Chiral Symmetry

An Approach to the Study of the Strong Interactions

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Contents

Abstract

The ideas behind chiral $SU(3)\otimes SU(3)$ and their relation to current algebra and PCAC have led to a suggested experimental program for using low energy experiments to probe the structure of the strong interactions.

I would like to discuss, in the next few lectures, the general nature of a program in which Roger Dashen and I have been engaged for the last two years. Our purpose is to see what one can learn about the general structure of the strong interactions, by pursuing to their logical conclusions the joint assumptions of 'PCAC' and 'current algebra'. Since, however, PCAC still means many things to many people part of my goal is to present you with what we feel is a completely consistent interpretation of this hypothesis. I hope to convince you, that the way of looking at PCAC which I will discuss has the virtue of clarifying the various confusing points which arise in the application of the PCAC-hypothesis; and simultaneously, it strongly suggests interesting new areas for theoretical and experimental investigation. Moreover, as far as experiment is concerned, we shall see that there is much to be learned from the exploitation of existing accelerators for careful low energy experiments. In these days of shrinking budgets, this is by no means an unimportant consideration.

Before proceding with a detailed discussion of what we feel PCAC really is; let me first list for you some of the things we are fairly sure PCAC *is not*. This list will serve as a framework for our subsequent discussion as explaining each point will naturally lead to the next; eventually culminating in the full development of the ideas which I hope to present.

I. PCAC is not an immediate consequence of the equation $\partial_\mu A^\mu_\pi \sim c\Phi_\pi$ even in a Lagrangian field theory in which Φ_π is the 'canonical pion field'.

II. Results following from PCAC such as the Goldberger-Treiman relation, are not a direct consequence of the accidental smallness of the pion mass coupled with the usual nearest singularities philosophy of dispersion theory.

III. For those familiar with some of the ideas of chiral $SU(2)\otimes SU(2)$ symmetry (and I hope all of you will be by the end of these talks) PCAC is not related to how the symmetry is broken; but rather, it is critically dependent upon the way in which the symmetry is realized.

I have listed these three points explicitly, because each of them corresponds to a popular misconception about the nature of the physics involved in understanding the PCAC hypothesis. I would now like to outline, briefly, the way in which I hope to procede.

A. Critically Discuss Old Results

1. Usual Derivation of the Goldberger-Treiman Relation a lá Dispersion Theory (and present alternative derivation.)
2. Show how this leads to idea of approximate symmetry.
3. Show how this applies to Pion-Nucleon Scattering Length Calculation.

B. Discuss the Differences between Goldstone Boson Type of Symmetry and the more Familiar Kind

1. σ-Model.
2. General discussion.
3. Nambu's Model with Pion as Composite Particle.
4. Generalized $G-T$ Relation, etc. when $\partial_\mu A^\mu_\alpha = 0$.

C. Phenomenological Lagrangians

1. Case in which $\partial_\mu A^\mu_\alpha = 0$.
2. The case of an approximately broken symmetry.

D. PCAC *in the Real World: Perturbation Theory Interpretation*

1. Recapitulation of some familiar results.
2. Going beyond the zeroth approximation.
3. New experiments.

E. Some Old 'Problems' with PCAC

1. K_{l_3}-Decays.
2. $\eta \rightarrow 3\pi$.
3. Electromagnetic Mass-Differences of Mesons.
4. PCAC and Electromagnetism in general.

Clearly, this is a rather long outline and not everything in it can be covered in more than an abbreviated fashion. Moreover, some of the things which I will have to say towards the end of these lectures will be more in the nature of raising questions to be answered by further work. However, I hope to be able to discuss most things in detailed enough fashion to allow you to get a good feeling for the general situation.

So much for introductory comments; let us now procede to the discussion of the various points listed in our outline.

1. Notation

Before really discussing the topics outlined, however, it pays to establish once and for all the notation I will use in this set of lectures.

Throughout our discussion we shall alternate between a discussion of the idea of an approximate $SU(2)\otimes SU(2)$ symmetry for the strong interactions and that of an approximate $SU(3)\otimes SU(3)$ symmetry.

The difference will lie primarily in whether or not we are considering predictions for processes involving the strange-pseudo scalar mesons. We shall therefore adopt the typical convention of letting the symbols $V_\alpha^\mu(x)$ and $A_\alpha^\mu(x)$ ($\alpha = 1, 2, \ldots 8$) stand for the eight vector and axial-vector currents encountered in the theory of the weak and electromagnetic interactions. The index $\mu = 0, 1, 2, 3$, is a space time index and my metric is such that $x^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2$. When we wish to restrict ourselves to the strangeness non-changing currents we shall restrict α to be either 1, 2, or 3.

The Gell-Mann current algebra hypothesis can now be summarized in the following set of equations

$$\begin{aligned}
[V_\alpha^0(x), V_\beta^\mu(y)]_{x^0=y^0} &= i f_{\alpha\beta\gamma} V_\gamma^\mu(x)\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) + \text{S.T.}\\
[V_\alpha^0(x), A_\beta^\mu(y)]_{x^0=y^0} &= i j_{\alpha\beta\gamma} A_\gamma^\mu(x)\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) + \text{S.T.}\\
[A_\alpha^0(x), V_\beta^\mu(y)]_{x^0=y^0} &= i f_{\alpha\beta\gamma} A_\gamma^\mu(x)\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) + \text{S.T.}\\
[A_\alpha^0(x), A_\beta^\mu(y)]_{x^0=y^0} &= i f_{\alpha\beta\gamma} V_\gamma^\mu(x)\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) + \text{S.T.}
\end{aligned} \tag{1.1}$$

where S.T. stands for Schwinger terms; (all time ordered products will be assumed to be defined to be covariant so that Schwinger terms can be ignored).

In order to agree with usual conventions we shall use symbols such as $A^{\mu}_{\pi^+}(x)$, $V^{\mu}_{\pi^0}(x)$, $A^{\mu}_{K^+}(x)$ etc., to stand for linear combinations of currents s.t.

$$[V_3^0(x), J^{\mu}_{\alpha}(y)]_{x^0=y^0} = ic_{\alpha}J^{\mu}_{\alpha}(x)\delta^3(\mathbf{x}-\mathbf{y}) + \text{S.T.}$$

where J^{μ}_{α} stands for some definite current.

$$\left(\begin{aligned}&\text{e.g. } [V_3^0(x), A^{\mu}_{\pi^+}(y)] = iA_{\pi^+}(x)\delta^3(x-y) + \text{S.T.}\\ &\qquad [V_3^0(x), A^{\mu}_{\pi^-}(y)] = -iA_{\pi^-}(x)\delta^3(x-y) + \text{S.T.}\end{aligned}\right)$$

and also s.t. $[(2/\sqrt{3})\,V_8^0(x), J^{\mu}_{\alpha}(y)] = ic'_{\alpha}J^{\mu}_{\alpha}\delta^3(x-y) + \text{S.T.}$ *i.e.*: We are classifying currents by their third component of isospin and their hypercharge. Thus $A_{\pi^+} = A_1 + iA_2$, $A_{\pi^0} = A_3$, $A_{K^+} = A_{4+i}A_5$ etc.

If we accept (as we shall) the usual current-current picture for the leptonic and semi-leptonic weak interactions the constants $(2f_{\alpha})^{-1}$ ($\alpha = 1, \dots 8$) defined by

$$\langle M_{\alpha}(q)|\, A^{\mu}_{\alpha}(0)\,|0\rangle = -iq^{\mu}/2f_{\alpha} \tag{1.2}$$

(where $|M_{\alpha}(q)\rangle$ stands for a pseudoscalar meson of momentum q) are experimentally determined by studying the decays

$M_{\alpha} \to$ lepton + neutrino.

It then follows, strictly from translation invariance and Eq. (1.2) that

$$\langle M_{\alpha}(q)|\, \partial_{\mu}A^{\mu}_{\alpha}(0)\,|0\rangle = m_{\alpha}^2/2f_{\alpha} \tag{1.3}$$

where m_{α}^2 = physical mass of the pseudo-scalar meson $|M_{\alpha}(q)\rangle$.

We shall also have occasion to use the following conventional, general expressions for the matrix element of the axial-vector current and its divergence between single nucleon states:

$$\langle N(p')|\, A^{\mu}_{\alpha}(0)\,|N(p)\rangle = \bar{u}(p')[(\gamma^{\mu}g_A(q^2) + q^{\mu}h_A(q^2))\gamma^5\, \tau_{\alpha}/2]\,u(p)$$

$$\langle N(p')|\, \partial_{\mu}A^{\mu}_{\alpha}(0)\,|N(p)\rangle = +i\bar{u}(p')[\gamma^5(\tau_{\alpha}/2)d(q^2)]\,u(p) \tag{1.4}$$

$$= -i\bar{u}(p')[(2m_N g_A(q^2) + q^2 h_A(q^2))\gamma^5\, \tau_{\alpha}/2]\,u(p) \tag{1.5}$$

(assuming the nucleon masses are the same) and the pion-nucleon vertex is written as:

$$\bar{u}(p')\{\gamma^5\tau_{\alpha}G_{\pi NN}\}\,u(p)\,. \tag{1.6}$$

With these notational question out of the way let us now proceed to discuss the question which probably started the whole game *i.e.* the Goldberger-Treiman relation.

2. The Goldberger-Treiman Relation

Let us first study the usual derivation of the Goldberger-Treiman relation given by dispersion theory. Consider Eq. (1.5). Following standard dispersion arguments (or L.S.Z. reduction formalism) we see that the function $d(q^2)$ has a pole contribution from diagrams in which $\partial_\mu A^\mu_\alpha$ creates a one meson state from vacuum and then continium contributions. Eq. (1.3) fixes the residue of the pole, so that we have

$$d(q^2)=\frac{m_\pi^2 G_{\pi NN}}{f_\pi(q^2-m_\alpha^2)}+\int_{9m_\pi^2}^{\infty}\frac{p(\mu^2)\,\mathrm{d}\mu^2}{(q^2-\mu^2)}. \tag{2.1}$$

Thus we have from Eqs. (1.5) and (2.1)

$$\frac{G_{\pi NN}}{f_\pi}\frac{m_\pi^2}{(q^2-m_\pi^2)}+\bar{\partial}(q^2)=-(2m_N g_A(q^2)+q^2 h_A(q^2)) \tag{2.2}$$

or setting $q^2=0$

$$2m_N g_A=(G_{\pi NN}/f_\pi)(m_\pi^2/m_\pi^2)-\bar{\partial}(0) \tag{2.3}$$

where $\partial(0)$ is identified with the integral in Eq. (2.1). If one now assumes that $\bar{\partial}(0)$ is negligible in comparison to the contribution of the pole term one then obtains the Goldberger-Treiman relation

$$g_A\cong G_{\pi NN}/2m_N f_\pi \tag{2.4}$$

which holds to (15%).

What argument tells us that $\bar{\partial}(0)$ is negligible in comparison to the pole term? The usual argument proceeds as follows: the pole term's denominator is $(m_\pi^2)^{-1}$ at $q^2=0$, while the denominator of the continuum integral is bounded above by $(9m_\pi^2)^{-1}$ so clearly one can expect the pole term to dominate. This argument, however, critically depends upon the fact that there is nothing anomalous about the pole residue. A glance at Eq. (2.2) shows that because it is the divergence of the axial-vector current which we are considering this is not the case. In fact, purely for kinematic reasons (i.e. Eqs. (1.2) and (1.3)) we must have a factor of m_π^2 in the numerator as well. Thus, there is no a-priori reason for the size of the pole-term to be related in any way to the small denominator $(m_\pi^2)^{-1}$.

There is another argument which makes this point even more clearly. Reconsider Eq. (2.2). The form-factor $h\ (q^2)$ also has a pion-pole and it can be written as:

$$q^2 h_A(q^2)=[q^2/(q^2-m_\pi^2)](G_{\pi NN}/f_\pi)+q^2\bar{h}_A(q^2) \tag{2.5}$$

in analogy to Eq. (2.1).

Combining Eqs. (2.2) and (2.5) we have

$$G_{\pi NN}/f_\pi = 2m_N g_A(q^2) + q^2 h_A(q^2) + \bar{\partial}(q^2) \tag{2.6}$$

and setting $q^2 = 0$ we again obtain

$$G_{\pi NN}/2f_\pi = 2m_N g_A(0) + \bar{\partial}(0)\,. \tag{2.7}$$

This derivation clearly shows that the validity of the Goldberger-Treiman relation has nothing to do a-priori with the magnitude of the meson mass or with the presence of a pole denominator, as it is cancelled completely. Thus, some other principle is needed to provide us with a reason to suspect that $\bar{\partial}(0)$ is in fact small.

Rewriting Eq. (2.7) gives us a solid hint as to what this principle might be. Divide both sides by $2m_N$ to get

$$G_{\pi NN}/2f_\pi m_N = g_A + \bar{\partial}(0)/2m_N \tag{2.8}$$

which implies that $\bar{\partial}(0)/2m_N$ must be $\ll 1$, if the Goldberger-Treiman relation is to be true. Heuristically, we realize that if the axial current were conserved then $\bar{\partial}(0)$ would vanish identically; thus, $\bar{\partial}(0)$ is related to the part of the strong interactions which violate conservation of the axial-current. The number $(2m_N)$ is a typical matrix element of the total strong interaction Hamiltonian and thus the ratio $\bar{\partial}(0)/2m_N$ is some rough measure of the ratio of the part of the strong Hamiltonian violation axial-current conservation to the total Hamiltonian. Thus if we think of this piece of the Hamiltonian as small, then the Goldberger-Treiman relation is a zero order in symmetry breaking sum rule.

Let us make this argument more precise. In order to do so consider the 'formal charges', $Q_\alpha(t)$ and $Q_\alpha^5(t)$ defined as follows:

$$\begin{aligned} Q_\alpha(t) &\equiv \int d^3x\, V_\alpha^0(t, \boldsymbol{x}) \\ Q_\alpha^5(t) &\equiv \int d^3x\, A_\alpha^0(t, \boldsymbol{x})\,. \end{aligned} \tag{2.9}$$

It follows from Eq. (1.1) that these charges, at equal times, close under commutation to form the Lie-algebra of $SU(3)\otimes SU(3)$ (again with the restriction that when $\alpha = 1, 2, 3$ we are dealing with $SU(2)\otimes SU(2)$). This is most easily seen if one forms the 'chiral' combinations

$$Q_\alpha^\pm(t) \equiv [Q_\alpha(t) \pm Q_\alpha^5(t)]/2 \tag{2.10}$$

then trivial to show that

$$\begin{aligned} [Q_\alpha^\pm, Q_\beta^\pm] &= i f_{\alpha\beta\gamma} Q_\gamma^\pm \\ [Q_\alpha^+, Q_\beta^-] &= 0 \end{aligned} \tag{2.11}$$

(N.B. When $\alpha = 1, 2, 3$, $f_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma}$).

Now, whereas the Hamiltonian of the theory in the Heisenberg picture is independent of time, the fact that the equal time charges form a closed algebra implies that we can always decompose H into two parts, i.e.:

$$H = H_0(t) + \varepsilon H_1(t) \tag{2.12}$$

such that,

$$[Q_\alpha^\pm(t), H_0(t)] = 0$$

and

$$[Q_\alpha^\pm(t), \varepsilon H_1(t)] = [Q_\alpha^\pm(t), H]\,. \tag{2.13}$$

In other words $H_0(t)$ is the largest piece of H which transforms as an $SU(2)\otimes SU(2)$ (or $SU(3)\otimes SU(3)$) scalar operator under commutation with the charges $Q_\alpha^\pm(t)$, and $\varepsilon H_1(t)$ transforms as same sum of irreducible tensor under the same algebra. I use the explicit parameter ε in front of $H_1(t)$ to symbolically represent a factor which explicitly fixes the scale of $H_1(t)$ relative to $H_0(t)$. Of course, this decomposition means nothing until we add to it the assumption that an expansion about the limit $\varepsilon \to 0$ makes sense. The reason this is an interesting hypothesis to make is that in any Lagrangian field theory it is true that

$$\begin{aligned}\partial_\mu V_\alpha^\mu(x) &= i\varepsilon[Q_\alpha(x^0), \mathscr{H}_1(x)]\\ \partial_\mu A_\alpha^\mu(x) &= i\varepsilon[Q_\alpha^5(x^0), \mathscr{H}_1(x)]\end{aligned} \tag{2.14}$$

where, by assumption $H_1(t) \equiv \int d^3x\, \mathscr{H}_1(t, \mathbf{x})$.

If we now assume $\varepsilon \ll 1$, this amounts to assuming that the term $(\bar{\partial}(0)/2m_N)$ appearing in Eq. (2.8) is smaller than 1. As we have already seen, this implies the Goldberger-Treiman relation.

N.B. There is a very important point to be made about matrix elements of the operator $\partial_\mu A_\alpha^\mu$; that is, that although they have an explicit power of ε in them one must be careful about saying that they are small. To see that this is the case one needs only to consider Eq. (1.2) which shows that $(m_\alpha^2/2f_\alpha)$ is on the order of ε. As we shall see- the model which is most interesting is the one in which m_α^2 is of order ε. In that case, Eq. (2.1) shows that $d(q^2)$ has a pole term whose denominator (for $q^2 \cong 0$) is of order ε^{-1}; so that near $q^2 = 0$, $d(q^2)$ is order 1. However, as you have seen we only needed to assert that $\bar{\partial}(0)$ (which corresponds to $d(0)$ minus its pole term) is of order ε and that is still true. The lesson here, is that one must be careful in handling single meson poles in all of our expressions; at least when one wishes to present arguments about the formal order in ε of a term under consideration. As we shall see in a few moments, consistent, careful consideration of exactly this point allows one to discover a rather interesting structure in the usual PCAC identities.

At this stage of our discussion we should notice that we have yet to completely specify the nature of the limit $\varepsilon \to 0$. In the next lecture we shall devote time to exactly this question. However, before we handle this point in detail it is worth spending a few moments of our time in order to convince ourselves, that our idea, that the consequences of PCAC are equivalent to calculating to zeroth order in ε, works in a more complicated situation. Therefore, let us consider the case of pion-nucleon scattering.

Our study of the pion-nucleon scattering amplitude begins with the expression

$$\begin{aligned}&\langle N'|\,T(\partial_\mu A^\mu_\alpha(q)\,\partial_\nu A^\nu_\beta(-k))\,|N\rangle \\ &\equiv \int \mathrm{d}^4x\,\mathrm{d}^4y\,\mathrm{e}^{+iq\cdot x}\mathrm{e}^{-ik\cdot y}\langle N(p')|\,T(\partial_\mu A^\mu_\alpha(x)\,\partial_\nu A^\nu_\beta(y))\,|N(p)\rangle\,.\end{aligned} \tag{2.15}$$

As before, in our study of the Goldberger-Treiman relation, we note that this matrix element has, as a function of q^2 and k^2, poles at $q^2 = m_\pi^2$ and $k^2 = m_\pi^2$. These poles, in the language of Feynman diagrams, are due to diagrams in which the divergence of the axial-vector current creates single meson states from the vacuum. Before we extract these pole terms by hand, it is necessary to make a few comments about the kinematics of this process. Clearly the function defined in Eq. (2.15) is a function of the four-momenta p', p, q and k. Being relativistically invariant it is in fact just a function of the kinematic invariants

$$\begin{aligned}s &= (p+k)^2 = (p'+q)^2 \\ t &= (p'-p)^2 = (k-q)^2 \\ u &= (p'-k)^2 = (p-q)^2\end{aligned} \tag{2.16}$$

and the quantities 'q^2' and 'k^2'. (Note, $p^2 = p^2 = m_\pi^2$, and translation invariance implies that $s+t+u = 2m_\pi^2 + q^2 + k^2$).

Since we shall be discussing the extraction of poles in the variables 'q^2' and 'k^2' from the function in question it is necessary to decide which variables to hold fixed when deriving our various identities. As we shall see, it doesn't really matter (except from the point of convenience of presentation) which variables we hold fixed as long as we are consistent. For our purposes it will turn out that the variables

$$\begin{aligned}&\nu = (p'+p)\cdot(q+k)/2 \\ &\text{and} \\ &x = q\cdot k\end{aligned} \tag{2.17}$$

are by far the most convenient variables to consider, along with 'q^2' and 'k^2' to be the independent variables in out problem. With these conventions we have

$$\langle N'|\,T(\partial_\mu A^\mu_\alpha(q)\,\partial_\nu A^\nu_\beta(-k))|N\rangle\,(q^2,k^2;\nu,x) \equiv +i$$
$$\left[\frac{m_\pi^4\langle N'\pi'_\alpha|S|N\pi_\beta\rangle(\nu,x)}{(2f_\pi)^2(q^2-m_\pi^2)(k^2-m_\pi^2)} + \frac{m_\pi^2\langle N\pi'_\alpha|\,\bar\partial_\beta(-k)|N\rangle\,(k^2;\nu,x)}{(2f_\pi)(q^2-m_\pi^2)}\right. \tag{2.18}$$
$$\left. + \frac{m_\pi^2\langle N|\,\bar\partial_\alpha(q)|N\pi_\beta\rangle\,(q^2;\nu,x)}{(2f_\pi)(k^2-m_\pi^2)} + \langle N'|\,T(\bar\partial_\alpha(q)\,\bar\partial_\beta(-k))|N\rangle\,(q^2,k^2;\nu,x)\right]$$

where the functions $\bar\partial(q^2;\nu,x)$ are defined in direct analogy to those in Eq. (2.1) and (2.2) with one of the external states replaced by a two particle state. $\langle N'\pi'_\alpha|S|N\pi_\beta\rangle$ stands for the appropriate S-matrix element.

What we have done up to this point can be considered to be nothing more than a series of definitions. We now continue by rewriting the time ordered product defined in Eq. (2.15) by pulling all derivatives through the time ordering instruction, (a step which I hope everyone is familiar with). Using the current algebra given in Eq. (1.1) in order to evaluate the resulting equal time commutators, we get

$$\langle N'|\,T(\partial_\mu A^\mu_\alpha(q)\,\partial_\nu A^\nu_\beta(-k))|N\rangle = +i^2[-q_\mu k_\nu\langle N'|\,T(A^\mu_\alpha(q)A^\nu_\beta(-k))|N\rangle$$
$$+\varepsilon_{\beta\alpha\varrho}q_\mu\langle N'|\,V^\mu_\varrho(q-k)|N\rangle - i\langle N'|\Sigma_{\alpha\beta}(q-k)|N\rangle] \tag{2.19}$$

where

$$-i\Sigma_{\alpha\beta}(q-k) \equiv \int d^4x\,d^4y\,e^{+iq\cdot x}e^{-ik\cdot y}\delta(x^0-y^0)\,[A^0_\alpha(x),\partial_\mu A^\mu_\beta(y)]$$
$$\equiv -i\int d^4x\,e^{+i(q-k)\cdot x}\Sigma_{\alpha\beta}(x)\,. \tag{2.20}$$

Now, in exact analogy with the G−T relation, we notice that the term

$$\langle N'|\,T(A^\mu_\alpha(q)A^\nu_\beta(-k))|N\rangle$$

also has poles at $q^2=k^2=m_\pi^2$. Exactly, as in the case of Eq. (2.18) we can write this as

$$\langle N'|\,T(A^\mu_\alpha(q)A^\nu_\beta(-k))|N\rangle(q^2,k^2;\nu,x) \equiv \left[\frac{-q^\mu k^\nu\langle N'\pi_\alpha|S|N\pi_\beta\rangle(\nu,x)}{(2f_\pi)(q^2-m_\pi^2)(k^2-m_\pi^2)}\right.$$
$$+\frac{-q^\mu\langle N'\pi'_\alpha|\bar A^\nu_\beta(-k)|N\rangle(k^2;\nu,x)}{(2f_\pi)(q^2-m_\pi^2)} + \frac{k^\nu\langle N'|\bar A^\mu_\alpha(q)|N\pi_\beta\rangle(q^2;\nu,x)}{(2f_\pi)(k^2-m_\pi^2)}$$
$$\left.+\langle N'|\,T(\bar A^\mu_\alpha(q)\bar A^\nu_\beta(-k))|N\rangle(q^2,k^2;\nu,x)\right]. \tag{2.21}$$

Combining Eqs. (2.2) and (2.18) and replacing q^2 by $((q^2-m_\pi^2)+m_\pi^2)$ and k^2 by $((k^2-m_\pi^2)+m_\pi^2)$ we get the following formula separated into its double, single and no-pole terms:

$$0=\left[\frac{\langle N'\pi'_\alpha|S|N\pi_\beta\rangle}{(2f_\pi)^2}(\nu,x)-\frac{q_\mu}{(2f_\pi)}\langle N'|\bar{A}^\mu_\alpha(q)|N\pi_\beta\rangle(q^2;\nu,x)\right.$$
$$+\frac{k_\nu}{(2f_\pi)}\langle N'\pi'_\alpha|\bar{A}^\nu_\beta(-k)|N\rangle(k^2;\nu,x)$$
$$-q_\mu k_\nu\langle N|T(\bar{A}^\mu_\alpha(q)\bar{A}^\nu_\beta(-k))|N\rangle(q^2,k^2;\nu,x)$$
$$-\langle N|T(\bar{\partial}_\alpha(q)\bar{\partial}_\beta(-k))|N\rangle(q^2,k^2;\nu,x)$$
$$+\varepsilon_{\beta\alpha\varrho}q_\mu\langle N'|V^\mu_\varrho(q-k)|N\rangle(q^2,k^2;x) \tag{2.22}$$
$$\left.-i\langle N'|\Sigma_{\alpha\beta}(q-k)|N\rangle(q^2,k^2;x)\right]+\frac{m_\pi^2}{(2f_\pi)(q^2-m_\pi^2)}\left[\frac{\langle N'\pi'_\alpha|S|N\pi_\beta\rangle}{2f_\pi}(\nu,x)\right.$$
$$\left.+k_\nu\langle N'\pi'_\alpha|\bar{A}^\nu_\beta(-k)|N\rangle(k^2;\nu,x)-\langle N'\pi'_\alpha|\bar{\partial}_\beta(-k)|N\rangle(k^2;\nu,x)\right]$$
$$+\frac{m_\pi^2}{(2f_\pi)(k^2-m_\pi^2)}\left[\frac{\langle N'\pi'_\alpha|S|N\pi_\beta\rangle}{(2f_\pi)}(\nu,x)-q_\mu\langle N'|\bar{A}^\mu_\alpha(q)|N\pi\rangle(q^2;\nu,x)\right.$$
$$\left.-\langle N'|\bar{\partial}_\alpha(q)|N\pi_\beta\rangle(q^2;\nu,x)\right]$$
$$+\frac{m_\pi^4}{(2f_\pi)^2(q^2-m_\pi^2)(k^2-m_\pi^2)}\left[\langle N'\pi'_\alpha|S|N\pi_\beta\rangle-\langle N'\pi'_\alpha|S|N\pi_\beta\rangle\right].$$

Now, the coefficient of the terms with two, one or no-poles must separately vanish. Thus we get four identities. The first, coming from setting the coefficient of the term $((q^2-m_\pi^2)(k^2-m_\pi^2))^{-1}$ equal to zero is trivial. The next two come from setting the coefficients of the terms $(q^2-m_\pi^2)^{-1}$ and $(k^2-m_\pi^2)^{-1}$ equal to zero, and they read;

$$\begin{aligned}(1/2f_\pi)\langle N'\pi'_\alpha|S|N\pi_\beta\rangle(\nu,x)=&-k_\nu\langle N'\pi'_\alpha|\bar{A}^\nu_\beta(-k)|N\rangle(k^2;\nu,x)\\&+\langle N'\pi'_\alpha|\bar{\partial}_\beta(-k)|N\rangle(k^2;\nu,x)\end{aligned} \tag{2.23a}$$

and

$$\begin{aligned}(1/2f_\pi)\langle N'\pi'_\alpha|S|N\pi_\beta\rangle(\nu,x)=&\,q_\mu\langle N'|\bar{A}^\mu_\alpha(q)|N\pi_\beta\rangle(q^2;\nu,x)\\&+\langle N'|\bar{\partial}_\alpha(q)|N\pi_\beta\rangle(q^2;\nu,x).\end{aligned} \tag{2.23b}$$

These two equations can be thought of as generalizations of the Goldberger-Treiman relation with one of the external states replaced by a two particle state. It is important to realize that Eqs. (2.23a) and (2.23b) are exact and therefore devoid of content. However, they do have the amusing feature that the left-hand side of both equations is independent of 'q^2', but the right hand side is not. Thus, we can evaluate the right hand side for any value of q^2, and we shall choose to do so at the point

$q^2=0$. If we write out the most general expressions consistent with Lorentz invariance and isospin – conservation, and go to the point $\nu=x=0$; once, we realize that due to the q_μ in the right hand side of the equation only the pole term in $q_\mu\langle N|\bar{A}^\mu_\alpha(q)|N'\pi'_\beta\rangle$ survives. We get the Adler PCAC – consistency conditions for pion-nucleon scattering if and only if we ignore the term $\bar{\partial}(0,0,0)$ which is order ε^1.

The final identity comes from setting the sum of terms with no pole at $q^2=k^2=m_\pi^2$ equal to zero and reads

$$\begin{aligned}(1/2f_\pi)^2\langle N'\pi'_\alpha|S|N\pi_\beta\rangle(\nu,x) = {} & q_\mu k_\nu\langle N'|T(\bar{A}^\mu_\alpha(q)\bar{A}^\nu_\beta(-k))|N\rangle(q^2,k^2;\nu,x)\\ & +(q_\mu/2f_\pi)\langle N'\pi'_\alpha|\bar{A}(q)|N\pi_\beta\rangle(q^2;\nu,x)\\ & -(k_\nu/2f_\pi)\langle N'\pi'_\alpha|\bar{A}^\nu_\beta(-k)|N\rangle(k^2;\nu,x)\\ & -q_\mu\varepsilon_{\alpha\beta\gamma}\langle N'|V^\mu_\gamma(q-k|N\rangle(q^2,k^2;x)\\ & +i\langle N'|\Sigma_{\alpha\beta}(q-k)|N\rangle(q^2,k^2;x)\\ & +i\langle N'|T(\bar{\partial}_\alpha(q)\bar{\partial}_\beta(-k))|N\rangle(q^2,k^2;\nu,x)\,.\end{aligned} \tag{2.24}$$

If we now choose to evaluate the right hand side of this formula at the point $q^2=k^2=\nu=x=0$, then to order ε^0 (recalling that the Σ-term is explicitly order ε^1 and $\langle N'|T(\bar{\partial}_\alpha(q)\bar{\partial}_\beta(-k))|N\rangle$ is order ε^2) we have that the amplitude vanishes. Rewriting this result in terms of invariant form factors retrieves the PCAC-calculation of the pion-nucleon scattering lengths. The important point to be noticed is that once again the usual approximations which give us the PCAC results are equivalent to calculating to order ε^0.

We shall return at a later point to give a more thorough discussion of Eq. (2.24) at which time we shall be interested in what one can learn about the terms of order ε^1; but for now, this is a natural place to stop trying to motivate our study of an approximate symmetry and to state carefully what such a symmetry would be like.

3. Symmetrics — Conventional vs. Goldstone

To this point we have discussed only vaguely the idea of an approximate chiral $SU(2)\otimes SU(2)$ (or $SU(3)\otimes SU(3)$) symmetry for the strong interactions. I have also hinted that we would find that this had to be an approximate symmetry of a rather unusual kind. What *I* would like to do in this lecture is discuss in some detail just what is unusual about this 'symmetry' and provide you with enough background material to make you somewhat comfortable with these ideas.

Before entering into this discussion however, let me briefly summarize what the results of our discussion will be.

1. We shall see that there is a perfectly well defined sense in which all of usual PCAC-results can be reinterpreted as zeroth order perturbation theory calculations.

2. That is, the Hamiltonian of the strong interactions can be divided into two pieces i.e.

$$H_0 + \varepsilon H_1$$

where H_0 commutes with the vector and axial-vector charges (Q_α and Q_α^5 and εH_1 is a 'small perturbation' of H_0. Then, the usual consequences of applying the PCAC-hypothesis can be thought of as theorems which hold exactly for the model defined by setting $\varepsilon = 0$ and letting $H = H_0$.

3. The symmetry of the Hamiltonian 'H_0' will be unusual in that, even though the 'charges Q_α and Q_α^5' commute with H_0 and close under commutation to the Lie algebra of $SU(2) \otimes SU(2)$; the eigenstates of H_0 do not fall into irreducible representations of $SU(2) \otimes SU(2)$. Instead the consequences of the identities $\partial_\mu A_\alpha^\mu(x) = 0$ are a set of low energy theorems for the scattering of zero-mass pions.

It is really point (3) in this list which requires the greatest elaboration at this time; and this will be the subject of most of the remainder of this lecture. If we go back to Eq. (2.14) we see, that, at least formally,

$$\begin{aligned} \partial_\mu V_\alpha^\mu(x) &= i\varepsilon[Q_\alpha(x^0), \mathscr{H}_1(x)] \\ \partial_\mu A_\alpha^\mu(x) &= i\varepsilon[Q_\alpha^5(x^0), \mathscr{H}_1(x)] \,. \end{aligned} \tag{3.1}$$

Since we will assume throughout these lectures that isospin is a good symmetry of the total strong interaction Hamiltonian, we have that $\partial_\mu V_\alpha^\mu = 0$ or $[Q_\alpha, \mathscr{H}, (x)] = 0$. However, such a statement is manifestly false for the axial current; since, as we have seen in Eq. (1.3)

$$\langle \pi_\alpha | \partial_\mu A_\alpha^\mu(0) | 0 \rangle = m_\alpha^2 / 2 f_\alpha$$

which is not zero. Thus, the decomposition of H into H_0 and εH_1 is a non-trivial step, and the limit $\varepsilon \to 0$ implies $\partial_\mu A_\alpha^\mu \to 0$ and therefore $m_\alpha^2 / 2 f_\alpha$ also goes to zero. As we shall see, this passage to the 'symmetry limit' can take place in one of two ways. The first, corresponds to definition of a symmetry which is familiar from elementary quantum mechanics. It is recognized by the fact that the particles group themselves into irreducible representations of the symmetry group in question and are degenerate in mass. The second way of realizing a 'symmetry' of the Hamiltonian is less familiar, but is nevertheless quite interesting. This kind of symmetry is recognized by the fact that even though the Hamiltonian is invariant under a certain group of transformations, none of the usual consequences of such a symmetry (such as there being multiplets of particless degenerate in mass) need hold. Such a symmetry is characterized by one additional feature; that is, the appearance of zero mass particles, i.e. 'Goldstone-bosons'. Thus, if in any theory the limit $\varepsilon \to 0$ is to correspond to a symmetry of the second kind there must be a set of states whose mass goes to zero as $\varepsilon \to 0$.

Let us now try to figure out in more detail the essential differences between these two different kinds of 'symmetries'. The best way to do this is to reinvestigate our usual notions of a symmetry and see what assumptions lead to the usual conclusions about, for example, the degeneracies of energy levels. While I am certain everyone here is quite familiar with these arguments, they are important, thus, at the risk of being pedantic, allow me to remind you of their salient features.

The usual definiton of a 'symmetry group' in quantum mechanics, is that there is a group of unitary transformations defined on the Hilbert space of states such that for any transformation 'Ug' in thes group, we have

$$U g H_0 U g^{\dagger} = H_0 . \tag{3.2}$$

In other words, the Hamiltonian is invariant under all transformations *induced* by the state transformation Ug. If we assume, that the symmetry is 'local' – or equivalently that the momentum of a state is unchanged by the transformation 'Ug' then it follows that

$$E_{Ug(\psi)} = (Ug(\psi); H\, Ug(\psi)) = (\psi, (U g^{\dagger} H\, Ug)\psi) = (\psi, H\psi) = E_\psi \tag{3.3}$$

or in other words, the state $Ug(\psi)$ and ψ necessarily have the same energy. Since the group representation is unitary, we immediately have that the space of states decomposes into mutually orthogonal subspaces, each one being an irreducible representation of the group in question; and, all the states in any one subspace must the same energy.

One then defines "conserved charges" or "hermitian generators" for the group by considering all one parameter subgroups; that is, sets of transformations of the form $U(\alpha)$ (for 'α' a real parameter) satisfying the assumptions

i) $U(\alpha_1)\, U(\alpha_2) = U(\alpha_1 + \alpha_2)$

ii) $U(0) = 1$

(where 1 denotes the identity transformation in the Hilbert space). To each such one-parameter subgroup one associates a 'charge' or 'generator' defined by

$$Q = i^{-1}\, \mathrm{d}U(\alpha)/\mathrm{d}\alpha|_{\alpha=0} . \tag{3.4}$$

It follows from our assumptions that Q defined in this way, is a hermitian operator (even though it is usually unbounded). Moreover, the following facts are true

$$U(\alpha) = e^{i\alpha Q} = \sum_{n=0}^{\infty} (i\alpha Q)^n/n! \tag{3.5}$$

and

$$U(\alpha) H U^{+}(\alpha) = H \Rightarrow [Q, H] = 0 . \tag{3.6}$$

Whenever the group is a compact and finite dimensional then it is true that there exists only a finite number of linearly independent charges Q_α ($\alpha = 1, \dots n$ where $n =$ dimension of the group) and every group transformation is written (not necessarily uniquely) as

$$Ug = \exp i \sum_{\alpha=1}^{n} v^\alpha Q_\alpha \tag{3.7}$$

and the charges satisfy some Lie algebra

$$[Q_\alpha, Q_\beta] = i f_{\alpha\beta\gamma} Q_\gamma . \tag{3.8}$$

The constants '$f_{\alpha\beta\gamma}$' are called structure constants for the group and completely specify the group in question.

For example, the group $SU(2)$ has three generators which we can call Q_i ($i = 1, 2, 3$) and

$$[Q_i, Q_j] = i\varepsilon_{ijk} Q_k . \tag{3.9}$$

The next step usually is to choose a maximal set of commuting generators and use their eigenvalues to label a basis for an irreducible representation. (e.g. for $SU(2)$ we use Q_3; for $SU(3)$, Q_3 and $Y = 2Q_8/\sqrt{3}$). These 'eigenvalues' define the interesting 'quantum numbers' for the problem and usually provide the framework which we use to discuss symmetry breaking.

The situation when we are doing Lagrangian field theory is a little different. In that case we start out with a Lagrangian, which we take to be some function of some 'fundamental fields' and their derivatives. We then look to see if the Lagrangian is invariant under certain 'local transformations' of the field involved. If it is, then Noether's theorem tells us that there is a current $j^\mu(x)$ associated with each invariance and furthermore

$$\partial_\mu j^\mu(x) = 0 \tag{3.10}$$

Eq. (3.10) then guarantees (again this is only formal) that the 'formal charge'

$$Q(t) = \int d^3x j^0(t, \boldsymbol{x}) \tag{3.11}$$

is independent of 't' (i.e. Q defines a conserved charge). Since Q is formally a hermitian operator, if the integral over all space in Eq. (3.11) really exists; then Eq. (3.5) provides us with a one-parameter family of unitary transformations and we have a usual kind of symmetry of the Hamiltonian since

$$i^{-1}\mathrm{d}Q(t)/\mathrm{d}t = [Q, H] = 0\,. \tag{3.12}$$

The usual situation corresponds to having a Lagrangian invariant under an n-parameter group of transformations and therefore 'n'-conserved currents. From these we form 'n'-formal charges '$Q\alpha$' which we show close to form the Lie algebra of the n-parameter symmetry group of the Lagrangian. Thus, if these 'formal charges' really exist (i.e. if Eq. (3.11) really is well defined in some operator sense) then there is no escaping the usual consequences of the assumption that there is a symmetry group for the Hamiltonian (e.g. degenerate multiplets of particles, etc.).

Fortunately, it can be shown that the 'formal charges' never exist; however, in some cases one can use them to define real unitary symmetry groups for the theory and in other cases this is patently impossible. These two cases correspond to the case of a usual symmetry or a Goldstone type symmetry, respectively. In the first case all of the familiar 'group theory results' hold in unmodified fashion; in the second, all bets are off, but there are *new consequences* of the current conservation equations.

Rather than discuss these points any further in an abstract way, let us first consider a concrete example of this kind of situation. The first model I will consider is the so called 'σ-model' but without nucleons.

The σ-model is an interacting field theory for a world composed of a sclar $I=0$ σ-meson and an $I=1$ triplet of pseudo-scalar pi-mesons. Our starting point is the very general Lagrangian

$$\mathscr{L} = (1/2)\left((\partial_\mu \sigma')^2 + (\partial_\mu \boldsymbol{\pi})^2\right) - B^2(\sigma'^2 + \boldsymbol{\pi}^2 - A)^2\,. \tag{3.13}$$

This Lagrangian (and its associated Hamiltonian $H = -L$) is easily seen to be invariant under the infinitessimal transformations

$$\begin{aligned} \boldsymbol{\pi} &\to \boldsymbol{\pi} + \delta\boldsymbol{\pi} = \boldsymbol{\pi} + \varepsilon \boldsymbol{v} \times \boldsymbol{\pi} \\ \sigma' &\to \sigma' + \delta\sigma' = \sigma' \end{aligned} \tag{3.14}$$

(where $\boldsymbol{v}$ is an arbitrary vector).

These transformation are merely statements of the isotopic spin properties assumed for the fields σ' and $\boldsymbol{\pi}$. It follows from Noether's theorem that the current

$$V^\mu = \boldsymbol{\pi} \times \partial_\mu \boldsymbol{\pi} \tag{3.15}$$

is conserved; and the formal charges Q_α ($\alpha = 1, 2, 3$) have property that (using canonical commutation relations)

$$\begin{aligned}[][\boldsymbol{v}\cdot\boldsymbol{Q}, \boldsymbol{\pi}] &= \varepsilon\boldsymbol{v}\times\boldsymbol{\pi}\\ [\boldsymbol{v}\cdot\boldsymbol{Q}, \sigma'] &= 0\end{aligned} \tag{3.16}$$

for $\boldsymbol{v}$ an arbitrary 3-iso-vector.

Our Lagrangian can also be seen to be invariant under the transformation

$$\begin{aligned}\boldsymbol{\pi} &\to \boldsymbol{\pi} - \varepsilon\boldsymbol{v}\sigma'\\ \sigma' &\to \sigma' + \varepsilon\boldsymbol{v}\cdot\boldsymbol{\pi}\end{aligned} \tag{3.17}$$

where as before $\boldsymbol{v}$ is an arbitrary C-number 3-iso-vector. Since these transformations mix even and odd-parity objects, the conserved current defined by Noether's theorem

$$A^\mu = \sigma'\partial_\mu\boldsymbol{\pi} - \partial_\mu\sigma'\boldsymbol{\pi} \tag{3.18}$$

is an axial-vector current. Associated with it are the 'axial-charges' Q_α^5, satisfying

$$\begin{aligned}[][\boldsymbol{v}\cdot\boldsymbol{Q}^5, \boldsymbol{\pi}] &= -\boldsymbol{v}\sigma'\\ [\boldsymbol{v}\cdot\boldsymbol{Q}^5, \sigma'] &= \boldsymbol{v}\cdot\boldsymbol{\pi}\,.\end{aligned} \tag{3.19}$$

Furthermore, usual canonical commutation relations imply that the Q_α's and Q_α^5's form the Lie algebra of $SU(2)\otimes SU(2)$ defined (essentially) in Eq. (1.1).

So far everything appears quite familiar, but let us investigate Eq. (3.13) in more detail. First, let us consider '$\mathscr{L}$' as defining a classical field theory. Then a glance at Eq. (3.13) suffices to convince us that the 'ground state' of the system corresponds to having 'σ'' and '$\boldsymbol{\pi}$' independent of position (so that $(\partial_\mu\sigma')^2$ and $(\partial_\mu\boldsymbol{\pi})^2$ are zero) and then requiring that the constant values of σ' and $\boldsymbol{\pi}$ are chosen so as to minimize the 'potential energy' terms. However, the solutions to the problem of finding such a minimum for the energy is very different for the two cases (i) $A < 0$ and (ii) $A > 0$. Let us therefore consider these two cases separately.

(i) $A < 0$

In this case the Lagrangian is equivalent to

$$\begin{aligned}\mathscr{L} &= (-1/2)\,((\partial_\mu\sigma)^2 + (\partial_\mu\boldsymbol{\pi})^2) - B^2(\sigma'^2 + \boldsymbol{\pi}^2 + |A|)^2\\ &= (-1/2)\,((\partial_\mu\sigma')^2 + m_\sigma^2) - (1/2)\,((\partial_\mu\boldsymbol{\pi})^2 + m_\pi^2) - B^2(\sigma'^2 + \boldsymbol{\pi}^2)^2\\ &\quad -(B^2A^2)\end{aligned} \tag{3.20}$$

where $m_\pi^2 = m_\varrho^2 = 4B^2|A|$. Furthermore, examination of Eq. (3.13) shows that the 'ground state' problem has a unique solution, namely $\sigma = \pi = 0$. Intuitively, we see that Eq. (3.20) defines a theory having degenerate σ and π mesons with arbitrary mass. (The masses $m_\sigma = m_\pi$ do however go to zero as $A \to 0$.)

There are no problems with assuming that we can quantize this theory in the usual way. Namely, assume there exists a unique vacuum state, $|0\rangle$, characterized by the fact that $\langle 0|\sigma|0\rangle$ and $\langle 0|\pi|0\rangle$ all vanish. If we do so, we find that the formal currents $V_\alpha^\mu(x)$ and $A_\alpha^\mu(x)$ defined in Eqs. (3.15) and (3.19) are conserved.

A careful study of the formal charges Q_α and Q_α^5 shows that they do not exist as defined, due to the fact that the spatial integral extends over all space. However, due to the fact that the zeroth components of the vector and axial-vector currents have 'local' commutation relations with the fields 'σ'' and 'π' the commutators of any number of 'formal charges' with the 'σ'' and 'π' can be unambiguously defined. This can be done by integrating the charges up over finite volumes, taking the commutators and letting the volume go to infinity. Clearly, since only a finite region of the volume of integration is tunelike separated from any fixed point 'x^μ' (i.e. that region contained inside the intersection of the space like-surface used to define Q_α or Q_α^5 and the light cone whose vertex is at 'x^μ') after the region of integration is large enough it doesn't matter how we let the volume go to infinity.

What this means is, that even though the 'formal charges' do not define transformations of the states they do define one parameter transformations of the fields; to do so one need only use formulas such as:

$$\sigma_v'(\alpha) = \sum_{n=0}^{\infty} ((i\alpha)^n/n!)\, [\boldsymbol{v}\cdot\boldsymbol{Q}[\ldots[\boldsymbol{v}\cdot\boldsymbol{Q}, \sigma' \underbrace{]\ldots]}_{n\text{-times}} \tag{3.21}$$

or

$$\sigma_v'^5(\alpha) = \sum_{n=0}^{\infty} ((i\alpha)^n/n!)\, [\boldsymbol{v}\cdot\boldsymbol{Q}^5[\ldots[\boldsymbol{v}\cdot\boldsymbol{Q}^5, \sigma' \underbrace{]\ldots]]}_{n\text{-times}}\,. \tag{3.21}$$

One may then ask the question, 'Is there a one-parameter group of unitary transformations $U(\alpha)$ such that, for example,

$$\sigma_v'(\alpha) \equiv \mathrm{e}^{+i\alpha\boldsymbol{v}\cdot\boldsymbol{Q}}\sigma'\mathrm{e}^{-i\alpha\boldsymbol{v}\cdot\boldsymbol{Q}}\,. \tag{3.22}$$

If so, we can use the generator of this one parameter group as the actual *conserved charge* corresponding to the formal charges defined by the Noether's theorem.

Note, since a dense set of states, in the Hilbert space of states, is generated by applying 'smeared' polynomials in the fields to the vacuum (smeared with functions of compact support, i.e. functions vanishing outside finite regions), Eq. (3.21) defines an action on the states of the theory if we assume that

$$U(\alpha)|0\rangle = |0\rangle \tag{3.23}$$

or, in other words that the 'real' charge operator annhilates the vacuum. In our present case, defined by Eq. (3.20), there is no inconsistency in this assumption and so we may assume that these 'charges' do exist: and in fact, we do see that the theory does have its particles fall in $SU(2) \otimes SU(2)$ multiplets. (Since $\partial_\mu A_\pi^\mu = 0$ and $m_\pi^2 \neq 0$ we have $\langle \pi | A^\mu | 0 \rangle = 0$ or $(2f_\pi)^{-1} = 0$). Thus Eq. (3.20) plus the usual canonical quantization procedure defines a theory possessing a normally realized $SU(2) \otimes SU(2)$ symmetry.

What of our second case $A > 0$?

(ii) $A > 0$

Going back to the case $A > 0$, we see that a new feature presents itself. No longer is the ground state of the theory defined uniquely by the condition that it be the state of lowest energy. Instead, we see that the lowest energy states form a degenerate three parameter family defined by the condition,

$$\sigma'^2 + \pi^2 = A\,. \tag{3.24}$$

How are we to proceed to canonically quantize such a theory? Any normal procedure requires that we assume that the vacuum state of the theory is unique; and if we are going to do perturbation theory, we prefer to work with fields whose vacuum expectation value vanish. But, any one of the states defined by Eq. (3.24) is a good ground state and thus to pick a unique vacuum state we need a subsidiary condition. It is suggestive, to rewrite Eq. (3.24) as follows,

$$\begin{aligned} \langle 0|\sigma'|0\rangle &= \sqrt{A} \\ \langle 0|\pi|0\rangle &= 0\,. \end{aligned} \tag{3.25}$$

This amounts to assuming that we have chosen the state $\sigma' = \sqrt{A}, \pi = 0$ as the 'ground state' of our classical theory. In this event, following our dictum that we prefer to work with fields whose vacuum expectation values vanish let us introduce a field 'σ' such that

$$\sigma' = \sigma + \sqrt{A}\,. \tag{3.26}$$

With this modification our Lagrangian, Eq. (3.13), becomes:

$$\begin{aligned}\mathscr{L} &= (-1/2)\,((\partial_\mu\sigma)^2 + 8B^2|A|\sigma^2) - (1/2)\,((\partial_\mu\boldsymbol{\pi})^2) \\ &\quad - B^2[(\sigma^2+\boldsymbol{\pi}^2)^2 - 4\sqrt{A}\,\sigma(\sigma^2+\boldsymbol{\pi}^2)] \\ &= (-1/2)\,((\partial_\mu\sigma)^2 + m_\sigma^2\sigma^2) - (1/2)\,(\partial_\mu\boldsymbol{\pi})^2 \\ &\quad - B^2[(\sigma^2+\boldsymbol{\pi}^2)^2 - 4\sqrt{A}\,\sigma(\sigma^2+\boldsymbol{\pi}^2)]\,.\end{aligned} \tag{3.27}$$

Inspection of this Lagrangian reveals a surprising thing. The Lagrangian is, as before invariant under the transformations –

$$\begin{aligned}&\boldsymbol{\pi}\to\boldsymbol{\pi}+\varepsilon\boldsymbol{v}\times\boldsymbol{\pi}\\ &\sigma\to\sigma\end{aligned} \tag{3.28}$$

and the additional transformations

$$\begin{aligned}&\boldsymbol{\pi}\to\boldsymbol{\pi}+\varepsilon\boldsymbol{v}(\sigma+\sqrt{A})\\ &\sigma\to\sigma+\varepsilon\boldsymbol{v}\times\boldsymbol{\pi}\end{aligned} \tag{3.29}$$

however, the particle states of the theory no longer belong to a degenerate, irreducible representation of $SU(2)\otimes SU(2)$ since the σ-meson now has a finite mass and the π-mesons are massless. Moreover, although the vector-current defined by Noether's theorem is still

$$V^\mu = \boldsymbol{\pi}\times\partial_\mu\boldsymbol{\pi} \tag{3.30}$$

the axial-current is now written as:

$$A^\mu = \sigma\partial_\mu\boldsymbol{\pi} - \partial_\mu\sigma\boldsymbol{\pi} + \sqrt{A}\,\partial_\mu\boldsymbol{\pi} \tag{3.31}$$

and thus it follows that in this theory, since $\langle\pi_\alpha|A_\alpha^\mu|0\rangle \neq 0$, the conservation of the axial-current forces $m_\pi^2=0$. One might wonder at this point, if this is not all deceptive. After all, suppose there still is an $SU(2)\otimes SU(2)$ symmetry group defined as acting upon the space of states as in Eqs. (3.21) and (3.23); then, the physical particle states would, perforce, lie in irreducible presentations of $SU(2)\otimes SU(2)$. This, however, is not possible. If one examines the consequences of Eq. (3.31) we see that

$$[\boldsymbol{v}\cdot\boldsymbol{Q}^5, \boldsymbol{\pi}] = \boldsymbol{v}(\sigma+\sqrt{A})\,. \tag{3.32}$$

This says that

$$\begin{aligned}\frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}t}\langle 0|\pi_v(t)|0\rangle &= \langle 0|[\boldsymbol{v}\cdot\boldsymbol{Q}^5,\boldsymbol{\pi}]|0\rangle \\ t=0 &= \boldsymbol{v}\sqrt{A}\,.\end{aligned} \tag{3.33}$$

Thus, it follows that in this theory the one particle states created by applying smeared pion fields to the vacuum, do not stay orthogonal to the vacuum under the transformation generated by the axial-charges; but, if the transformations were unitary this would be necessary, since they start out orthogonal to the vacuum.

Thus, we see that a theory such as the one defined by Eq. (3.27) has conserved currents whose 'formal-charges' form the algebra of $SU(2) \otimes SU(2)$; but, none of the usual consequences of such a 'symmetry' are true. It is easy to see that such a violation of the usual symmetry results is intimately bound to the existence of zero mass particles, Goldstone-bosons, in the theory. The simplest argument can be briefly stated as follows.

In order that the 'symmetry' generated by a charge 'Q' be not realizable by unitary transformation, it is sufficient that there exist a local operator such that

$$\langle 0 | [Q^5, \varphi(0)] | 0 \rangle \neq 0 \,. \tag{3.34}$$

If we define Q as:

$$Q = \lim_{q \to 0} \int d^3x\, j^0(t, x) \tag{3.35}$$

put this into Eq. (3.34), and expand the commutator by inserting a complete set of intermediate states; then, translation invariance of Q implies that only zero mass states can contribute to the vacuum expectation value in Eq. (3.34) as we let $q \to 0$. Thus, if Eq. (3.34) holds there must be zero mass particles in the theory whose coupling to the conserved current $j^\mu(x)$ does not vanish.

As we have just seen, the case $A > 0$ provides us with an example of the second type of symmetry a field theory may possess. This is the symmetry we call a Goldstone-type of symmetry, or a 'spontaneously broken symmetry' (a nomenclature I shall henceforth avoid as it will cause unnecessary confusion when we wish to discuss real symmetry breaking terms added to the Hamiltonian).

The foregoing example can be expanded to include nucleons by using the Lagrangian.

$$\begin{aligned}\mathscr{L} = \bar{N}(i\partial)N + g\bar{N}(\sigma' + i\tau\pi\gamma_5)N - (1/2)\left((\partial_\mu\sigma')^2 + (\partial_\mu\pi)^2\right) \\ - B^2(\sigma'^2 + \pi^2 - A)^2 \,.\end{aligned} \tag{3.36}$$

The discussion of this theory proceeds very much as before. Again we divide our study into the cases (i) $A < 0$ and (ii) $A > 0$. As before, case (i) $A < 0$ corresponds to a theory of the conventional type, in which the particles group themselves into degenerate $SU(2) \otimes SU(2)$ multiplets.

The nucleon mass is forced to be zero by γ^5-invariance and the σ-meson and π-meson masses are degenerate and of arbitrary magnitude.

The case (ii) $A>0$ corresponds to a Goldstone type of symmetry, in which the nucleon obtains a non-zero mass, the σ-meson becomes massive and the π-mesons (our Goldstone-Bosons) are massless.

Finally, in response to those persons who are afraid that a Goldstone particle must be an 'elementary particle' and cannot be 'composite', I would like to refer you to a paper by Y. Nambu and G. Jona-Lascino [1]. In this paper they present a model in which the basic Lagrangian is a non-linear Lagrangian involving only nucleons. They then show, within the framework of perturbation theory with a finite cut-off that there is a solution of this theory of the Goldstone-type. The 'pion' in this theory is described as a complicated nucleon-anti-nucleon bound state. Although the mathematics of this paper is not rigorous, it is very suggestive that the Goldstone interpretation of their results is a correct one.

With these examples behind us, let us now go back and discuss the possible structure of the real world. In the first lecture, we discussed the fact that in the real world the following equations are exact

$$\begin{aligned} \langle\pi_\alpha|A^\mu_\alpha|0\rangle &= -iq^\mu/2f_\pi \\ \langle\pi_\alpha|\partial_\mu A^\mu_\alpha|0\rangle &= m_\pi^2/2f_\pi \\ G_{\pi NN}/f_\pi &= 2m_N g_A + \bar{\partial}(0)\,. \end{aligned} \tag{3.37}$$

Moreover, we saw that the term $m_\pi^2/2f_\pi$ and $\bar{\partial}(0)$ were explicitly of the order of $SU(2)\otimes SU(2)$ symmetry breaking. Let us then enquire as to what the symmetry limit $\varepsilon\to 0$ might be. There are clearly three possibilities

(i) as $\varepsilon\to 0$; $m_\pi^2\to 0$, $(2f_\pi)^{-1}\not\to 0$; $m_N\not\to 0$, $g_A\not\to 0$, $\partial\to 0$ and the Goldberger-Treiman relation becomes an exact, non trivial identity.

(ii) as $\varepsilon\to 0$; $m_\pi^2\not\to 0$ which necessitates that $(2f_\pi)^{-1}\to 0$ and therefore either

(iia) $m_N\to 0$ and $g_A\not\to 0$ or

(iib) $m_N\not\to 0$, $g_A\to 0$ and there must be an opposite parity partner for the nucleon. (The last statement follows from the fact that $\langle N|[Q_1^5+iQ_2^5, Q_1^5-iQ_2^5]|N\rangle=+1/2$, and therefore taking the nucleon-states at rest we get our result).

Now, while it is obvious that case (i) is consistent with the idea of an approximate $SU(2)\otimes SU(2)$ symmetry as being the explanation of all PCAC-results, neither case (ii) nor case (iii) is. This is true since in neither case do we have the Goldberger-Treiman relation since the last identity in Eq. (3.37) becomes zero equals zero. Moreover, in the real world the nucleon mass is not small, $g_A\sim 1$ and there is no opposite parity partner to the nucleon nearly degenerate in mass.

It is obvious that the symmetry limit should be of the Goldstone type, if the approximate symmetry explanation of PCAC is to be correct, since we do not see any approximately degenerate $SU(2)\otimes SU(2)$ multiplets of particles in nature.

Therefore, we have reached the point of being able to propose the following alternative to the usual PCAC hypothesis.

Basic Assumptions

1. The total strong interaction Hamiltonian can be decomposed into two parts H_0 and εH_1. H_0 is assumed to possess an $SU(2)\otimes SU(2)$ symmetry, the axial part of which is realized in the Goldstone way.

2. (εH_1) is assumed to be small, in the sense that perturbation theoretic results about the symmetry limit are reliable.

3. In the symmetrical limit $\varepsilon\to 0$ the pi-mesons become the zero mass 'Goldstone-bosons' for the theory.

4. In the limit $\varepsilon\to 0$ the equation

$$\langle \pi_\alpha | A_\alpha^\mu | 0\rangle = -iq^\mu/2f_\pi$$

becomes

$$\langle \pi_\alpha | A_\alpha^\mu | 0\rangle = -iq^\mu/2f_0$$

where $(f_\pi - f_0)$ is first order in ε.

Within this framework the PCAC *approximation* is *defined* to be the calculation of all physical quantities to zeroth order in ε.

Thus, in the derivation of the Goldberger-Treiman relation, the PCAC approximation tells us to drop the $\bar{\partial}$ terms, while the combination of Eqs. (2.23) and (2.24) together with the PCAC-approximation gives us

$$\begin{aligned}(1/2f_\pi)^2 \langle N'\pi'_\alpha | S | N\pi_\beta\rangle (v=0; x=0) \\ = (-q_\mu k_\nu \langle N' | T(\bar{A}_\alpha^\mu(q)\bar{A}_\beta^\nu(-k)) | N\rangle (0,0;0,0))\end{aligned}$$

which is just the standard PCAC result.

Since all the PCAC results are, by our hypothesis, equivalent to calculating in the symmetry limit $(\varepsilon=0)$, it is interesting to derive a simple procedure for getting all of these low energy theorems. As we shall see in the next lecture the attempt to do so establishes a very beautiful connection between this definition of PCAC and Weinberg's 'phenomenological Lagrangian' formalism.

4. Phenomenological Lagrangians

I would now like to discuss the way in which one derives the consequences of a symmetry of the Goldstone type. A really detailed treatment of this point is given in a paper written by Roger Dashen and myself [2]. I would like, however, to use this lecture in order to familiarize you with the most important aspects of our results. Fortunately, almost all of the important physical insight into what is going on, can be obtained from the study of a few simple cases.

First let us restudy the Goldberger-Treiman relation and the pion-nucleon scattering amplitudes in a world in which $\varepsilon=0$. We shall assume throughout that the isospin symmetry, corresponding to the vector changes, is realized in the conventional way. That is, particles will fall into isotopic multiplets degenerate in mass. The remainder of the symmetry group, $SU(2)\otimes SU(2)$, will be realized in the Goldstone way. Thus, we assume that the pions correspond to the three massless Goldstone-bosons of the theory and the equation $\partial_\mu A^\mu_\alpha=0$ is exact, We, of course, will make extensive use of the kinematic identity

$$\langle\pi_\alpha|A^\mu_\alpha(0)|0\rangle=-iq^\mu/2f_0\,. \tag{4.1}$$

One further point which should be made, is that quantities not required to be zero by the assumption that the axialvector currents are conserved will be assumed to be not *too* different from their physical values.

The derivation of the Goldberger-Treiman relation proceeds as follows. One start with the equation.

$$0=\langle N'|\partial_\mu A^\mu_\alpha(q)|N\rangle=-iq_\mu\langle N'|A^\mu_\alpha(q)|N\rangle(q^2)\,. \tag{4.2}$$

Now, as before, we separate the right hand side of this equation into a pole term and a term having no poles in q^2. The difference is, that since the pion mass is zero the pole in 'q^2' due to the creation of a single meson state from vacuum has its pole at $q^2=0$. Eq. (4.1) and the general form for $\langle N'|A^\mu_\alpha(q)|N\rangle(q^2)$ adopted in Eq. (1.5) allows us to rewrite Eq. (4.2) as

$$\begin{aligned}0=(-q^2/2f_0q^2)\bar{u}(p')\{\gamma^5\tau_\alpha G_{\pi NN}\}\,u(p)+\bar{u}(p')\{(2m_N g_A(q^2)\\ +q^2\bar{h}_A(q^2))\gamma^5\tau_\alpha/2\}\,u(q)\end{aligned} \tag{4.3}$$

or

$$G_{\pi NN}/f_0=2m_N g_A(q^2)+q^2\bar{h}_A(q^2)\,. \tag{4.4}$$

Eq. (4.4) is an exact identity, whose most interesting feature (as before) is that the left-hand side is independent of q^2 while the right-hand side is not. Thus, we are free to evaluate the right-hand side anywhere we wish. Clearly, unless we know a great deal about the form factors

$g_A(q^2)$ and $\bar{h}_A(q^2)$ this is not a very useful thing to do; however, the point $q^2=0$ is an interesting one as by going to $q^2=0$ we get the Goldberger-Treiman relation

$$G_{\pi NN}=2m_N f_0 g_A\,. \tag{4.5}$$

So much for the Goldberger-Treiman relation, let us now turn our attention to the question of pion-nucleon scattering.

As in the case of the Goldberger-Treiman relation our starting point is the identity

$$0=\langle N'|\,T(\partial_\mu A^\mu_\alpha(q)\,\partial_\nu A^\nu_\beta(-k))\,|N\rangle\,(q^2,k^2;\nu,x)\,. \tag{4.6}$$

Pulling all derivatives through the time-ordered product and using the algebra of currents assumed in Eq. (1.1) we get

$$\begin{aligned}0=(+i)^2\,[&-q_\mu k_\nu\langle N'|\,T(A^\mu_\alpha(q)A^\nu_\beta(-k))\,|N\rangle\,(q^2,k^2;\nu,x)\\ &+q_\mu\varepsilon_{\beta\alpha\gamma}\langle N'|\,V^\mu_\gamma(q-k)\,|N\rangle\,(q^2,k^2;x)]\,.\end{aligned} \tag{4.7}$$

If we now note that the term $T(A^\mu_\alpha A^\nu_\beta)$ has double, single, and no-pole terms (just as we saw in the first lecture) except that now they occur at $q^2=k^2=0$, we can rewrite Eq. (4.7) as:

$$\begin{aligned}&(q^2k^2/q^2k^2)(1/(2f_0)^2)\langle N'\pi'_\alpha|\,S\,|N\pi_\beta\rangle(\nu,x)\\ &\quad-(k^2/k^2)(1/2f_0)\langle N'|\,\bar{A}^\mu_\alpha(q)\,|N\pi_\beta\rangle\,(q^2;\nu,x)\\ &\quad+(q^2/q^2)(1/2f_0)\cdot k_\nu\langle N'\pi'_\nu|\,\bar{A}^\nu_\beta(-k)\,|N\rangle\,(k^2;\nu,x)\\ &\quad+q_\mu k_\nu\langle N'|\,T(\bar{A}^\mu_\alpha(q)\,\bar{A}^\nu_\beta(-k))\,|N\rangle\,(q^2,k^2;\nu,x)\\ &\quad+q_\mu\varepsilon_{\beta\alpha\gamma}\langle N'|\,V^\mu_\gamma(q-k)\,|N\rangle\,(q^2,k^2;x)=0\,.\end{aligned} \tag{4.8}$$

This can be simplified, if we notice that the equation

$$0=\langle N'\pi'_\alpha|\,\partial_\mu A^\mu_\beta(-k)\,|N\rangle\,(k^2,\nu,x) \tag{4.9}$$

implies, that (exactly as for the G − T relation except holding 'y' and 'x' fixed)

$$(1/2f_0)\langle N'\pi'_\alpha|\,S\,|N\pi_\beta\rangle(\nu,x)=-k_\nu\langle N'\pi'_\alpha|\,\bar{A}^\nu_\beta(-k)\,|N\rangle\,(k^2,\nu,x) \tag{4.10}$$

we also have

$$0=\langle N'|\,\partial_\mu A^\mu_\alpha(q)\,|N\pi\beta\rangle\,(q^2,\nu,x) \tag{4.11}$$

which similarly implies that

$$(1/2f_0)\langle N'\pi'_\alpha|\,S\,|N\pi_\beta\rangle(\nu,x)=q_\nu\langle N'|\,\bar{A}^\mu_\alpha(q)\,|N\pi_\beta\rangle\,(q^2,\nu,x)\,. \tag{4.12}$$

(N.B. By necessity, the right-hand sides of both Eq. (4.10) and Eq. (4.12) are really independent of 'q^2' since the left-hand sides are. Moreover, in this case the point $q^2=0$ and $k^2=0$ are on mass-shell values for these variables.)

Using Eqs. (4.10) and (4.12) to simplify Eq. (4.8) we get:

$$(1/2f_0)^2\langle N'\pi'_\alpha|S|N\pi_\beta\rangle(\nu,x)=+q_\mu k_\nu\langle N'|T(\bar{A}^\mu_\alpha(q)\bar{A}^\nu_\beta(-k))|N\rangle(\nu,x) \\ +q_\mu\varepsilon_{\alpha\beta\gamma}\langle N'|V^\mu_\gamma(q-k)|N\rangle(x) \tag{4.13}$$

(setting $q^2=k^2=0$).

This is, of course, the same result as obtained from Eqs. (2.23) and (2.24) if we ignore terms of order ε and higher.

At this point we have rederived enough results having something to say about the symmetrical world to see a general pattern emerging. It is clear, that the starting point for every identity relating pi-meson scattering amplitudes to matrix elements of the axial-vector and vector currents between the same states with some of the pions removed is an identity of the form

$$0=\langle A|T(\partial_\mu A^\mu_{\alpha 1}(q_1)\ldots\partial_\mu A^\mu_{\alpha n}(q_n))|B\rangle\,. \tag{4.14}$$

The next step is to rewrite this by moving all derivatives through the time ordering instruction and using the algebra of currents to evaluate the non-vanishing equal time commutators. After this has been accomplished, the basic identities we derive follow from extracting the various $(q_i^2)^{-1}$ pole terms as we did in Eq. (4.8) and simplifying the results by using lower order identities such as those given in Eqs. (4.10) and (4.11). What I would like to do now is state a pair of general results which can be proven without too much difficulty, and then (rather than proving the results) discuss how one uses them to obtain phenomenological Lagrangians for calculating low energy theorems in the world defined by setting $\varepsilon=0$.

I choose to do this, because I feel it is more instructive to see what the limitations of these phenomenological Lagrangians must be, than to follow the straightforward but tedious details leading up to them. Besides, all of the details appear in the previously-mentioned paper by Roger Dashen and myself.

Before stating these two basic results, however, let me first spend a few moments convincing you that in order to know how to evaluate the most general expression of the form

$$\int d^4x_1\ldots d^4x_n e^{+iq_1\cdot x_1}\ldots e^{+iq_n\cdot x_n}\langle A|T(\partial_\mu A^\mu_{\alpha 1}(k_1)\ldots\partial_\mu A^\mu_{\alpha n}(x_n))|B\rangle \tag{4.14}$$

it suffices to know how to evaluate any expression of the form

$$\int d^4x_1\ldots d^4x_n\langle A|T(\varphi\cdot\partial_\mu A^\mu(x_1)\ldots\varphi\cdot\partial_\mu A^\mu(x_n))|B\rangle \tag{4.15}$$

for an arbitrary c-number function $\varphi(x)$.

The proof of this statement can be given in many ways, but perhaps the easiest is to choose for the function $\varphi(x)$ appearing in Eq. (4.15) a function of the form

$$\varphi(x) = \sum_{i=1}^{n} \lambda_i \varepsilon_i \, e^{+iq_1 \cdot x} \tag{4.16}$$

where λ_i are arbitrary parameters and ε_i is an isovector all of whose components except its α_i^{th} component is equal to zero. ($\alpha_i = 1, 2, 3$).

It is then clear that Eq. (4.15) defines a homogenous polynomial of degree 'n' in the variables λ_i ($i = 1, \ldots n$) which by assumption is going to be equal to some other homogenous polynomial of degree 'n' (after doing all commutators), thus the coefficients of the different terms are separately equal. In particular this implies that since the coefficient of the term ($\lambda_1 \cdot \lambda_2 \cdot \lambda_n$) is just $n!$ times the term in Eq. (4.14) we can read off the proper identity for Eq. (4.14) once we know how to handle Eq. (4.15).

With these preliminaries out of the way, we can state the following general result, (which follows from making a canonical transformation on the term $\exp(i2f_0 \int d^4x \boldsymbol{\varphi} \cdot \partial_\mu A^\mu)$ and cancelling poles from the resulting identity.

Theorem: If we let S_0 stand for the S-matrix in the symmetrical theory then

$$\langle A + \pi(\varepsilon_1, q_1) + \cdots + \pi(\varepsilon_n, q_n) | S_0 | B \rangle = (f_\pi)^n \langle A | U^n(q_1 \ldots q_n) | B \rangle \tag{4.17}$$

where $U^n(q_1 \ldots q_n)$ is defined by taking the coefficient of f_π^n in the exponential

$$\begin{aligned} T\{\exp[&+i \textstyle\int d^4x(-2f_\pi)(\partial_\mu \boldsymbol{\varphi} \cdot \bar{A}^\mu(x)) \\ &+ [f_\pi/(1+f_\pi^2)]\varphi^2\{(\boldsymbol{\varphi} \times \partial_\mu \boldsymbol{\varphi}) \cdot V^\mu - f_\pi \varphi \bar{\partial}_\mu \boldsymbol{\varphi} \cdot A^\mu\})]\} \end{aligned} \tag{4.18}$$

and letting $\boldsymbol{\varphi}(x) = \sum_{i=1}^{n} \varepsilon_i e^{+iq_i \cdot x}$ keeping only those terms in which each of the q_i's appears exactly once. (As in previous cases, the bar above the axial-vector current indicates that its single-meson poles in any one of the q_i^2's have been removed).

Note that crossing symmetry allows us to get any other meson amplitude by merely changing the sign of the q_i's, which in effect gives us the amplitude for the i^{th} – pion inserted in state $|B\rangle$.

It is easy to see that the cases $n = 1$ and $n = 2$ give us the formulas obtained in Eqs. (4.10) and (4.13), namely

$$\langle A + \pi(\varepsilon, k) | S_0 | B \rangle = f_\pi[-2i \langle A | \textstyle\int d^4x \, \partial_\mu \boldsymbol{\varphi} \bar{A}^\mu | B \rangle]$$

and

$$\langle A+\pi(\varepsilon_1,q)+\pi(\varepsilon_2,k)|S_0|B\rangle = f_\pi^2[2(+i)^2\langle A|T(\int\partial_\mu\varphi\cdot\bar{A}^\mu \int\partial_\nu\varphi\bar{A}^\nu)|B\rangle - 2i\langle A|\int(\varphi\times\partial_\mu\varphi)\cdot V^\mu|B\rangle].$$

This general identity already seems to be practically a phenomenological Lagrangian in that we get the appropriate S-matrix elements for scattering processes by treating as if it were the pion field and calculating to the relevant order in f_π. This is not quite true, however. We shall soon see the reason for this, and the remedy; but first there is a point to be made.

The identities given by Eqs. (4.17) and (4.18) are exact, but useless. I say useless, because we do not know any reliable way of evaluating matrix elements of time-ordered products of currents. Nevertheless, as with the case of the G – T relation and pi-nucleon scattering amplitudes useful results can be obtained when we let all of the momenta $q_i\to 0$. Let us now see how this works.

In order to discuss this point it becomes convenient to introduce a scaling parameter 'ξ' into the problem by choosing fixed four momenta Q_i and letting $q_1=\xi Q_i$. The so-called soft-pion limit, which is what we are interested in, then corresponds to letting $\xi\to 0$. It can be shown in general that one can derive (in a very straight forward manner) a 'phenomenological Lagrangian' which can be used to correctly calculate the coefficient of the lowest power of ξ appearing in an expansion of the amplitude in question about the point $\xi=0$. While we do not have the time to go through the general treatment of this problem we can see all of the important considerations exemplified by studying the process $\pi+N\to 2\pi+N$.

In order to see how everything works we use (4.17) and (4.18) to derive:

$$\begin{aligned}\langle N+3_\pi|S_0|N\rangle = f_\pi^3\,[&(4/3)i\langle N|T(\partial_\mu\varphi\cdot\bar{A}\,\partial_\nu\varphi\bar{A}\,\partial_\sigma\varphi\bar{A}^\sigma)|N\rangle\\ &-2\langle N|T(\partial_\mu\varphi\bar{A}^\mu(\varphi\times\partial_\mu\varphi)\cdot V^\mu)|N\rangle\\ &+2i\langle N|\varphi^2\partial_\mu\varphi\cdot A^\mu|N\rangle].\end{aligned}\tag{4.19}$$

Calculating the term of order ξ is not difficult at this point, provided we observe that every factor of $\partial_\mu\varphi$ gives rise to an explicit factor of ξ, multiplying any term in which it appears. Thus the first term in Eq. (4.19) is of order ξ^3, the second ξ^2 and the last of order ξ.

If the explicit ξ dependence were everything, then only the last term could contribute to the leading order of ξ; however, this is not the case.

For example, the first term in Eq. (4.19) has a piece which goes as ξ^{-2}, coming from diagrams of the sort shown in Fig. (4.1a), corresponding to insertions of the axial-vector current on the external nucleon lines, since the propagators (which are of the form $(p-q)$ go as ξ^{-1}. Note, there is another diagram which gives rise to a factor ξ^{-2}, coming from diagrams in which a single pion intermediate state of momentum $(q_1+q_2+q_3)$ is created from vacuum. This state can be created since we have removed poles in q_1^2, q_2^2 and q_3^2 but not in $(q_1+q_2+q_3)^2$.

This diagram is shown in Fig. (4.1b), and we shall have more to say about it in a moment. For the moment, however, let us emphasize that these two diagrams (and only these two diagrams) contribute to the behavior of the going as ξ. Let us now go on to consider the second term in Eq. (4.19).

Again, this term contributes to order ξ^2, except for the two terms shown in Figs. (4.1c) and (4.1d); finally, we see that the diagrams in Figs. (4.1c) and (4.1f) are the only important contributors to the last term in Eq. (4.19) in the limit $\xi \to 0$. Now, the diagrams given in Figs. (4.1a), (4.1c) and (4.1d) are easily evaluated in the limit $\xi \to 0$ in terms of the known axial-vector and vector coupling constants; however, the same is not true for Figs. (4.1b), (4.1d) and (4.1f). All is not lost however; although we cannot say much about these terms individually, we can say what the coefficient of the leading power of ξ in their sum must be when $\xi = 0$. The reason for this being that unitarity tells us the residue of the one-pion pole term which appears in the scattering amplitude in question must factor into the product of the on-mass-shell $\pi-\pi$ scattering amplitude times the $\pi - N$ vertex function. It can be shown by using an analysis analogous to this for divergence of four axial-currents taken between vacuum states (here I refer you to the paper I previously mentioned) that the $\pi-\pi$-amplitude goes as ξ^2. Thus, it cancels the ξ^{-2} coming from the pion propagator and therefore once we know how to compute the ξ^2 part of the $\pi-\pi$-amplitude we can calculate the coefficient of the sum of the three terms.

Note this last point well, for it is very important. In order to know how to calculate the behavior of scattering amplitudes involving mesons in the limit $\xi \to 0$ it says we can pretend that the basic identity given in Eq. (4.18) can be looked upon as a 'phenomenological Lagrangian' and all calculations are to be done in the '*tree approximation*' (i.e. allow no diagrams having internal loops) *only after we have explicitly separated out a phenomenological term which correctly gives the ξ^2 part of $\pi-\pi$ scattering and explicitly added it to our Lagrangian.*

Let us see how the previous discussion gives us such a Lagrangian. Clearly the vertices in Fig. (4.1a) are given by an expression of the form

$$\bar{u}_N \{g_A \gamma^\mu \gamma_5 \boldsymbol{\tau}\} u_N \tag{4.20}$$

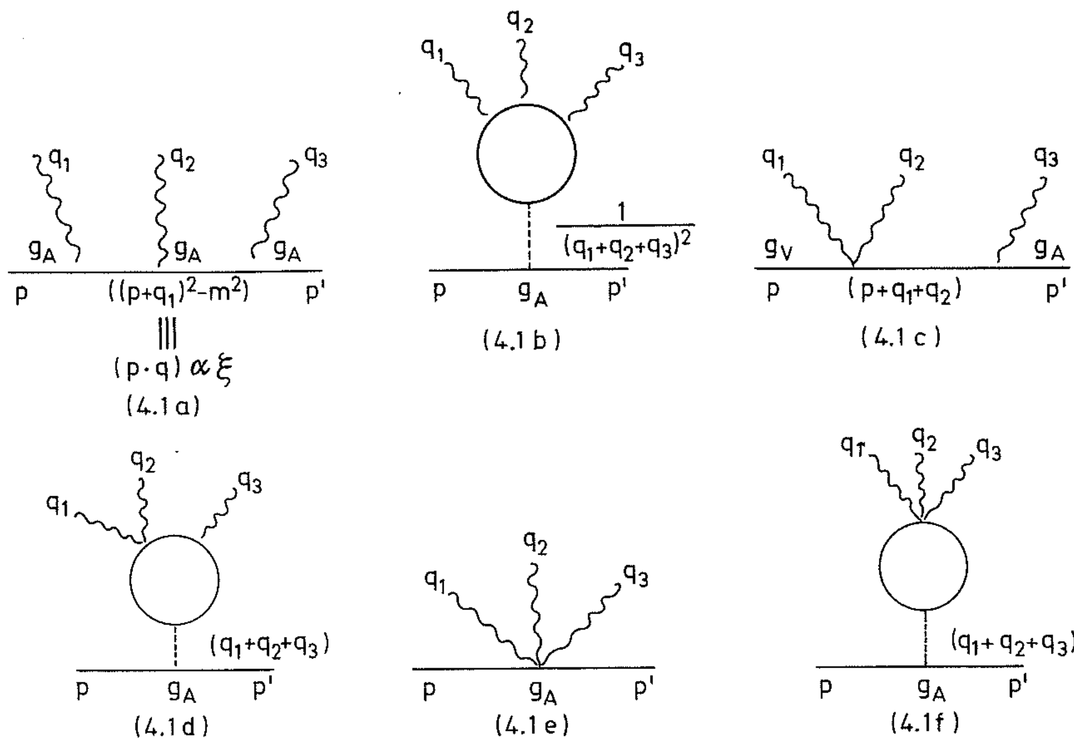

Fig. 4.1

and Eq. (4.19) tells us to multiply each such factor by a $\partial_\mu \boldsymbol{\varphi}$, so that the coefficient of the term of order ξ' arising from the first term in Eq. (4.19) is the same as if we calculated to order f_π^3 with an effective pion-nucleon coupling of the form

$$\mathscr{L}_1 = -f_\pi \partial_\mu \boldsymbol{\varphi} \cdot \overline{\psi}_N (g_A \gamma^\mu \gamma_5 \boldsymbol{\tau}) \psi_N(x) \tag{4.21}$$

where $\varphi(x)$ and $\psi_N(x)$ are the pion and nucleon fields respectively. Similarly, the vector vertex appearing in Fig. (4.1c) contributes to the order ξ' piece of the second term in Eq. (4.19) a term of the general form

$$\overline{u}_N (\gamma^\mu \boldsymbol{\tau}) u_N \tag{4.22}$$

and a glance at Eq. (4.19) tells us that this term, when it appears, must be multiplied by a factor $\boldsymbol{\varphi} \times \partial_\mu \boldsymbol{\varphi}$. Again, it is clear that we get the same results by adding to our Lagrangian a term of the form

$$\mathscr{L}_2 = -f_\pi^2 (\boldsymbol{\varphi} \times \partial_\mu \boldsymbol{\varphi}) \cdot \overline{\psi}_N (\gamma^\mu \boldsymbol{\tau}) \psi_N(x)\,. \tag{4.23}$$

By using $\mathscr{L}_1 + \mathscr{L}_2$ and calculating to order f_π^3 we generate the contributions of Figs. (4.1a) and (4.1c).

Next, it is obvious that the last term in Eq. (4.19) corresponds to a term:

$$\mathscr{L}_3 = f_\pi^3 \varphi^2 \partial_\mu \boldsymbol{\varphi} \cdot \overline{\psi}_N (g_A \gamma^\mu \gamma_5 \boldsymbol{\tau}) \psi_N \tag{4.24}$$

and now $\mathscr{L}_1 + \mathscr{L}_2 + \mathscr{L}_3$ used to calculate tree-diagrams of order f_π^3 gives us all of the amplitude in question except the contribution of $\pi - \pi$-scattering.

If I may quote a result without proof, it can be shown that the ξ^2-part of the four pion vertex can be written as

$$\mathscr{L}_4 = f_\pi^2 \varphi^2 \partial_\mu \boldsymbol{\varphi} \partial^\mu \boldsymbol{\varphi} \tag{4.25}$$

and now $(\mathscr{L}_1 + \mathscr{L}_2 + \mathscr{L}_3 + \mathscr{L}_4)$ gives the same answers for the behaviour of the physical amplitude as that required by PCAC and current algebra.

If one carries out this analysis for all possible pion-nucleon processes we find that the non-linear Lagrangian

$$\begin{aligned}\mathscr{L}_{\text{eff}} = &-\overline{\psi}_N[i\gamma^\mu \partial_\mu + m_N]\psi_N - \overline{\psi}_N \left[\frac{\gamma^\mu}{1 + f_\pi^2 \varphi^2}(f_\pi g_A \gamma_5 \boldsymbol{\tau} \partial_\mu \boldsymbol{\varphi}\right. \\ &\left. + f_\pi^2 \boldsymbol{\tau} \cdot (\boldsymbol{\varphi} \times \partial_\mu \boldsymbol{\varphi}))\right]\psi_N - \frac{1}{2}\frac{(\partial_\mu \boldsymbol{\varphi} \cdot \partial^\mu \boldsymbol{\varphi})}{(1 + f_\pi^2 \varphi^2)^2}\end{aligned} \tag{4.26}$$

can be used to correctly calculate the coefficient of the lowest non-vanishing order of ξ in any process involving only pions and nucleons.

These considerations can be extended to other processes involving particles other than nucleons and the results of such considerations are to reproduce the most general phenomenological Lagrangian first considered by Weinberg.

Clearly, our discussion already makes clear the fact that the Lagrangians are only reliable when used to calculate the low energy behaviour of meson amplitudes and in general there is no compelling reason to believe anything they have to say about terms involving the next order in ξ.

The primary usefulness of these Lagrangians lies in the fact that whenever there is a result which follows from our assumption of a Goldstone-symmetry of this type, one is guaranteed that there exists a phenomenological Lagrangian (of sufficiently complex nature) which we can use to get the same result. Very often this is the most economical way of doing things. A more important use of such Lagrangians, however, is as a source of counterexamples to conjectured theorems one hopes he might prove as a consequence of PCAC + current algebra. Since, if one can find a Lagrangian which invalidates his proposed idea, then one need not continue to try to prove it, by more cumbersome means.

As far as the real world goes, since we have identified the PCAC-results with calculation of the terms in an amplitude which are order ε^0 (i.e. zeroth order in chiral-symmetry breaking) we see that phenomenological Lagrangians provide an alternative way to calculate these coefficients; provided, of course, that we use a sufficiently complicated Lagrangian.

Though there is much more which can be said about what happens in the symmetrical world and many formal identities which hold for a world in which $\varepsilon > 0$ but is small, I shall not talk about these points at all. Instead in the next lecture I would like to leave such general considerations behind and talk about specific *'new experimental predictions'* which one is led to when one begins to try to understand how to study the behaviour of terms of order ε^1.

5. Beyond the Zeroth Approximation

I would like to devote this lecture to discussing how this idea of an approximate chiral symmetry of the Goldstone type leads one to consider any entirely new set of predictions (which are subject to experimental verification in the near future) which follow from trying to go beyond the PCAC-approximation. That is, beyond the calculation of terms of order ε^0.

The most interesting possibilities arise when we stop restricting ourselves to an approximate Goldstone symmetry for pions, and postulate instead that the entire octet of pseudo-scalar mesons (π's, K's and η) form a set of would-be Goldstone bosons. In other words, that the approximate symmetry of interest is $SU(3)\otimes SU(3)$. Of course, except for purely technical complications none of the results discussed up to this point are seriously modified by enlarging the group.

Allow me to remind you of the basic idea which we are discussing. Our starting point is a picture of the *strong interactions* in which

1. The total strong interaction Hamiltonian can be decomposed into an $SU(3)\otimes SU(3)$-symmetric part H_0, plus a symmetry breaking term εH_1.

2. ε is assumed to set the scale of H_1 relative to H_0 and by assumption ε is to be thought of as small enough so that most predictions (for processes which do not vanish in the symmetry limit) are correct to 20%

3. The limit $\varepsilon \to 0$ corresponds to one which the $SU(3)$-symmetry generated by the vector charges is realizable in the conventional way; that is, by a unitary group of Hilbert space transformations which leave the vacuum invariant. However, the remainder of the symmetry transformations are not realized in this way, but rather in the Goldstone manner. This, of course, implies that the particles in this theory group themselves into $SU(3)$ multiplets of particles degenerate in mass, and that there exists an octet of massless pseudoscalar mesons (which we assume to be the $\varepsilon \to 0$ limit of the π's, K's and η in the real world).

To these we add the independent hypothesis:

4. For the purposes of calculating the major deviations from the predictions of symmetric theory it is sufficient to work to lowest order in εH_1.

In order to illustrate how one utilizes assumptions 1–4 in order to make predictions about the real world I would like to discuss several easily derivable results. Except for one such prediction (related to the form factors in K_{l_3}-decay) all of these results depend upon assuming specific $SU(3)\otimes SU(3)$ transformation properties for εH_1. There is, however, nothing surprising about this, as one always has to make some assumptions about the transformation properties of symmetry breaking before deriving any results. Nevertheless, I feel that the kind of results I am about to describe represent an improvement in the state of the art in that one makes no ad-hoc dynamical assumptions such as assuming that dispersion relations are saturated by a few low lying states, in order to make predictions. For this reason, the predictions which one makes are relatively clean and subject to experimental verification.

The first such prediction, which I would like to discuss in some detail is a generalized version of the Goldberger-Treiman relation. One virtue to starting with this prediction is that it is easy to discuss and yet provides us with an interesting example of how one implements hypothese 1–4 which we just discussed.

For pedagogical reasons, and because one obtains the largest number of interesting predictions in this way, we shall assume the simplest $SU(3)\otimes SU(3)$ transformation properties for εH_1; namely, that εH_1, belongs to the $(3,\bar{3})\oplus(3,\bar{3})$ representation of $SU(3)\otimes SU(3)$. Since we shall make use of the general properties of this model throughout the remainder of this discussion, I will first briefly summarize the important facts that follow from this assumption.

(N.B. All of the properties to be listed in the next few moments follow directly once one observes that the $(3,\bar{3})\oplus(3,\bar{3})$ representation of $SU(3)\otimes SU(3)$ is equivalent to that provided by the set of 3×3 matrices where the most general transformation due to the vector charges is given by

$$e^{i\eta\cdot Q}(A)\equiv e^{+i\eta\cdot\lambda}Ae^{-i\eta\cdot\lambda} \tag{5.1}$$

where λ's are the usual lambda matrices giving the three dimensional representation of $SU(3)$ and the $\eta_i\,(i=1,\dots 8)$ are eight arbitrary real numbers; and the most general transformation generated by the axial charges is given by

$$e^{i\eta\cdot Q^5}(A)\equiv e^{+i\eta\cdot\lambda}Ae^{+i\eta\cdot\lambda}\,. \tag{5.2}$$

As before it is convenient to introduce the two commuting sets of charges $Q\pm$ by;

$$Q_\alpha^\pm(t) = [Q_\alpha(t) \pm Q_\alpha^5(t)]/2. \tag{5.3}$$

If we now add the requirement that the Q_α's are even-parity operators and the Q_α^5's are odd parity operators, then:

$$\mathscr{P} Q^\pm \mathscr{P} = Q^\mp. \tag{5.4}$$

Adopting these conventions and assuming that εH_1, is given by

$$\varepsilon H_1(t) \equiv \varepsilon \int d^3x\, \mathscr{H}_1(t, \boldsymbol{x}) \tag{5.5}$$

one has that

$$\begin{aligned} \partial_\mu V_\alpha^\mu(t, \boldsymbol{x}) &= i\varepsilon[Q_\alpha(t), \mathscr{H}_1(t, \boldsymbol{x})] \\ \partial_\mu A_\alpha^\mu(t, \boldsymbol{x}) &= i\varepsilon[Q_\alpha^5(t), \mathscr{H}_1(t, \boldsymbol{x})]. \end{aligned} \tag{5.6}$$

The principal importance of Eq. (5.6) lies in the fact that once one assumes that $\mathscr{H}_1(t, \boldsymbol{x})$ belongs to a given irreducible representation of $SU(3) \otimes SU(3)$ generated by the $Q_\alpha(t)$'s, then it follows that the divergences belong to the same irreducible representation of $SU(3) \otimes SU(3)$. Much of what we shall discuss makes use of this fact.

Let us now specialize to the case in which $\varepsilon\mathscr{H}_1$ belongs to a Gell-Mann-Oakes-Renner type of model (i.e. $(3, \bar{3}) \otimes (\bar{3}, 3)$). If we decompose this representation into irreducible representations of the $SU(3)$ subgroup generated by the vector charges, we see that the eighteen dimensional space decomposes into a set U_0, U_α ($\alpha = 1, \ldots 8$) of even parity operators, and V_0, V_α of odd parity operators. The transformation of these operators under commutation with the charges Q_α and Q_α^5 are

$$\begin{aligned} &[Q_\alpha, u_\beta] = i f_{\alpha\beta\gamma} u_\gamma \qquad (\alpha, \beta, \gamma = 1, \ldots 8) \\ &[Q_\alpha, v_\beta] = i f_{\alpha\beta\gamma} v_\gamma \\ &[Q_\alpha^5, u_\beta] = -i d_{\alpha\beta\gamma} v_\gamma - i\sqrt{2/3}\, \delta_{\alpha\beta} v_0 \\ &[Q_\alpha^5, v_\beta] = -i d_{\alpha\beta\gamma} u_\gamma - i\sqrt{2/3}\, \delta_{\alpha\beta} u_0 \\ &[Q_\alpha, u_0] = [Q_\alpha, v_0] = 0 \\ &[Q_\alpha^5, u_0] = -i\sqrt{2/3}\, v_\alpha \\ &[Q_\alpha^5, v_0] = -i\sqrt{2/3}\, u_\alpha. \end{aligned} \tag{5.7}$$

If we now require that εH_1, conserves isospin, hypercharge and parity, so that Q_1, Q_2, Q_3, Q_8 and $\mathscr{P}$ commute with εH_1, we see that the most general form for $\varepsilon\mathscr{H}(t, x)$ is

$$\varepsilon\mathscr{H}_1(t, \boldsymbol{x}) \equiv \varepsilon[c_0 u_0(t, \boldsymbol{x}) + c_8 u_8(t, \boldsymbol{x})]. \tag{5.8}$$

It then follows directly from Eq. (5.6) that

$$\partial_\mu V^\mu_\alpha(t,\mathbf{x}) = \varepsilon c_8 \sum_{\gamma=1}^{8} f_{\alpha 8\gamma} u_\gamma(t,\mathbf{x})$$

$$\partial_\mu A^\mu_\alpha(t,\mathbf{x}) = -\varepsilon[\sqrt{2/3}\,\delta_{\alpha 8} c_8 v_0 + c_0\sqrt{2/3}\,v_\alpha + \sum_{\gamma=1}^{8} c_8 d_{8\alpha\gamma} v_\gamma]\,. \tag{5.9}$$

Since the matrix $(d_8)_{\alpha\beta}$ is diagonal it follows that the last line of Eq. (5.9) can be written as

$$\partial_\mu A^\mu_\alpha = -\varepsilon[c_\alpha v_\alpha + \sqrt{2/3}\,\delta_{\alpha 8} c_8 v_8] \tag{5.10}$$

where the constant c_α is defined as

$$c_\alpha \equiv (c_8 d_{8\alpha\alpha} + c_0\sqrt{2/3})\,. \tag{5.11}$$

There is a very important point which should be made at this time. Namely, that the fact that

$$\langle M_\alpha | \partial_\mu A^\mu_\beta | 0\rangle = (m^2_\alpha / 2f_\alpha)\delta_{\alpha\beta} \tag{5.12}$$

implies that

$$(m^2_\alpha/2f_\alpha)\delta_{\alpha\beta} = \varepsilon c_\alpha \langle M_\alpha | v_\beta | 0\rangle \equiv \varepsilon c_\alpha \delta_{\alpha\beta} ||V|| \tag{5.13}$$

or in other words

$$\varepsilon c_\alpha \propto (m^2_\alpha/2f_\alpha)\,. \tag{5.14}$$

What Eq. (5.14) says is that the divergences $\partial_\mu A^\mu_\alpha$ do not necessarily form a properly normalized basis for an irreducible representation of $SU(3)$; that is, the relation

$$[Q_\alpha, \partial_\mu A^\mu_\beta] = i f_{\alpha\beta\gamma} \partial_\mu A^\mu_\gamma \tag{5.15}$$

is not always true (since $\partial_\mu A^\mu_\alpha = \varepsilon c_\alpha v_\alpha$). However, it is true that the v_α's, where $v_\alpha = c_\alpha^{-1} \partial_\mu A^\mu_\alpha$, do (by assumption) satisfy Eq. (5.15). In many applications this is a very important point to remember because it implies a relation between the way one chooses to break $SU(3)\otimes SU(3)$ and the pattern of meson masses.

Note, within the framework of our assumptions deviations of meson-mass ratios from unity is explained by the fact that the meson mass vanishes in the limit $\varepsilon = 0$ and thus (m^2_α/m^2_β) can be anything it likes.

Eqs. (5.7) and (5.9) also allow us to write for the Σ-term:

$$\Sigma_{\beta\alpha}(t,\mathbf{x}) = -\varepsilon[Q^5_\alpha(t,\mathbf{x}), [Q^5_\beta(t), \mathscr{H}_1(t,\mathbf{x})]] \tag{5.16}$$

the equality

$$\Sigma_{\beta\alpha} = \varepsilon[((2/3)c_0\delta_{\alpha\beta} + \sqrt{2/3}\,c_8 d_{8\alpha\beta})u_0 + (c_0\sqrt{2/3}\,d_{\alpha\beta\gamma} + c_8 d_{8\beta\sigma}d_{\sigma\alpha\gamma} + (2/3)\delta_{\beta 8}\delta_{\alpha\gamma})u_\gamma]\,. \tag{5.17}$$

There is one additional formula relating the meson masses to the Σ-term which is easily derived as follows. Begin with the identity

$$\langle 0|\,T(\partial_\mu A^\mu_\alpha(q)\,\partial_\nu A^\nu_\beta(-q))\,|0\rangle = (+i)^2[+q_\mu q_\nu\langle 0|\,T(A^\mu_\alpha(q)\,A^\nu_\beta(-q))\,|0\rangle - i\langle 0|\,\Sigma_{\alpha\beta}(0)\,|0\rangle]\,. \tag{5.18}$$

(N.B. $q_\mu\langle 0|\,V^\mu_\varrho\,|0\rangle = 0$ by isospin invariance of the vacuum). Now if we note that isospin invariance implies that $\alpha=\beta$, both sides of the expression are functions of q^2 alone we get by extracting the $(q^2-m^2)^{-1}$ pole term explicitly from both sides of the equation

$$\left[\frac{m_\alpha^2}{2f_\alpha}\,\frac{i}{(q^2-m_\alpha^2)}\,\frac{m_\alpha^2}{2f_\alpha} - \langle 0|\,T(\bar{\partial}_\alpha(q)\,\bar{\partial}_\alpha(-q))\,|0\rangle\right] = \left[\left(\frac{q^2}{2f_\alpha}\right)\frac{(+i)}{(q^2-m_\alpha^2)}\left(\frac{q^2}{2f_\alpha}\right) - q_\mu q_\nu\langle 0|\,T(\bar{A}^\mu_\alpha(q)\,\bar{A}^\nu_\alpha(-q))\,|0\rangle + i\langle 0|\,\Sigma_{\alpha\alpha}(0)\,|0\rangle\right] \tag{5.19}$$

so, by recombining pole terms and setting $q^2=0$, we get

$$(m_\alpha/2f_\alpha)^2 = \langle 0|\,\Sigma_{\alpha\alpha}(0)\,|0\rangle + i\langle 0|\,T(\bar{\partial}_\alpha(0)\,\bar{\partial}_\alpha(0))\,|0\rangle\,. \tag{5.20}$$

Since, the last term in Eq. (5.20) is second order in symmetry breaking we have that to lowest order in symmetry breaking

$$(m_\alpha/2f_\alpha)^2 \cong \langle 0|\,\Sigma_{\alpha\alpha}(0)\,|0\rangle$$

or by Eq. (5.17)

$$m_\alpha^2 \cong (4f_\alpha^2)\,[(2/3)c_0 + \sqrt{2/3}\,c_8 d_{8\alpha\alpha}]\,\langle 0|\,u_0\,|0\rangle\,. \tag{5.20'}$$

Besides giving us a relationship between the Σ-term and the meson masses. Eq. (5.20) also makes it clear that there is a choice for εH_1, which leaves the $SU(2)\otimes SU(2)$ symmetry of H_0 unbroken. (i.e. s.t. $m_\pi^2=0$). Namely,

$$c_0 = -2^{-1/2}\,. \tag{5.21}$$

With these points behind us, let us proceed to study a sum rule for the derivations from Generalized Goldberger-Treiman relations.

Let us begin by writing the identity

$$\langle B'|\,\partial_\mu A^\mu_\alpha(q)\,|B\rangle = -iq_\mu\langle B'|\,A^\mu_\alpha(q)\,|B\rangle\,. \tag{5.22}$$

If we remove the single meson pole terms by hand and define $\bar{\partial}_\alpha$ and $\bar{A}_\alpha$ as before, this gives us the Generalized Goldberger-Treiman relation:

$$G_{B'B\alpha}/f_\alpha = (m_{B'} + m_B) g_{B'B\alpha} + \delta_{B'B\alpha} \tag{5.22'}$$

where we have adopted the conventions

$$\langle B'| A^\mu_\alpha(0)|B\rangle \equiv \bar{u}_{B'}(p')\{(1/2)[\gamma^\mu\gamma^5 g_{B'B\alpha}(q^2) + q^\mu\gamma^5 h_{B'B\alpha}(q^2)]\} u_B(p)$$

$$\langle B'|\partial_\mu A^\mu_\alpha(0)|B\rangle \equiv i\bar{u}_{B'}(p')\{(1/2)\gamma^5 \partial_{B'B\alpha}(q^2)\} u_B(p)$$

and

$$\partial_{B'B\alpha}(q^2) \equiv G_{B'B\alpha} m_\alpha^2/(2f_\alpha)(q^2 - m_\alpha^2) - \delta_{B'B\alpha}(q^2)\,. \tag{5.23}$$

Now, if we assume εH_1 belongs to a $(3,\bar{3})\oplus(\bar{3},3)$ representation of $SU(3)\otimes SU(3)$ then the divergence of the axial-vector current (with a factor of $(m_\alpha^2/2f_\alpha)$ removed) is the sum of an $SU(3)$ singlet operator and an $SU(3)$ octet operator. This means that to lowest order in $SU(3)$ *breaking* the term $\delta_{B'B\alpha}$ has the general form.

$$\delta_{B'B\alpha} = b[(1-a) f_{B'B\alpha} + iad_{B'B\alpha}](m_\alpha^2/2f_\alpha)(\alpha, B', B = 1, \ldots, 8) \tag{5.24}$$

where '$f_{B'B\alpha}$' and '$d_{B'B\alpha}$' are the $SU(3)$ 'f' and 'd' symbols and 'a' determines the amount of 'f' to 'd' mixing. Thus we have,

$$\delta_{B'B\alpha} = (m_\alpha^2 b/2f_\alpha)[(1-a) f_{B'B\alpha} + ad_{B'B\alpha}]\,. \tag{5.25}$$

Or, in other words the deviations from the generalized Goldberger-Treiman relation are (to lowest order in $SU(3)$) determined by two parameters. Thus, for example if we measure $g_{n\varrho\pi}$, $g_{N\Sigma K}$ and $g_{\Lambda NK}$ as well as $G_{N\Lambda K}$ and $G_{N\Sigma K}$ ($G_{NN\pi}$ is already known to the required accuracy); then, we have three equations in two unknowns – or, in other words there is a sum rule for these differences.

Note that if for some reason $SU(2)\otimes SU(2)$ remained an exact symmetry the Goldberger-Treiman relation for pions and nucleons would be exact and one would need to measure one more strange process in order to get an overdetermined set of equations for 'a' and 'b'.

So much for this, let us now go on to study the case of meson-baryon scattering. In lecture 1 we discussed this process for the case $\alpha = 1, 2, 3$. Clearly, there is no difference if we allow 'α' to range from one to eight, except that '$\varepsilon_{\alpha\beta\gamma}$' in Eq. (2.4) is replaced by '$f_{\alpha\beta\gamma}$'.

Thus, if in Eq. (2.24) we set $q^2 = k^2 = \nu = x = 0$

$$\begin{aligned}\frac{\langle B'M'_\alpha|S|BM_\beta\rangle(0,0)}{(2f_\alpha)(2f_\beta)} &\equiv q_\mu k_\nu \langle B'|T(\bar{A}^\mu_\alpha(q)\bar{A}^\nu_\beta(-k))|B\rangle(0,0;0,0)\\ &+ (q_\mu/2f_\beta)\langle B'|\bar{A}^\mu_\alpha(q)|BM_\beta\rangle(0;0,0) - (k_\nu/2f_\alpha)\langle B'M_\alpha|(\bar{A}^\nu_\beta(-k))|B\rangle(0;0,0)\\ &- q_\mu f_{\alpha\beta\gamma}\langle B'|V^\mu_\gamma(q-k)|B\rangle(0,0;0) + i\langle B'|\Sigma_{\alpha\beta}(q-k)|B\rangle(0,0;0)\\ &+ \langle B'|T(\bar{\partial}_\alpha(q)\bar{\partial}_\beta(-k))|B\rangle(0,0;0,0)\,.\end{aligned} \tag{5.26}$$

In order to get a useful result from Eq. (5.26) let us adopt the following conventions.

First, let us write the most general expression for the left hand side of Eq. (5.26), namely,

$$\langle B' M_\alpha | S | B M_\beta \rangle (\nu, x) \equiv \bar{u}_{B'}(p') [A_{\alpha\beta}(\nu, x) + (\not{q} + \not{k}) D_{\alpha\beta}(\nu, x)] u_B(p)$$

where

$$\nu = (1/2)(p_{B'} + p_B) \cdot (q + k)$$
$$x = q \cdot k . \tag{5.27}$$

Next, we note that we similarly have

$$\langle B' | \Sigma_{\alpha\beta}(0) | B \rangle (t) \equiv \bar{u}_{B'}(p') u_B(p) \chi_{\alpha\beta}(t) \tag{5.28}$$

where $t = (p' - p)^2$.

Let us now define the 'spin-averaged' amplitude by

$$M_{\alpha\beta}(\nu, x) \equiv (1/2) \mathrm{Tr} \left[\frac{(\not{p}'_B + m'_B)}{2m_{B'}} (A_{\beta\alpha}(\nu, x) + (k + q) D_{\beta\alpha}(\nu, x)) \frac{(\not{p}_B + m_B)}{2m_B} \right] \tag{5.29}$$

and the 'spin-averaged σ-term' by

$$\sigma_{\alpha\beta}(t) \equiv (1/4) \mathrm{Tr} \left[\frac{(\not{p}'_B + m_{B'})}{2m_{B'}} \frac{(\not{p}_B + m_B)}{2m_B} \right] \chi_{\alpha\beta}(t) . \tag{5.30}$$

If one adopts these conventions, and notes that by spin-averaging the terms $\langle B' | T(\bar{A}^\mu_\alpha(q) \bar{A}^\nu_\beta(-k)) | B \rangle$, $\langle B' | \bar{A}^\mu_\alpha(q) | B M_\beta \rangle$ and $\langle B' M_\alpha | \bar{A}^\nu_\beta(-k) | B \rangle$ these become functions only of the appropriate four-vectors $p_{B'}, p_B, q$ and K. We can (by taking the limit $q^2 = k^2 = \nu = x = 0$ and carefully studying the terms which survive) prove that [3]

$$(2f_\alpha)^{-1} (2f_\beta)^{-1} M_{\alpha\beta}(0, 0) \equiv (\mathrm{poles})_{\alpha\beta} - (1/2)(\sigma_{\alpha\beta}(0) + \sigma_{\beta\alpha}(0)) + O(\varepsilon^2) \tag{5.31}$$

where to order ε^2, $(\mathrm{poles})_{\alpha\beta}$ is readily calculated – as it is the contribution of those diagrams with single baryon states – using the measured value of g_A; and the 'f' to 'd' ratio. Of course, g_A stands for the usual axial-vector coupling constant; and the 'f' to 'd' ratio for the process in question is given in terms of a parameter $(1-c)/c$, which we can determine by finding the value of 'c' giving the best fit to the equation

$$G_{B'B\alpha} \cong G_{NN\pi} [c d_{B'B\alpha} + i(1-c) f_{B'B\alpha}] \tag{5.32}$$

Eq. (5.16) allows us to determine $(\sigma_{\alpha\beta} + \sigma_{\beta\alpha})$ from an assumption about $SU(3) \otimes SU(3)$ transformation properties of $\varepsilon \mathscr{H}_1$; for example for the case $(3, \bar{3}) \otimes (\bar{3}, 3)$, $(\sigma_{\alpha\beta} + \sigma_{\beta\alpha})$ is given by Eq. (5.17). Thus, Eq. (5.32) provides us with an extremely interesting theorem. Namely, it says that the 'one-mass shell' meson-nucleon scattering amplitude at the unphysical point $\nu = x = 0$ is determined up to terms of order ε^2 by making a purely group theoretic statement about the transformation properties of $\varepsilon\mathscr{H}_1$. Since the point $\nu = x = 0$ corresponds to a non-zero value of $t = (q' - k)^2 = (m_\alpha^2 + m_\beta^2)$, $S = (p + k)^2 = (m_B^2 - (m_\alpha^2 + m_\beta^2)/2)$ 't' is in the physical region for the process but (s) is below threshold. Thus, $M_{\alpha\beta}(0, 0)$ can be evaluated using 'on mass shell fixed 't' dispersion relations' which do not suffer from the most of ambiguities encountered by trying to go off-mass shell. At present, the experimental data seems to be at the point where one should, with some numerical work, be able to check this prediction.

There is one additional point which one should discuss at this point which has to do with the possible effects of the existence of the y_0^*; however, time does not permit us to go into this point in detail, so let me refer you once again to Ref. [3].

I would now like to conclude these talks by making some comments about the subject of chiral $SU(3)\otimes SU(3)$ symmetry and the weak interactions.

6. $SU(3)\otimes SU(3)$ and Weak and Electromagnetic Processes

Besides the application of the idea of an approximate $SU(3)\otimes SU(3)$ symmetry to strong interaction processes, one can try to go further and try to discuss the application of these ideas to weak and electromagnetic processes. The general idea behind all of the applications made to date is the following. Assume that the Hamiltonian describing the process in question is of the form

$$H = H_0 + \varepsilon H_1 + \alpha H_2 \tag{6.1}$$

where αH_2 stands for an arbitrary small term added to the Hamiltonian; which, for example, violates isospin and hypercharge conservation. It might be an effective weak interaction term or a second-order electromagnetic interaction term, etc. One then tries to prove theorems to lowest order in (αH_2) and various orders in 'ε'.

As an example let us study the decay $K^+ \to \pi^0 + l + \nu_e$ ($l = e$ or μ). If one accepts the usual current-current picture as giving a satisfactory phenomenological picture of the weak interactions one then has that the hadronic part for the amplitude for this process is given by the matrix element.

$$\langle K^+ | V^\mu_{K^+}(0) | \pi^0 \rangle (t) . \tag{6.2}$$

Phenomenologically, this matrix element is written in its most general form as

$$\langle K^+|V^\mu_{K^+}(0)|\pi^0\rangle(t) = (1/2)[(p_k+p_\pi)^\mu f_+(t) + (p_K - p_\pi)^\mu f_-(t)]\,. \tag{6.3}$$

By following the techniques outlined in the previous lectures we can prove the following two theorems about the divergence $\partial_\mu V^\mu_{K^+}$ whose matrix element from K^+ to π^0 is by Eq. (6.3) given by the general expression

$$\langle K^+|\partial_\mu V^\mu_{K^+}(0)|\pi^0\rangle(t) = (1/2)[(m_K^2 - m_\pi^2) f_+(t) + t f_-(t)] \tag{6.4}$$

where $t=(p_K-p_\pi)^2$.

Theorem 1: If we expand $\langle K^+|\partial_\mu V^\mu_{K^+}(0)|\pi^0\rangle(t)$ defined by Eq. (6.4) in powers of t, as follows:

$$\langle K^+|\partial_\mu V^\mu_{K^+}|\pi^0\rangle(t) \equiv a_0 + a_1 t + a_2 t^2 + \dots$$

Then, it can be shown that

$$\begin{aligned} a_0 &= (m_c^2 - m_\pi^2)/2 + O(\varepsilon^3) \\ a_1 &= (f_\pi/f_K - f_K/f_\pi)/2 + O(\varepsilon^2) \end{aligned} \tag{6.5}$$

which implies that the parameter 'ξ' defined by

$$\xi \equiv f_-(0)/f_+(0) \tag{6.6}$$

is given by

$$\xi = (f_\pi/f_K - f_K/f_\pi)/2 - ((m_K^2 - m_\pi^2)/m_\pi^2)\lambda_+ + O(\varepsilon^2) \tag{6.7}$$

where

$$\lambda_+ = m_\pi^2 \left.\frac{\mathrm{d}\ln f_+(t)}{\mathrm{d}t}\right|_{t=0}.$$

This theorem is an 'on-mass shell' theorem and provides an evaluation of ξ at $t=0$ correct to order ε^2. As we previously mentioned its importance lies in the fact that this particular prediction is independent of the nature of εH_1 and only depends upon εH_1 being *small enough* so that a perturbation expansion makes sense.

The proof of this theorem starts out by considering the matrix element $\langle 0|T(\partial_\mu A^\mu_{K^-}(q)\partial_\mu V^\mu_{K^+}(0)\partial_\mu A^\mu_{\pi^0}(k))|0\rangle$ and cancelling all single meson poles between this term and the equivalent term obtained by moving all derivatives through the T-ordering instruction. One then takes the limit $q^2=k^2=t=0$ and studies the behavior of the term linear in 't' (since the theorem on 'a_0' is just the Ademollo-Gatto theorem). Since the manipulations are a bit tedious, I leave them to you as an exercise [4].

The second theorem which we shall discuss somewhat more fully is:
Theorem 2: Assuming only $\varepsilon\mathscr{H}_1$ is small and that we have $\partial_\mu A^\mu_\alpha$ belonging to an $SU(3)$ octet of operators

$$[1-m_\pi^2/m_K^2]+\xi(m_K^2)=(1/f_+(m_K^2))(f_\pi/f_K)(1-c_\pi/c_K) \\ +O(SU(2)\otimes SU(2)\text{ breaking}) \tag{6.8}$$

where by assumption $\partial_\mu A^\mu_\alpha = c_\alpha P_\alpha$ (where the P_α's are a properly normalized octet of operators) and

$$\frac{c_\pi}{c_K}=\frac{m_\pi^2}{m_K^2}\left(\frac{f_K}{f_\pi}\right)\frac{\langle K^+|P_K+(0)|0\rangle}{\langle\pi^0|P_{\pi^0}(0)|0\rangle}. \tag{6.9}$$

The derivation of this equation is quite simple and we shall sketch how it goes. Begin with the matrix element

$$D(q^2,t)\equiv\langle K^+|T(\partial_\mu A^\mu_{\pi^0}(q)\partial_\mu V^\mu_{K^+}(0))|0\rangle \tag{6.10}$$

which, if we separate its one pi-meson pole term, can be written as:

$$D(q^2,t)=+i\left[\frac{m_\pi^2}{(2f_\pi)(q^2-m_\pi^2)}\langle K^+|\partial_\mu V^\mu_{K^+}(0)|\pi^0\rangle(t)+\bar{\partial}(q^2,t)\right]. \tag{6.11}$$

If we now extract the derivatives from the T-ordering instruction we get

$$D(q^2,t)=q_\mu(p_K-q)_\nu\langle K^+|T(A^\mu_{\pi^0}(-q)V^\nu_K+(0))|0\rangle \\ -(1/2)((q\cdot p_K)/2f_\pi)-\int d^4x\,e^{-iq\cdot x}\delta(x_0)\langle K^+|[A^0_{\pi^0}(x),\partial_\nu V^\nu_{K^+}(0)|0\rangle. \tag{6.12}$$

If we note that the first term on the right hand side has a term going as $(q^2-m_\pi^2)^{-1}$, whose residue is

$$(q^2/2f_\pi)\langle K^+|\partial_\nu V^\nu_{K^+}(0)|\pi^0\rangle(t) \tag{6.13}$$

then by separating this off explicitly and rearranging Eqs. (6.17) and (6.12) we get

$$\langle K^+|\partial_\nu V^\nu_{K^+}(0)|\pi^0\rangle(t)=+iq_\mu(p_K-q)_\nu\langle K^+|T(\bar{A}^\mu_{\pi^0}(-q)V^\nu_{K^+}(0))|\pi^0\rangle \\ +(1/2)[(q\cdot p_K)/2f_\pi]-i\int d^4x\,e^{-iq\cdot x}\delta(x_0)\langle K^+|[A^0_{\pi^0}(x),\partial V_{K^+}]|0\rangle \\ +\bar{\bar{\partial}}(q^2,t). \tag{6.14}$$

Since the left hand side is independent of q^2 we can evaluate Eq. (6.14) at the point $q^2=0$, $t=m_K^2$ (which is equivalent to taking $q^\mu\to 0$) and we get

$$\langle K^+|\partial_\nu(0)|\pi^0\rangle(m_K^2)=-i\langle K^+|[Q^5_{\pi^0},\partial_\nu V^\nu_{K^+}(0)|0\rangle+\bar{\bar{\partial}}(0,m_K^2). \tag{6.15}$$

If we now combine Eq. (6.4) with the equations

$$\partial_\mu A^\mu_\alpha = c_\alpha P_\alpha \tag{6.16}$$

$$\partial_\mu V^\mu_\alpha(0) = +i[Q_\alpha, \varepsilon\mathscr{H}_1(0)] \tag{6.17}$$

$$\partial_\mu A^\mu_\alpha(0) = +i[Q^5_\alpha, \varepsilon\mathscr{H}_1(0)] \tag{6.18}$$

and

$$[Q_\alpha, P_\beta] = i f_{\alpha\beta\gamma} P_\gamma$$

we get

$$\begin{aligned}(1 - m_\pi^2/m_K^2) + \xi(m_K^2) = (1/f_+(m_K^2))\,(f_\pi/f_K)\,(1 - c_\pi/c_K) \\ + (2f_\pi/m_K^2 f_+(m_K^2))\bar{\partial}(0, m_K^2)\,.\end{aligned} \tag{6.19}$$

If one then notes that $\bar{\partial}(0, m_K^2)$ is first order in $SU(2)\otimes SU(2)$ breaking while the other terms are order zero in this parameter we get the required result.

With this discussion out of the way we are left with the interesting question of how well these formulae agree with experiment. In order to make this comparison allow me to use the preliminary results of a UCLA-John Hopkins-SLAC collaboration reported at the last Washington meeting of the American Physical Society. My reason for choosing this experiment is simply that due to the intense kaon beams which they had to work with their studies of the processes

$$\mathrm{K} \to \pi + l + \nu$$

allowed them to amass an order of magnitude more events. Their preliminary results for ξ and λ_+ were

$$\lambda_+ = 0.08 \pm 0.01$$

$$\xi = -0.45 \pm 0.09\,.$$

If one uses their value for λ_+; and determines the ratio

$$(f_+^{(\pi)}(0)/f_+^{(K)}(0))(f_\pi/f_K) = x$$

from the experiments;

$$K^+ \to \pi^0 + l + \nu$$

$$\pi^+ \to \pi^0 + l + \nu$$

$$\pi \to \mu + \nu$$

and

$$K \to \mu + \nu$$

(since this ratio is independent of the Cabibbo angle) we get substituting these results into Eq. (6.8)

$$(1-0.08)+\xi(m_K^2)\cong\frac{1.4}{1+(0.08)12} \tag{6.20}$$

or

$$\xi\cong-0.03\,M_0$$

where everything has been corrected for the change in λ_+. Moreover, Eq. (6.7) gives $\xi(0)=-0.5$ so the agreement is very good indeed.

Thus, if this new experiment is to be believed, things look very good indeed for the subject of K_{l3} decays, although it means that $SU(3)$ violation is on the order of 30% or 40% in 'x'.

The other favorite puzzle that people think of as a trouble which arises when one combines chiral and current algebra has to do with the decay $\eta\to3\pi$. The problem here, succinctly stated, is that one can prove that under certain assumptions the process does not go to order α (where α is the fine structure constant) but it is suppressed and is of order $\varepsilon\alpha$. In fact of course the rate experimentally seems to be 50 times larger than it should be even if it goes in order α.

If, in fact, as I hope-all other predictions of chiral $SU(3)\otimes SU(3)$ are going to be fairly good; then, there is reason to think that it is our understanding of how one should calculate processes involving both $SU(3)\otimes SU(3)$ breaking and electromagnetism that is at fault. In fact, there is some reason to believe this might be true. But, that it a story for another time and another place.

To sum up then, I have tried to present to you what I believe to be a very pretty picture of the strong interactions-along with a possible experimental program which might in the near future show how correct these ideas are. There is clearly much more room for trying to suggest further, and possibly better, experimental checks of these ideas-and there is still much to be understood about how a chiral symmetry of this type works.

References

1. Nambu, Y., Jona-Lasinio, G.: Phys. Rev. **122**, 345 (1961).
2. Dashen, R., Weinstein, M.: Phys. Rev. **183**, 1261 (1969).
3. Weinstein, M.: Phys. Rev. **188**, 2330 (1969).
4. — Phys. Rev. Lett. **22**, 1337 (1969).

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