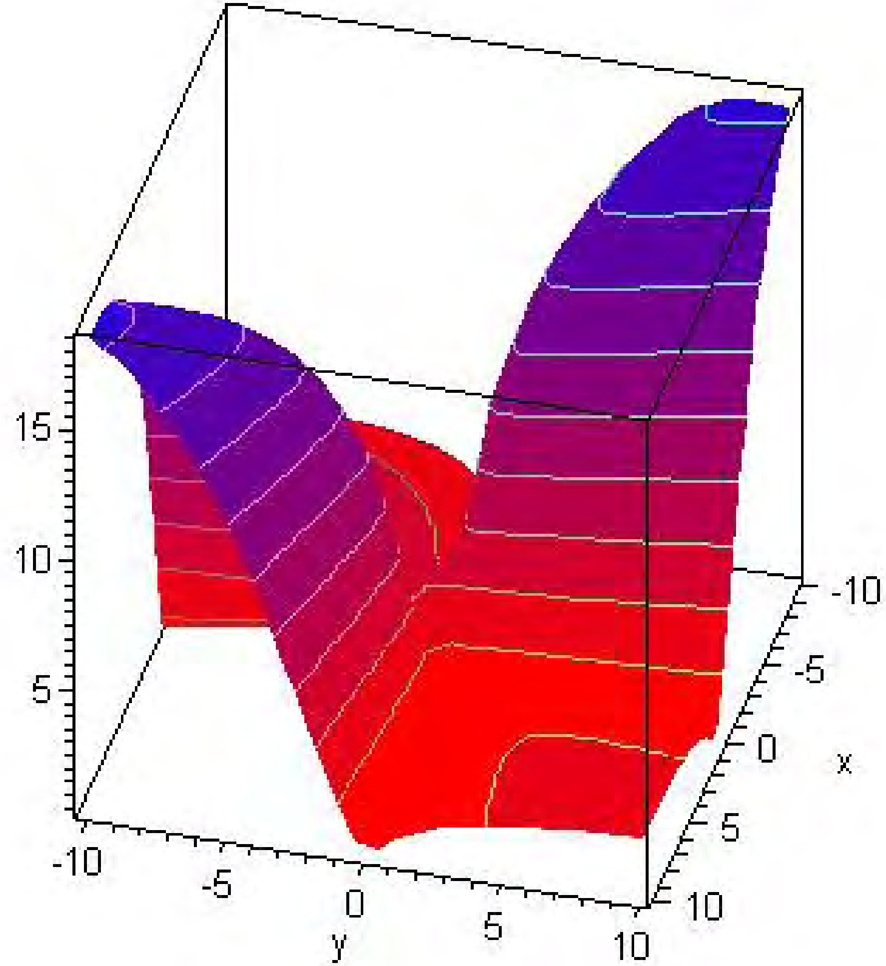
The Effective Scale

$Q^2_{eff}\left(10\,\mathrm{GeV}^2, p^2, p^2\right)$

$Q^2_{eff}\left(-10\,\mathrm{GeV}^2, p^2, p^2\right)$

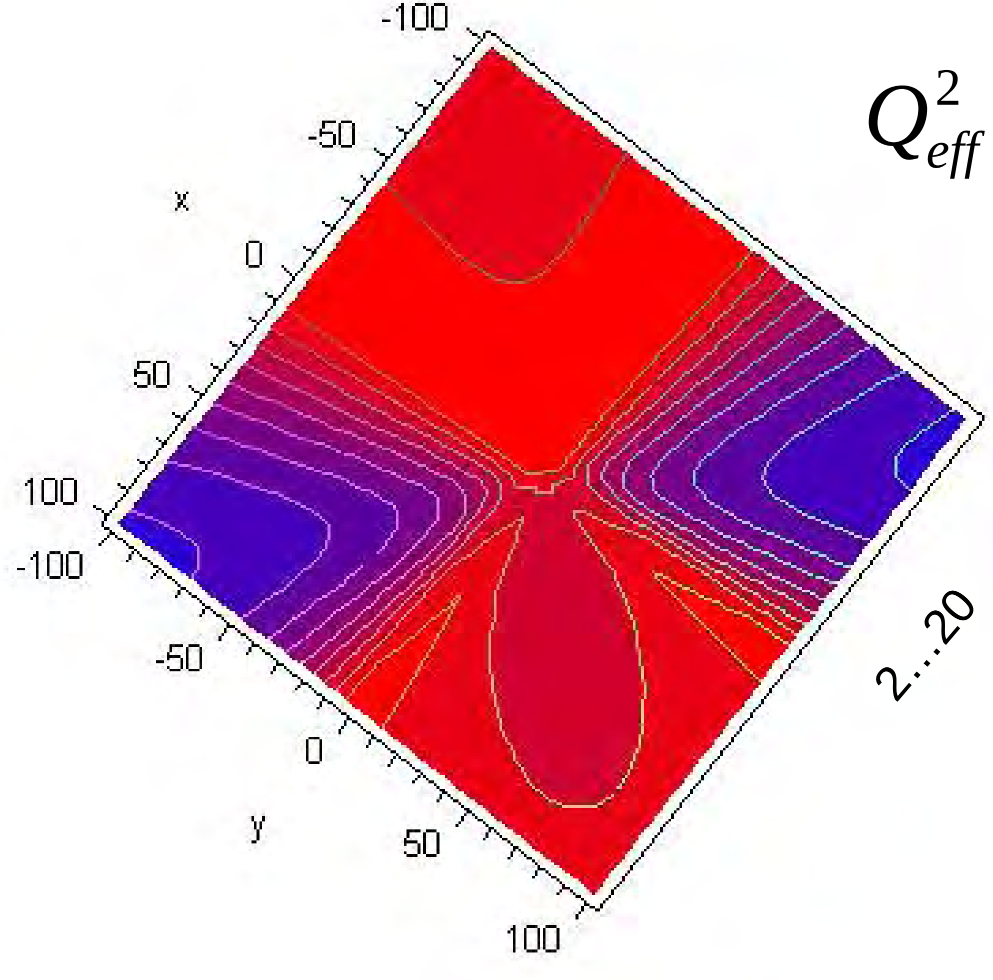


The Effective Scale



$$Q^2_{eff}(1, x, y)$$

The Effective Scale



$$Q^2_{eff}(1, x, y)$$

2…20

Mass Effects

Calculated for all form factors

SUSY relations $F_{MG} + 4F_{MQ} + (9-d)F_{MS} = 0$

FF of tree level tensor structure ⟺ Effective Charge

Massive “log-like” function : $L_{MQ}\left(\frac{a}{M^2},\frac{b}{M^2},\frac{c}{M^2}\right)$

$$L_{MQ}\left(\frac{a}{M^2},\frac{b}{M^2},\frac{c}{M^2}\right) \approx 5.125 \ \text{ for } \ M^2 >> |a|,|b|,|c|$$

$$L_{MQ}\left(\frac{a}{M^2},\frac{b}{M^2},\frac{c}{M^2}\right) \approx L(a,b,c) - \log M^2 \ \text{ for } \ M^2 << |a|,|b|,|c|$$

Massive Log-Like Function

$$L_{MQ}\left(\frac{a}{M^2},\frac{b}{M^2},\frac{c}{M^2}\right)=\frac{1}{\mathrm{K}}\left(\alpha\gamma\Lambda(a)+\alpha\beta\Lambda(b)+\beta\gamma\Lambda(c)-abc\overline{J}_M(a,b,c)\right)+\Omega$$

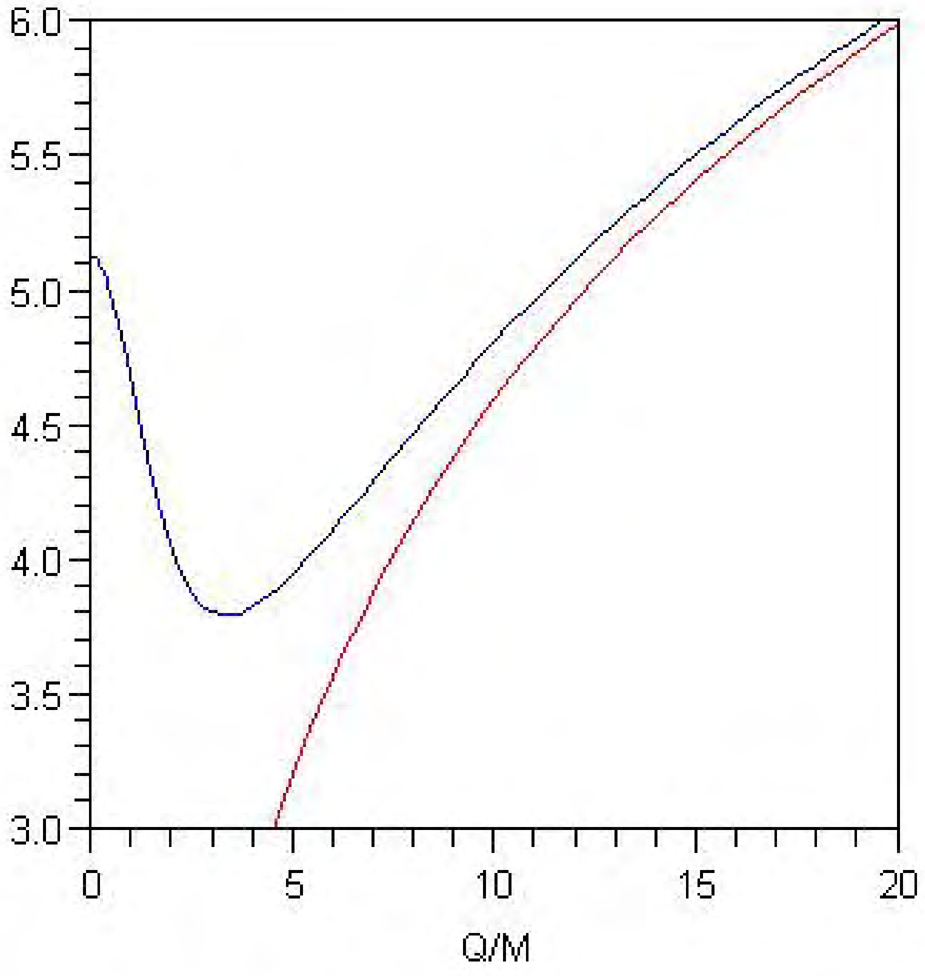
$$+2M^2\left(\frac{\Lambda(a)-2}{a}+\frac{\Lambda(b)-2}{b}+\frac{\Lambda(c)-2}{c}-\overline{J}_M(a,b,c)\right)$$

$$\Lambda(a)=\left\{\begin{array}{l}2v\tanh^{-1}\left(v^{-1}\right)\\ 2\overline{v}\tan^{-1}\left(\overline{v}^{-1}\right)\\ 2v\tanh^{-1}(v)-iv\pi\end{array}\right\}\text{ for }\left\{\begin{array}{l}a<0\\ 0<a<4M^2\\ a>4M^2\end{array}\right\}$$

$$v=\sqrt{1-\frac{4M^2}{a}}\qquad\qquad \overline{v}=\sqrt{\frac{4M^2}{a}-1}$$

Massive Master Triangle Integral (very complicated)

Symmetric Spacelike

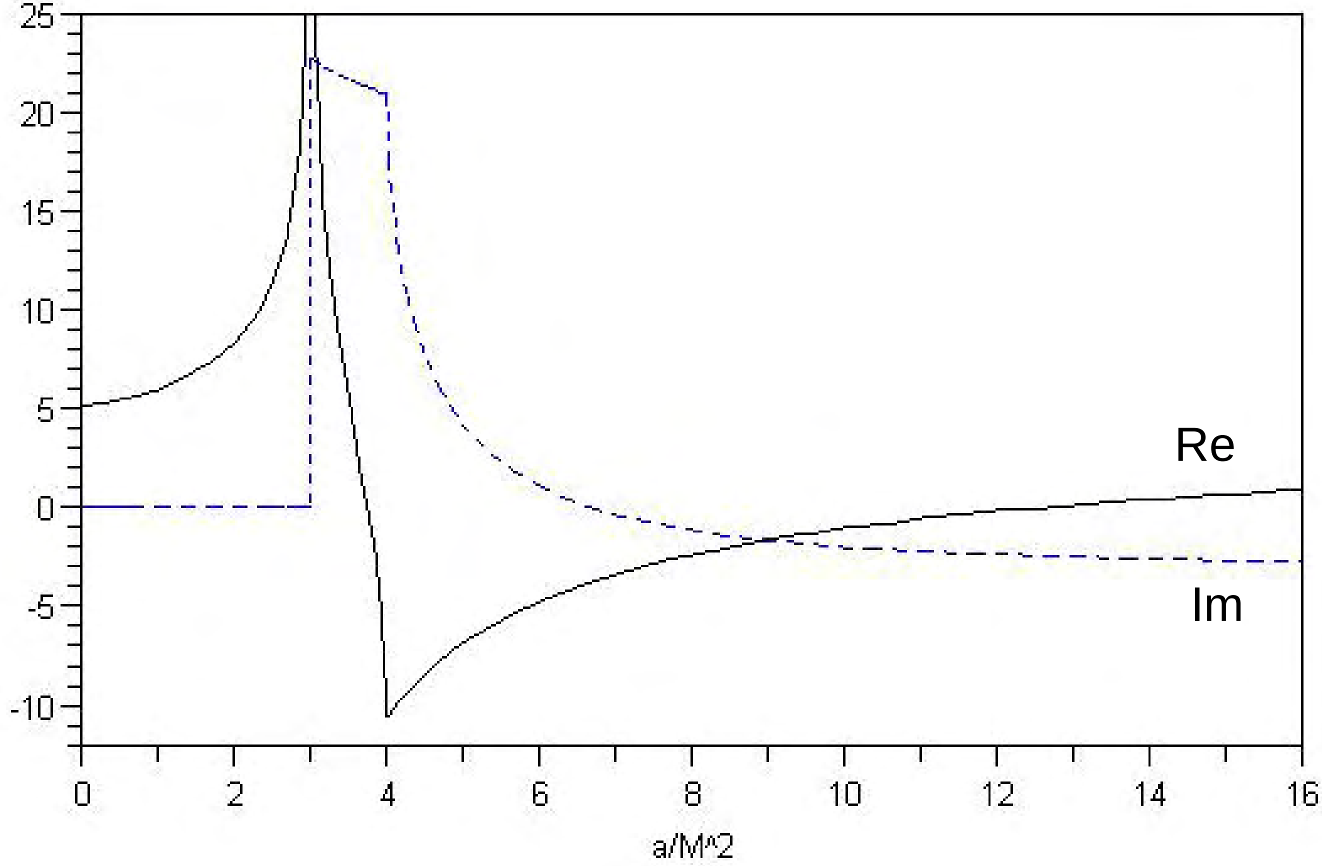


$$L_{MQ}\left(-\frac{Q^2}{M^2},-\frac{Q^2}{M^2},-\frac{Q^2}{M^2}\right)$$

$$\log\left(\frac{Q^2}{M^2}\right)$$

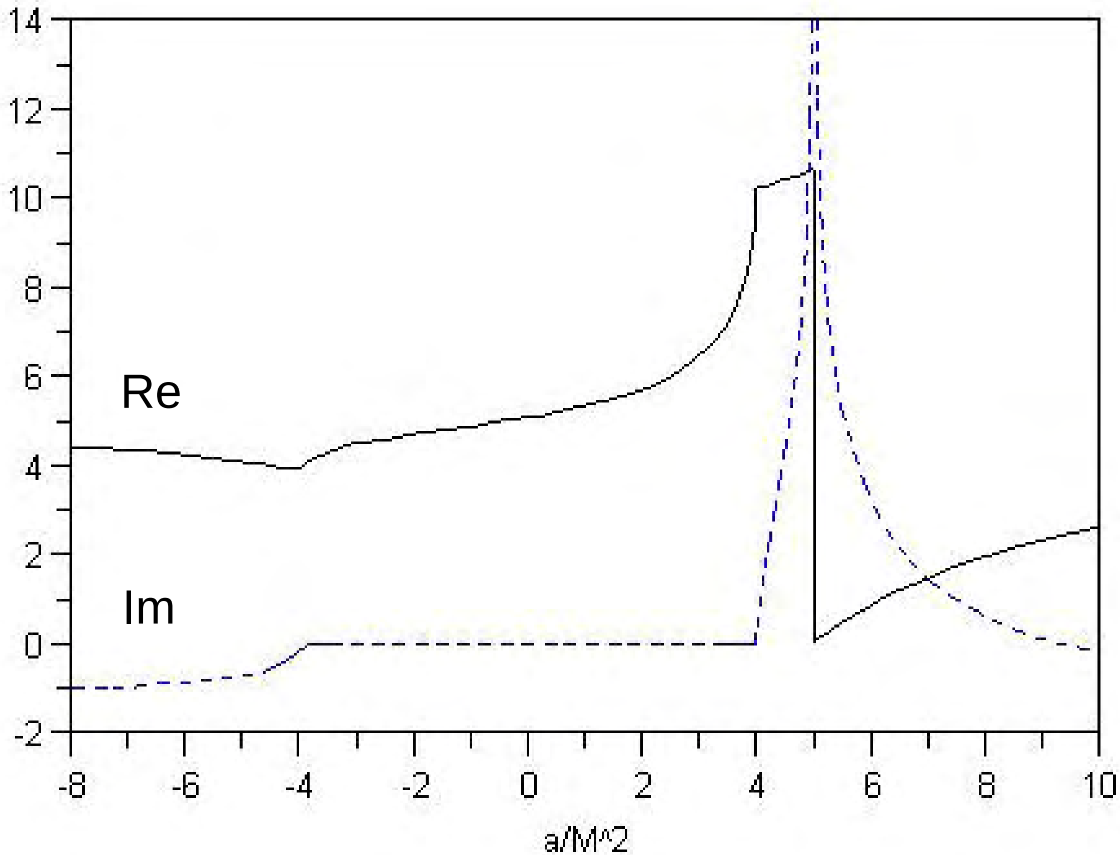
Symmetric Timelike

$$L_{MQ}\left(\frac{a}{M^2},\frac{a}{M^2},\frac{a}{M^2}\right)$$



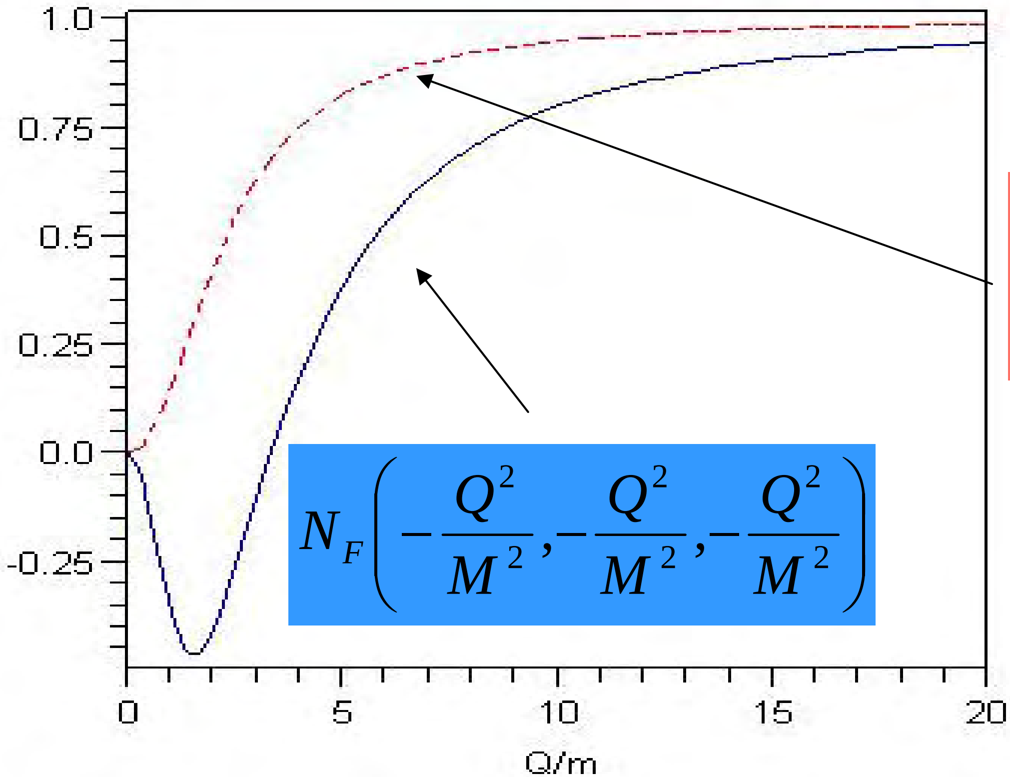
Symmetric Mixed Signature

$$L_{MQ}\left(\frac{a}{M^2},\frac{a}{M^2},-\frac{a}{M^2}\right)$$



Effective Number of Flavors

$$N_F\left(\frac{a}{M^2},\frac{b}{M^2},\frac{c}{M^2}\right)=-\frac{d}{d\log M^2}L_{MQ}\left(\frac{a}{M^2},\frac{b}{M^2},\frac{c}{M^2}\right)$$

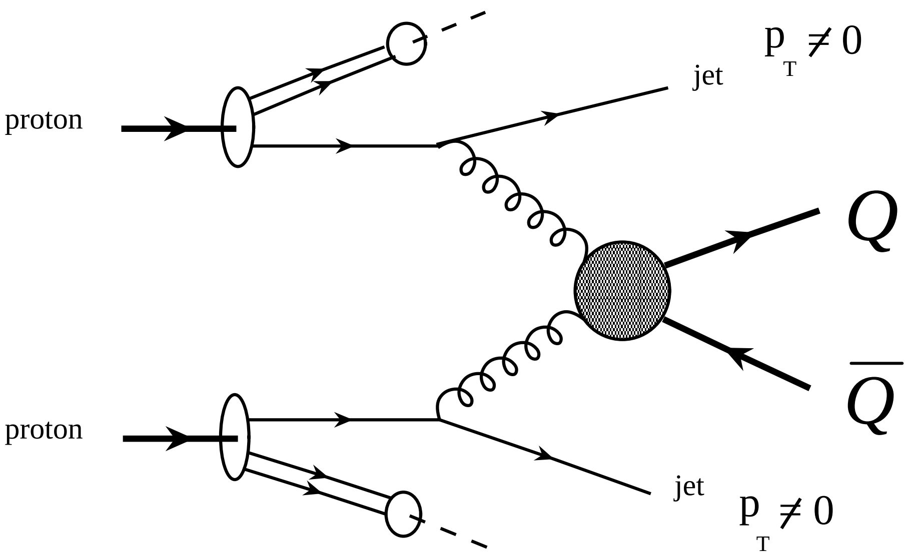


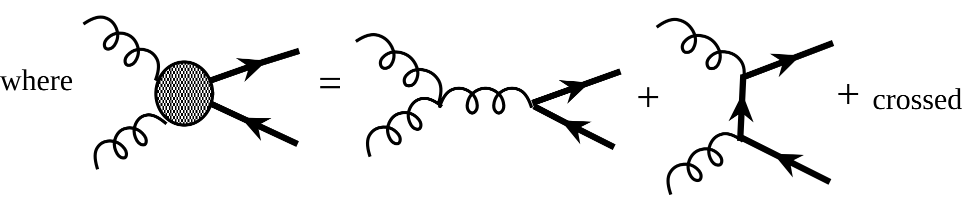
$$N_F\left(-\frac{Q^2}{M^2},-\frac{Q^2}{M^2},-\frac{Q^2}{M^2}\right)$$

$$n_f\left(\frac{Q^2}{M^2}\right)=-\frac{d}{d\log M^2}L_{1/2}\left(\frac{Q^2}{M^2}\right)$$

$$\approx\frac{1}{1+\frac{M^2}{Q^2}e^{5/3}}$$

Heavy Quark Hadro-production





- Preliminary calculation using (massless) results for tree level form factor
- Very low effective scale

→ much larger cross section than $\overline{MS}$ with scale $\mu_R = M_{Q\overline{Q}}$ or M_Q

- Future : repeat analysis using the full mass-dependent results and include all form factors

Expect that this approach accounts for most of the one-loop corrections

Outline

I. Preliminaries (13)

1. Motivations for Physical Renormalization
2. Lessons from QED
3. Applying these lessons to non-abelian theories

II. Gauge Coupling Unification in Physical Schemes (3)

III. The Gauge-Invariant Three Gluon Vertex (31)

1. General structure and symmetries
2. Supersymmetric relations
3. The form factors in physical processes
4. Effective charge and effective scale
5. Mass effects

$$\Rightarrow \quad \alpha_{eff}(p_1^2, p_2^2, p_3^2)$$
$$Q_{eff}^2(p_1^2, p_2^2, p_3^2)$$

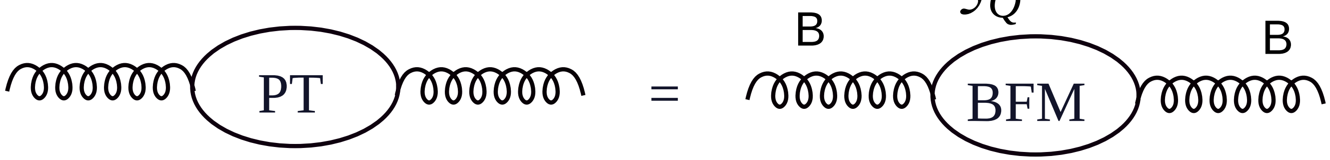
IV. Pinch-Technique Effective Charge at 2-loops (2)

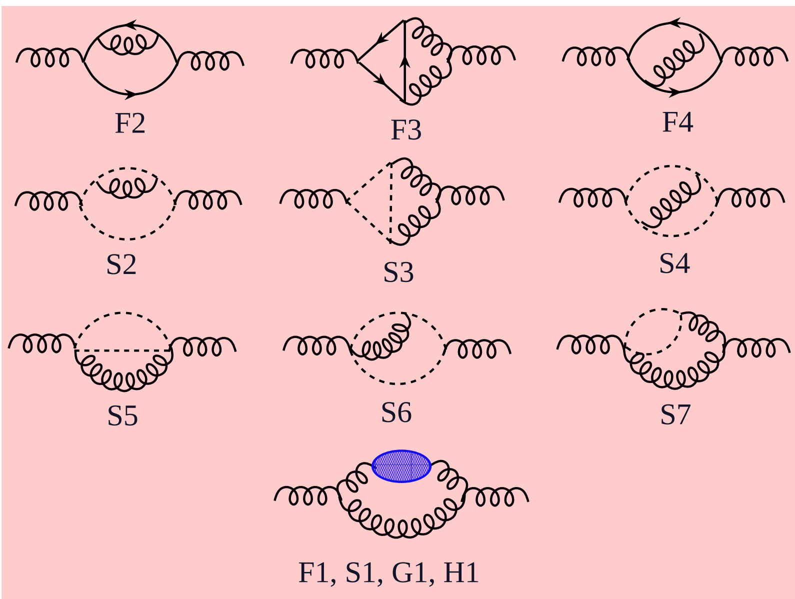
V. Future Directions and Summary (4)

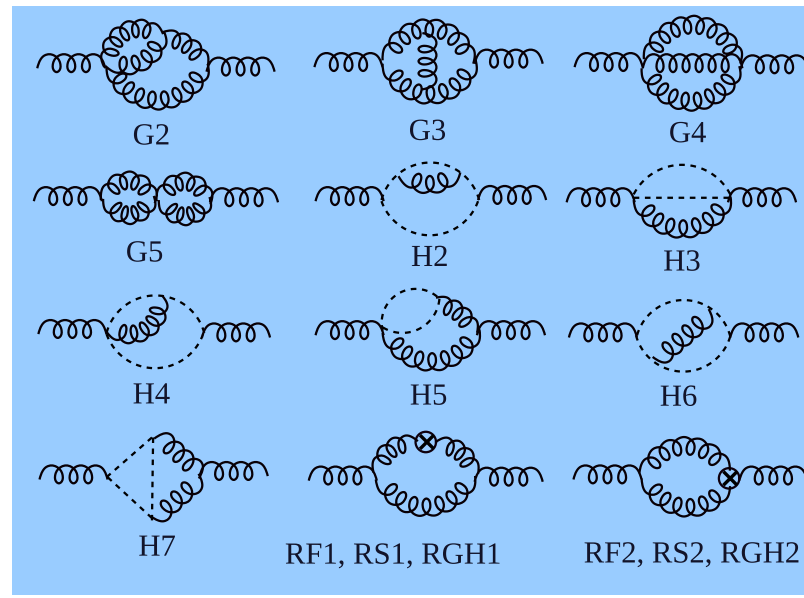
PT Self-Energy at Two-Loops

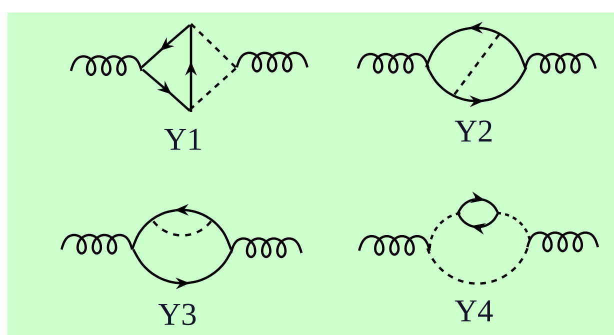
Papavassiliou showed :

$$\xi_Q = 1$$









PT Self-Energy at Two-Loops

- Finite terms give relation between
  $$\alpha_{PT}(Q^2) \text{ and } \alpha_{\overline{MS}}(Q^2)$$
- 3-loop beta function
- 2-loop longitudinal form factors of the three-gluon vertex (via the Ward ID)
- N=4 Supersymmetry gives a non-zero but UV finite contribution

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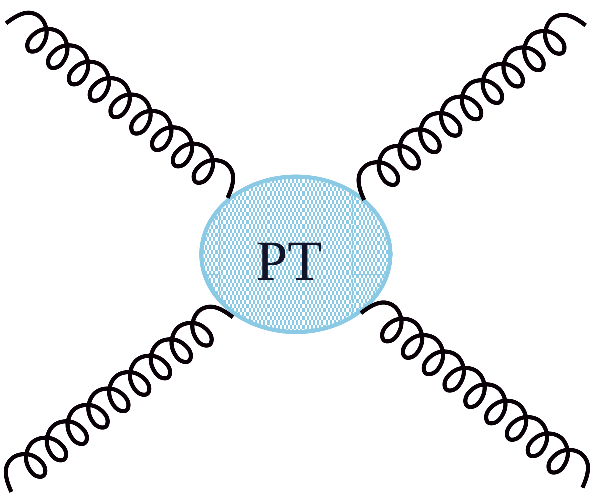
$\Rightarrow$ $\alpha_{eff}(p_1^2, p_2^2, p_3^2)$, $Q_{eff}^2(p_1^2, p_2^2, p_3^2)$

IV. Pinch-Technique Effective Charge at 2-loops (2)

V. Future Directions and Summary (4)

Future Directions

Gauge-invariant four gluon vertex

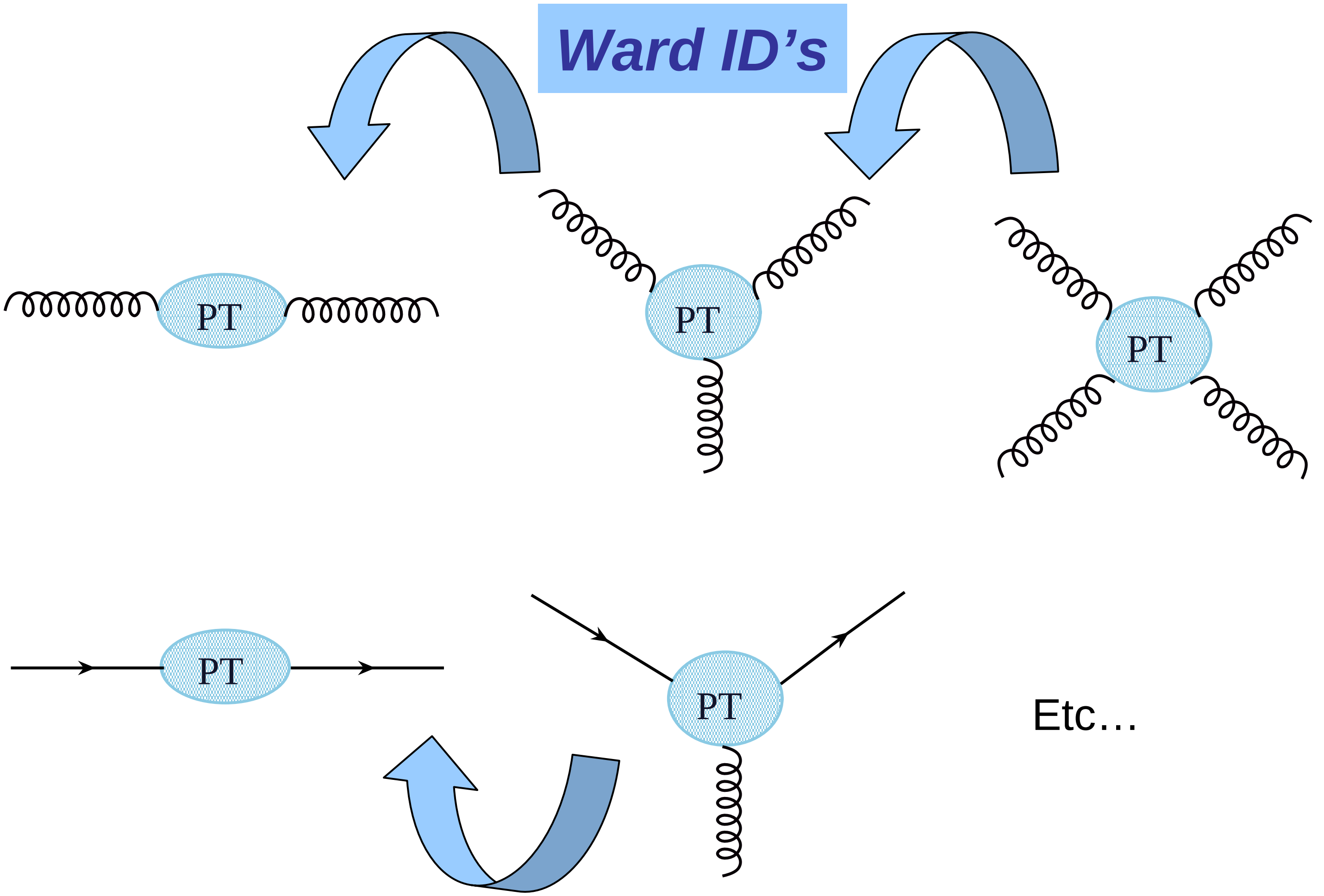


$$L_4(p_1, p_2, p_3, p_4)$$

$$Q^2_{4\,eff}(p_1, p_2, p_3, p_4)$$

Hundreds of form factors!

The Gauge-Invariant Family of Green's Functions



Future Directions

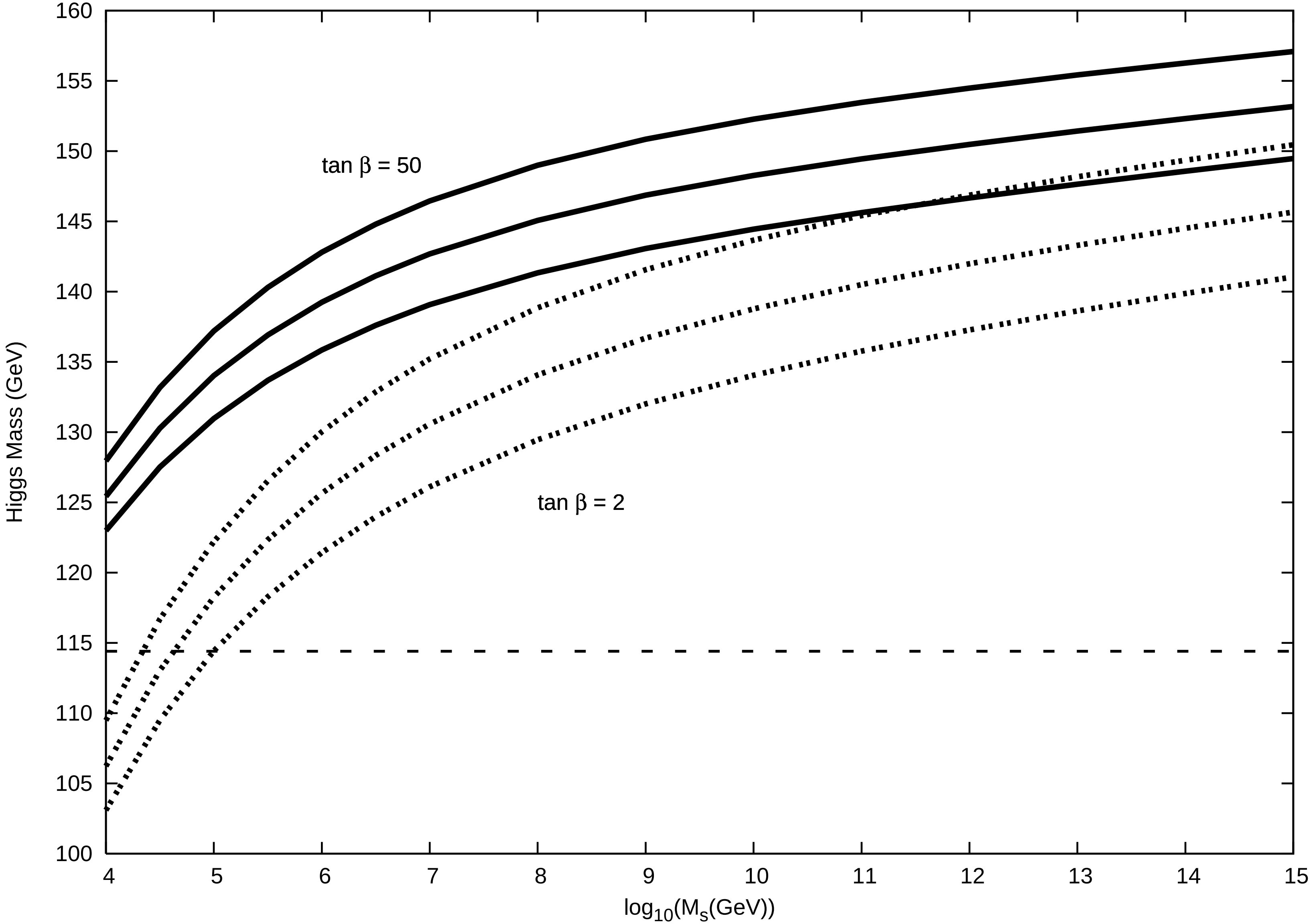
- Implement in Monte Carlo generator
- Gauge-invariant Standard Model triple gauge boson vertices
- Schwinger-Dyson Equations

Summary and Future

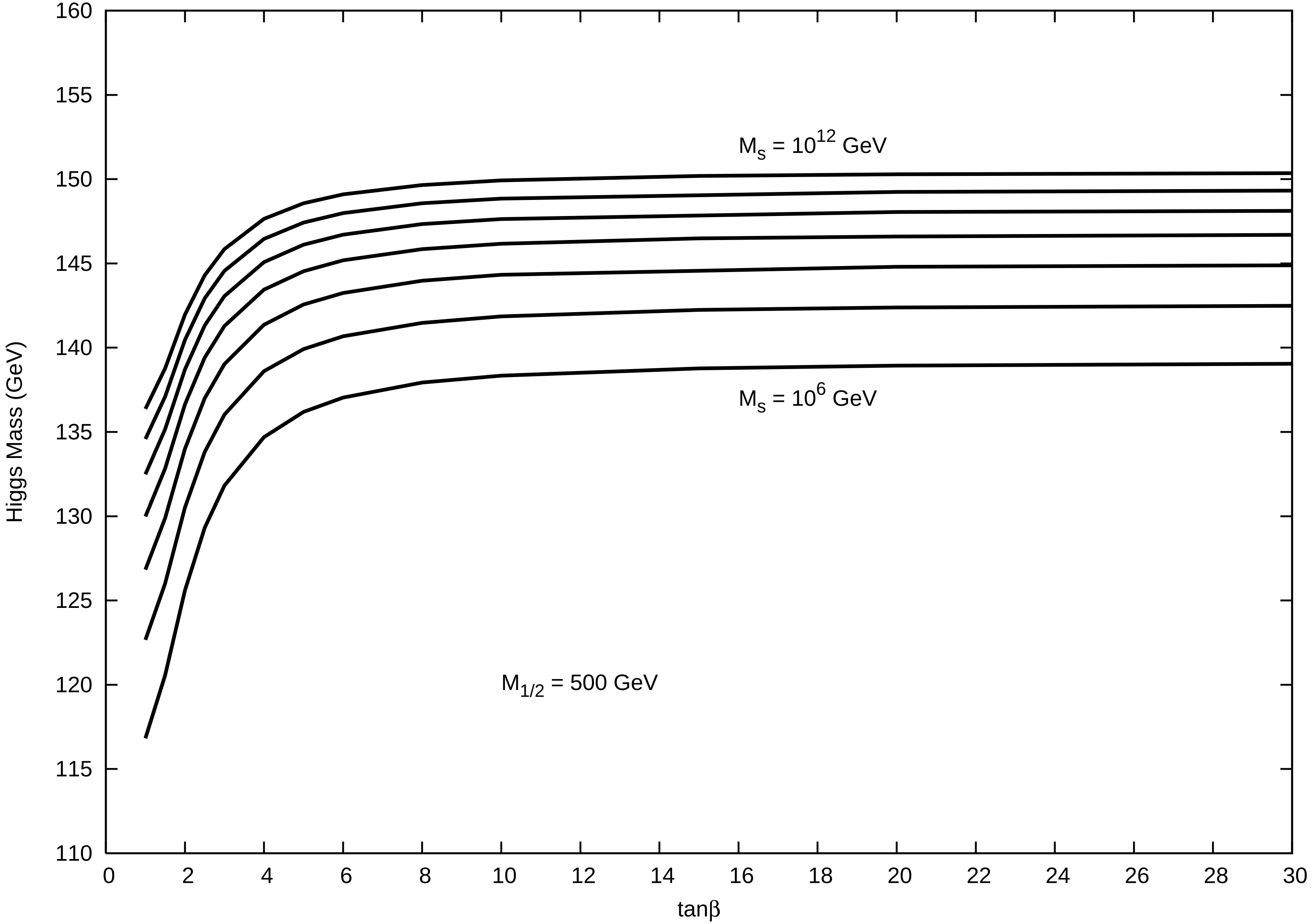
- ***Multi-scale analytic*** renormalization based on ***physical, gauge-invariant*** Green's functions

- ***Optimal*** improvement of perturbation theory with ***no scale-ambiguity*** since physical kinematic invariants are the arguments of the (multi-scale) couplings

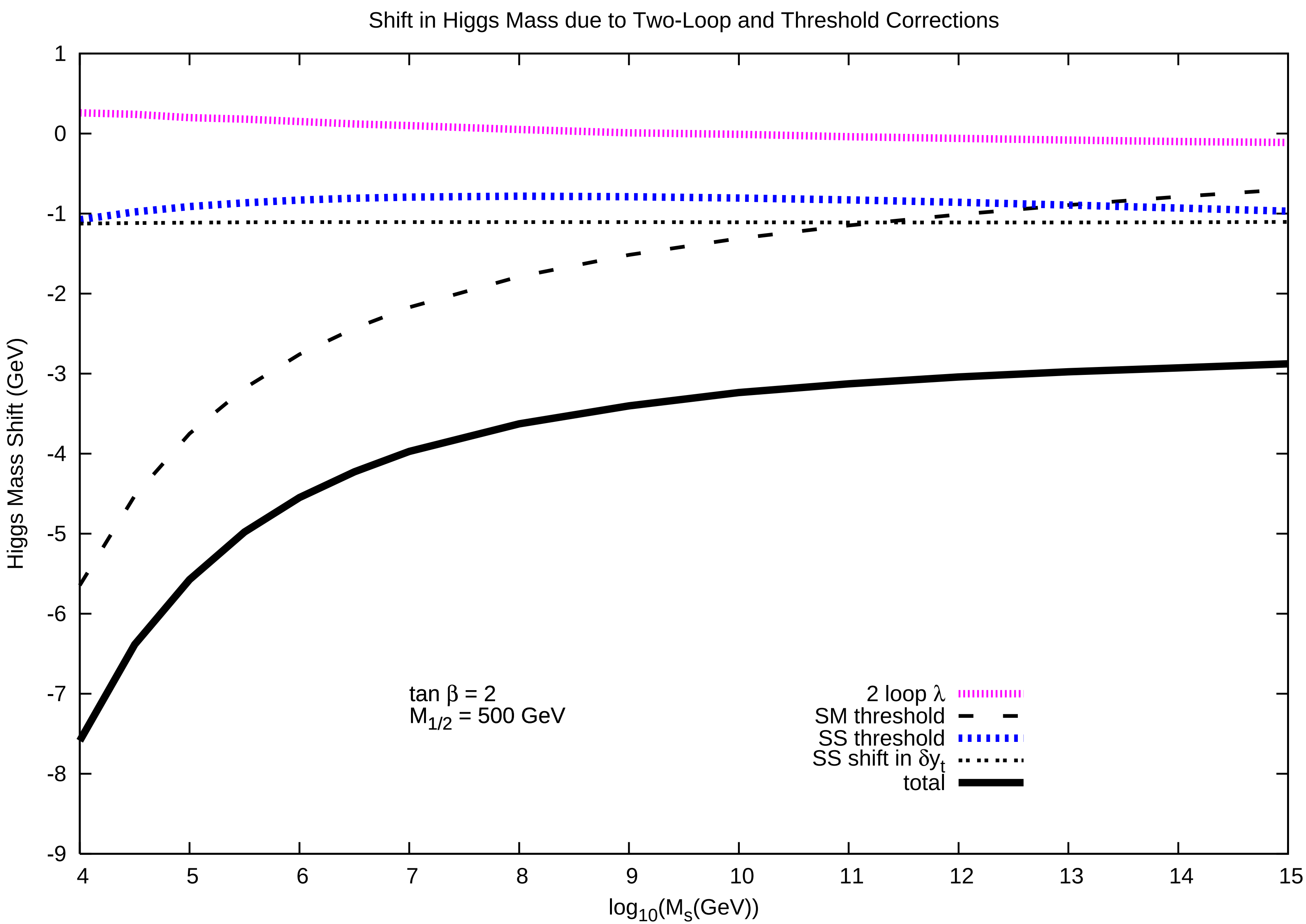
Two-loop Higgs Mass in Split Supersymmetry



Two-loop Higgs Mass in Split Supersymmetry

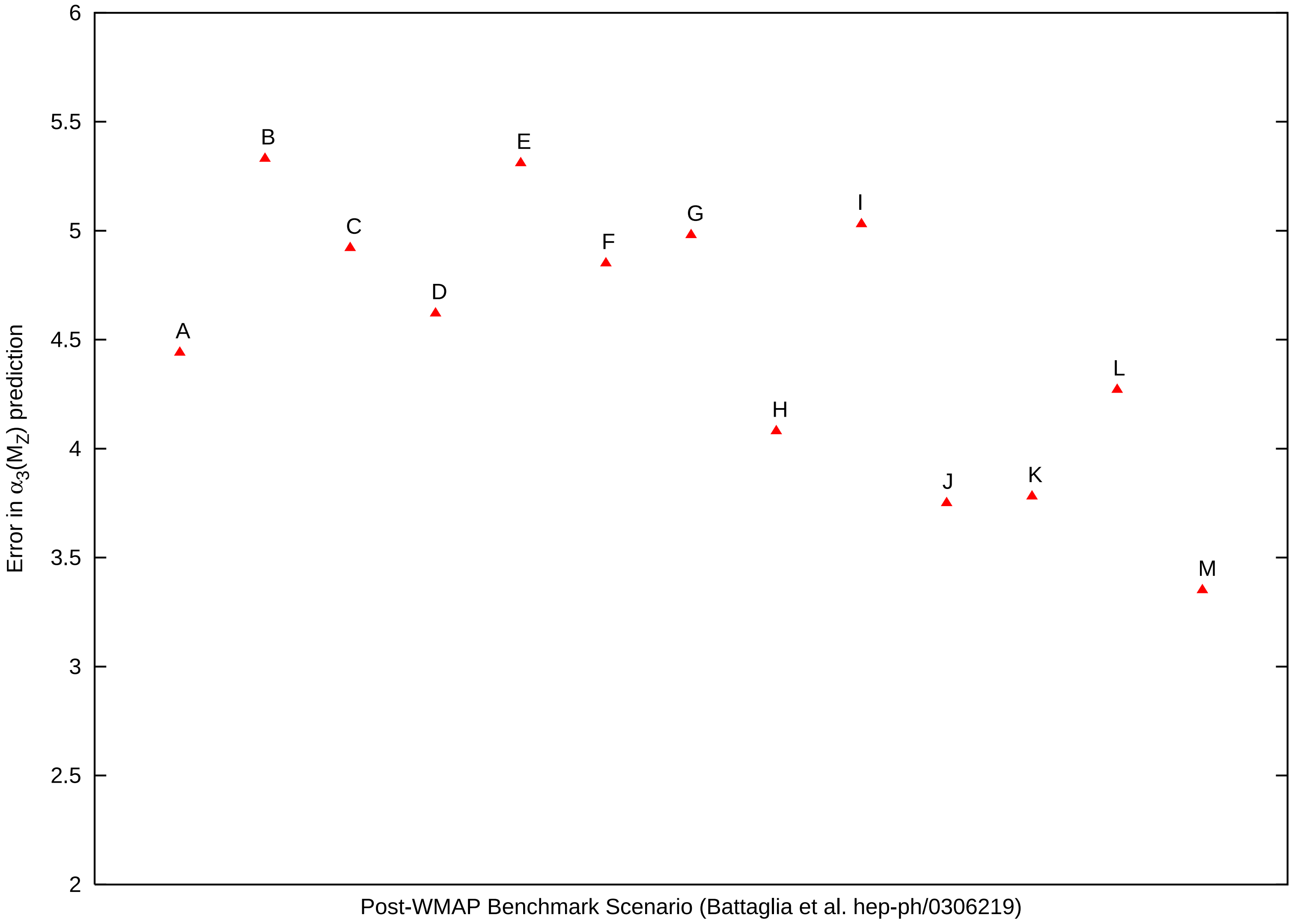


Two-loop Higgs Mass in Split Supersymmetry

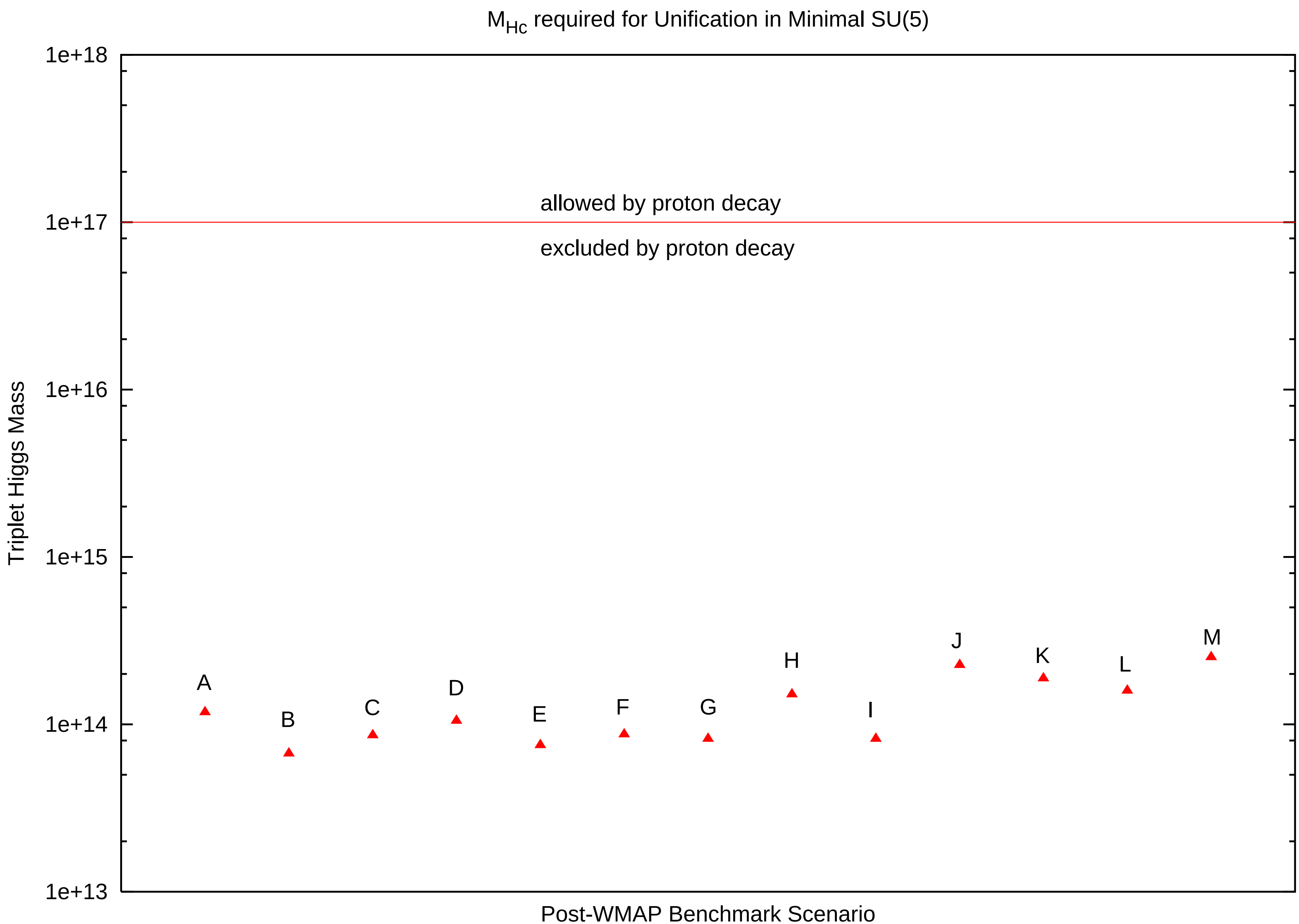


Two-loop Higgs Mass in Split Supersymmetry

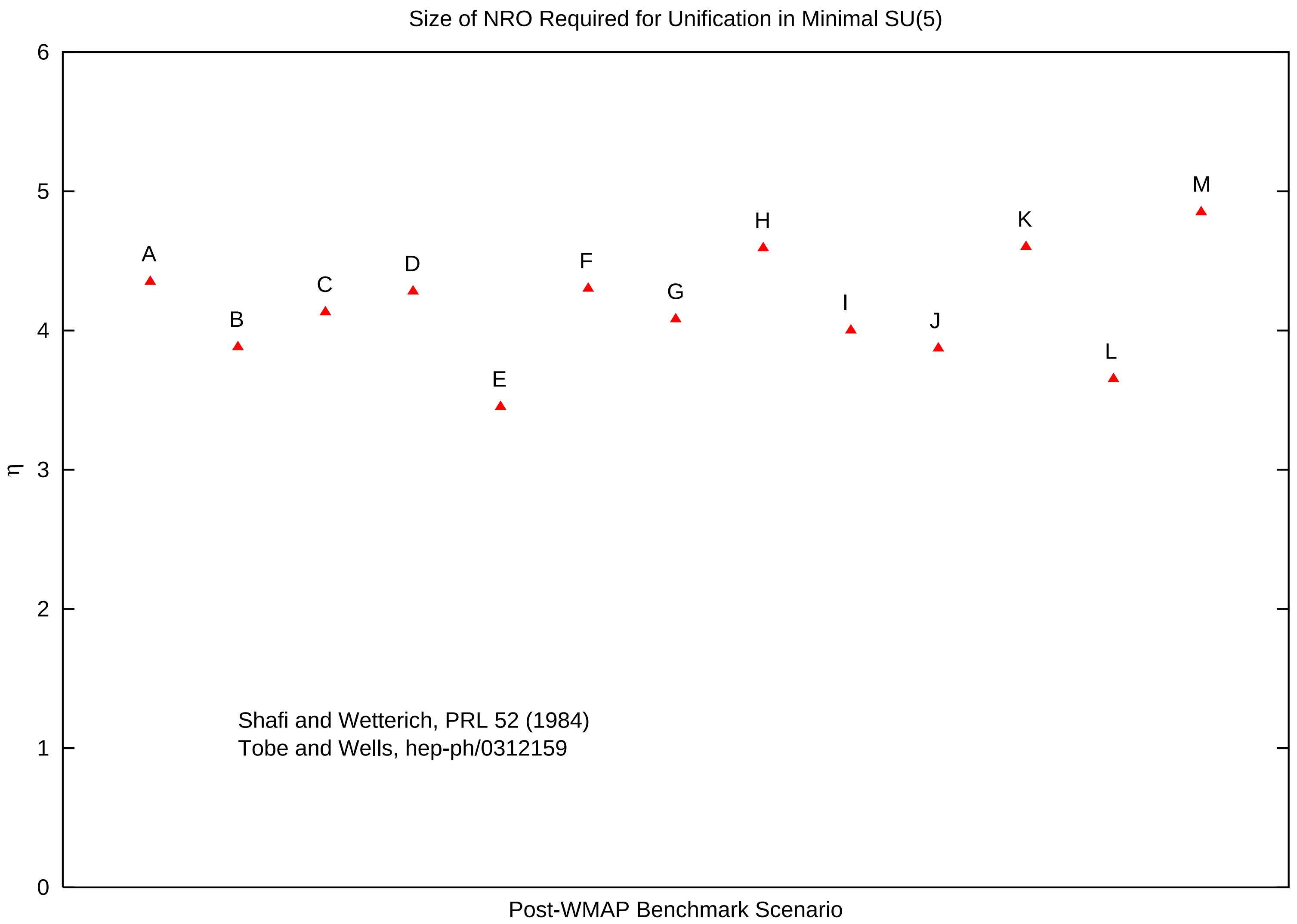
MSSM Benchmarks



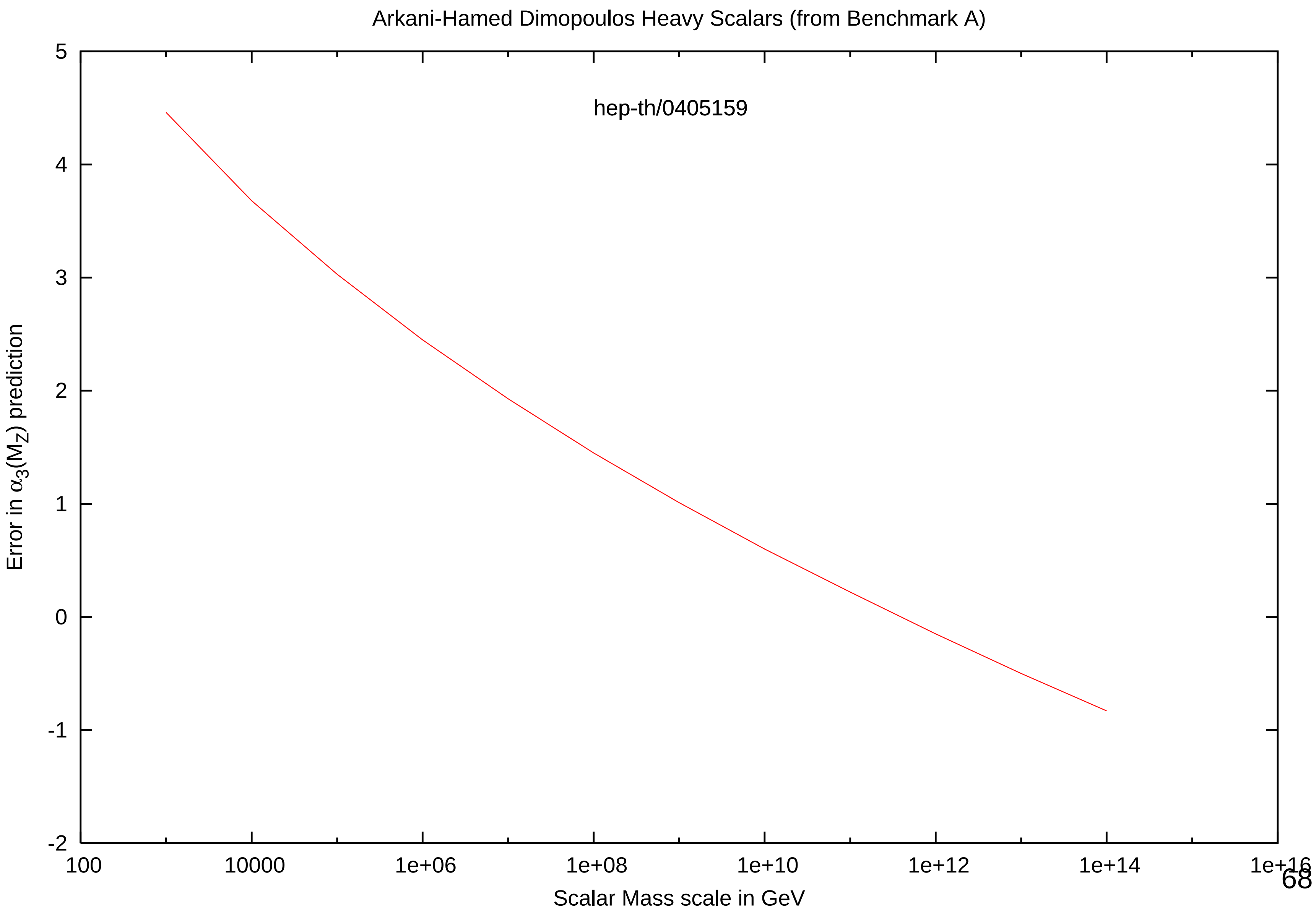
Minimal Supersymmetric SU(5)



Gravitationally Induced NRO's



Split Supersymmetry

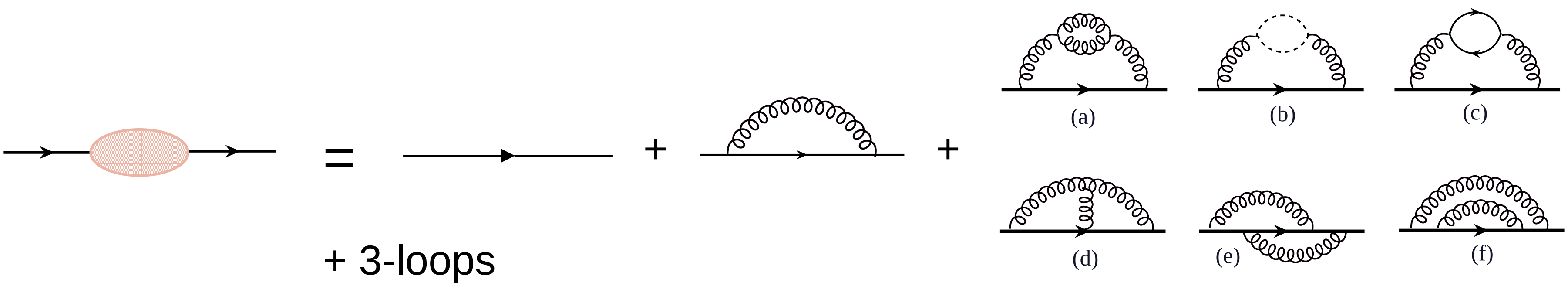


On Renormalons and the Structure of Perturbation Theory

Investigate the relation between :

1. Renormalons
2. BLM Scale Fixing
3. Effective Charges Running Inside of Loops

Laboratory : Higher order corrections to the quark propagator



(Gray, Broadhurst, Grafe, Schilcher and Chetyrkin, Steinhauser)

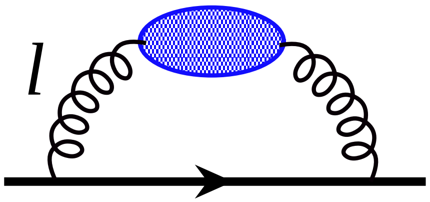
Relation between quark pole mass $\overline{MS}$ mass

On Renormalons and the Structure of Perturbation Theory

Test whether the "renormalon-like" contribution is dominant at 2-loops

Renormalon Inspired Approximation (RIA)

$\tilde{\Pi}(l)$ → Various ansatz for gauge-invariant Self-energy insertion



"PT / BFMFG" : 30-50% accuracy

$"\overline{MS}"$: very poor approx

Not at all clear that "renormalon-like" graphs are dominant

Huge Caveat :

There is no known way to consistent have the coupling run inside of loops

BLM Methods

- Predicts 3-loop term with an accuracy of 3-4%
- Conformal term is very small

Not associated with running coupling

Expect that almost all of the loop corrections are "associated with" the running coupling

Seems to be very much in contrast to what we found using the RIA

Perhaps the success of BLM is not tied to a hypothetical skeleton expansion with running charges inside of loops